

A Nodal Decic Counterexample to the Generic Projection Incidence Bound

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Abstract

We construct a degree-ten hypersurface in complex projective four-space with exactly sixty ordinary double points whose general plane projection lies on a sextic. Thus the projected nodes violate the proposed bound $k(n-1)$, since $60 > 6(10-1) = 54$, even though $60 < (10-1)^2$. This answers negatively the projection question in Cheltsov's Problem 18 in an AIM workshop list and gives a counterexample to Cheltsov–Park Conjecture 1.6. The node set satisfies the stronger ambient incidence bound $7k$ for curves of degree k . We use the known sextic-projection property of the projectivized H_4 roots, realize those points as nodes of an explicit symmetric decic surface, and remove every unwanted singularity after extending to a threefold. Exact Hessian, Gröbner-basis and interpolation certificates accompany the construction. The threefold is factorial; the separate sharp factoriality theorem is not contradicted.

1 The projection assertion

We work over $\mathbb{C}$. In the AIM list on numerical invariants of singularities, Cheltsov's Problem 18 asks, among other questions, whether for a nodal hypersurface $X_n \subset \mathbb{P}^4$ and a general linear projection $\pi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ one has

$$\#(\pi(\text{Sing } X_n) \cap B) \leq k(n-1) \quad \text{for every plane curve } B \text{ of degree } k. \quad (1)$$

The frozen Hugging Face record AIM-ALGEBRAIC_GEOMETRY-0318 transcribes this as item 8; it is item 18 on page 3 of the original list [1]. The same list also asks about factoriality below $(n-1)^2$ nodes. That threshold was proved by Cheltsov [3]; it is not an open theorem claimed here.

A related assertion appears in [2, Conjecture 1.6]. For a finite set $\Sigma \subset \mathbb{P}^4$, it assumes

$$\#\Sigma \leq (n-1)^2 - 1, \quad \#(\Sigma \cap C) \leq k(n-1) \quad \text{for every curve } C \subset \mathbb{P}^4 \text{ of degree } k, \quad (2)$$

and predicts the same incidence bound after a general plane projection. The realization by nodes is not one of its hypotheses. Our construction satisfies both the geometric realization requirement and the numerical hypotheses (2).

Theorem 1. *There is a reduced irreducible hypersurface $X \subset \mathbb{P}^4$ of degree ten whose singular locus Z consists of exactly sixty ordinary double points, with the following properties.*

1. *For a general linear projection $\pi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$, the sixty distinct points of $\pi(Z)$ lie on a plane sextic.*

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2. Every projective curve $C \subset \mathbb{P}^4$ satisfies $\#(Z \cap C) \leq 7 \deg C$.

3. The threefold X is factorial.

Consequently (1) is false, including under the additional hypothesis $\# \text{Sing } X_n < (n-1)^2$, and [2, Conjecture 1.6] is false.

The point configuration and its sextic-projection property are known [4, §3.7], [5, §5.4]. The construction below uses them as inputs. Its additional step is a nodal hypersurface realization with exactly those points as the singular locus. No assertion about a complete classification of factorial nodal hypersurfaces is made.

2 The known sixty-point configuration

Put $\phi = (1 + \sqrt{5})/2$. Let $Z \subset \mathbb{P}^3$ consist of the projective classes of the following vectors:

$$\begin{aligned} & (1, 0, 0, 0) \quad \text{and its coordinate permutations,} \\ & (1, \epsilon_1, \epsilon_2, \epsilon_3), \quad \epsilon_i \in \{1, -1\}, \\ & \text{even coordinate permutations of } (0, \epsilon_1, \epsilon_2\phi, \epsilon_3(\phi-1)), \quad \epsilon_i \in \{1, -1\}. \end{aligned} \tag{3}$$

Opposite vectors in the last row are identified. The three rows give 4, 8 and 48 points respectively. These are the projectivized roots of H_4 . For comparison with the coordinates in [4, §3.7], multiplication of $(0, 1, \phi, \phi-1)$ by ϕ gives $(0, \phi, \phi^2, 1)$.

The result from [4, §3.7] needed here is that, for a general point $P \in \mathbb{P}^3$, a sextic cone with vertex P contains Z . Equivalently, the general plane projection of Z lies on a sextic. In fact the projected set is a complete intersection of degrees 6 and 10; this is the $(6, 10)$ -geproci property discussed in [5, §5.4, pp. 67–68]. We use only the existence of the sextic.

Embed this $\mathbb{P}^3$ as the hyperplane $H = \{w = 0\} \subset \mathbb{P}^4$. It remains to realize Z as the full singular set of a nodal decic, while retaining the numerical incidence hypotheses.

3 An explicit auxiliary decic surface

For a four-part exponent tuple λ , write $m_\lambda(y)$ for the sum of the distinct monomials obtained by permuting its exponents among y_0, y_1, y_2, y_3 . Define

$$\begin{aligned} F(y) &= -m_{4100}(y) + m_{3200}(y) + 17m_{3110}(y) \\ &\quad - 24m_{2210}(y) + 21m_{2111}(y), \\ f(x) &= F(x_0^2, x_1^2, x_2^2, x_3^2). \end{aligned} \tag{4}$$

Thus f is homogeneous of degree ten and invariant under all signed coordinate permutations.

Lemma 2. *The surface $S = V(f) \subset H$ has finite singular locus T , and every point of Z is an ordinary double point of S .*

Proof. Signed permutation symmetry reduces the nodality assertion to three representatives. Substitution in f and its four first derivatives gives zero. The determinants of the three-variable affine Hessians are as follows; the named chart coordinate is fixed equal to one.

Point	Chart	Hessian determinant
$(1, 0, 0, 0)$	$x_0 = 1$	-8
$(1, 1, 1, 1)$	$x_0 = 1$	4194304
$(0, 1, \phi, \phi - 1)$	$x_1 = 1$	-13107200

These exact computations over $\mathbb{Q}(\sqrt{5})$ prove nondegeneracy. Using all signed permutations for this check does not assert that every such permutation belongs to the specified H_4 subset; it does cover every point of that subset.

For finiteness, stratify projective space by the number m of nonzero coordinates. By a coordinate permutation and projective scaling we may write the squared coordinates as

$$y_0 = 1, \quad y_m = \cdots = y_3 = 0.$$

For an active coordinate, $\partial f / \partial x_i = 2x_i F_{y_i}$, so singularity requires $F_{y_i} = 0$ for $0 \leq i < m$. The squaring map has finite fibers. It is enough that the resulting affine solution set in $y_1, \dots, y_{m-1}$ be finite.

For $m = 1$ there is just one point on a fixed coordinate support. For $m = 2, 3, 4$, reduced Gröbner bases of the ideals generated by the active F_{y_i} , after the above substitution, are respectively

$$\{y_1 - 1\}, \tag{5}$$

$$\left\{ \begin{array}{l} y_2^4 - 10y_2^3 + 23y_2^2 - 10y_2 + 1, \\ y_1 y_2^2 - 3y_1 y_2 + y_1 - \frac{1}{3}y_2^3 + \frac{2}{3}y_2^2 + \frac{2}{3}y_2 - \frac{1}{3}, \\ y_1^2 - 2y_1 y_2 - 2y_1 + y_2^2 - 2y_2 + 1 \end{array} \right\}, \tag{6}$$

and

$$\{y_1 - 1, y_2 - 1, y_3 - 1\}. \tag{7}$$

The order is graded reverse lexicographic, with $y_1 > \cdots > y_{m-1}$, and the coefficient field is $\mathbb{Q}$. In (6) the leading monomials are $y_2^4, y_1 y_2^2, y_1^2$. In every case the leading ideal contains a positive power of each variable, so the solution set is finite over $\mathbb{C}$. This is not a finite-field point-count argument. Euler's homogeneous identity makes the equation $f = 0$ redundant when its gradient vanishes. The finitely many coordinate supports therefore account for a finite singular locus. $\square$

The proof needs neither the cardinality of T nor the singularity types of its points outside Z . The remaining singularities will be removed in the next ambient dimension.

4 Interpolation and the spatial incidence bound

Lemma 3. *Degree-six forms on $\mathbb{P}^3$ impose independent values at the sixty points of Z .*

Proof. Here is an exact nonzero-minor certificate. Reduce the coordinates in $\mathbb{Z}[\phi]$ modulo 101, choosing $\phi = 23$, for which $23^2 - 23 - 1 = 0$ in $\mathbb{F}_{101}$. Normalize each projective vector so that its first nonzero coordinate is one. These normalizations are valid before and after reduction, since $1, \phi, \phi - 1$ are units. The reduced configuration still has sixty distinct points.

Order the reduced normalized vectors lexicographically, using residues $0, \dots, 100$, and order the degree-six monomials by increasing lexicographic exponent tuple (a_0, a_1, a_2, a_3) . Their evaluation matrix has size 60 by 84. The submatrix with the following zero-based column indices has determinant 39 modulo 101:

$$\begin{aligned} &0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, \\ &24, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 45, 49, \\ &50, 51, 52, 53, 54, 55, 56, 57, 58, 60, 64, 65, 66, 67, 68, 70, 74, 83. \end{aligned} \tag{8}$$

The accompanying program constructs the matrix directly from (3) and recomputes the determinant by two exact implementations. Its nonzero reduction proves that the lifted minor is nonzero over $\mathbb{Q}(\sqrt{5})$, hence over $\mathbb{C}$. Surjectivity follows. $\square$

We record an elementary consequence to control the base locus, rather than inferring basepoint-freeness just from a dimension count.

Lemma 4. *Let Z be a finite reduced subset of $\mathbb{P}^r$ over an algebraically closed field. If degree- d forms interpolate on Z , then the common zero set of $I_Z(d+1)$ is exactly Z .*

Proof. Fix $q \notin Z$ and choose a linear form ℓ nonzero at $Z \cup \{q\}$. In the affine chart $\ell = 1$, interpolation supplies polynomials L_i of degree at most d with $L_i(p_j) = \delta_{ij}$. Suppose every polynomial of degree at most $d+1$ vanishing on Z also vanished at q . Applying this to $\sum_i L_i - 1$ gives $\sum_i L_i(q) = 1$. Applying it to $(u_j - (p_i)_j)L_i$ gives

$$(q_j - (p_i)_j)L_i(q) = 0 \quad \text{for all } i \text{ and all affine coordinates } j.$$

Some $L_i(q)$ is nonzero, forcing $q = p_i$, a contradiction. Homogenizing with ℓ proves the assertion. $\square$

By Lemmas 3 and 4, $I_Z(7)$ has base locus exactly Z in H , and the same holds for $I_Z(9)$. For the latter statement, multiply a degree-seven separating form by the square of a linear form nonzero at the point to be separated. In $\mathbb{P}^4$ the degree-seven forms vanishing on Z also have base locus exactly Z : extend forms from H to separate points in H , and use w^7 to separate points outside H .

Let $C \subset \mathbb{P}^4$ be any projective curve. For each irreducible component of its support, some degree-seven form in I_Z does not vanish identically on that component. A finite union of proper linear subspaces cannot fill $I_Z(7)$, so one such form avoids all components simultaneously. Bézout's theorem on each component gives

$$\#(Z \cap C) \leq 7 \deg C. \tag{9}$$

Allowing nonreduced curves does not weaken this inequality for the number of distinct points, since their degree is at least that of their reduced support.

5 A threefold with exactly the prescribed nodes

Proposition 5. *There are a degree-nine form g on H and a degree-eight form h on $\mathbb{P}^4$ such that*

$$X = V(f(x) + wg(x) + w^2h(x, w)) \tag{10}$$

has singular locus exactly Z , and all its singularities are ordinary double points.

Proof. The finite set $T = \text{Sing } S$ contains Z . Since $I_Z(9)$ has no base point outside Z , choose $g \in I_Z(9)$ nonzero at every point of $T \setminus Z$. This only excludes finitely many proper linear subspaces of the space of possible g .

Fix this g and let h vary freely. On H , equation (10) restricts to $f = 0$. At a smooth point of S , some x derivative is nonzero. At a point of $T \setminus Z$, the w derivative is $g(p) \neq 0$. Thus no point of $H \setminus Z$ is singular on X .

At $p \in Z$, use an affine chart within which H has local coordinates u_1, u_2, u_3 and its normal coordinate is w . Both f and g vanish at p , with $df(p) = 0$. The quadratic term of the equation is

$$q_p(u) + w\lambda_p(u) + h(p)w^2, \tag{11}$$

where q_p is a nondegenerate quadratic form by Lemma 2 and λ_p is linear. The determinant of its symmetric matrix is affine in $h(p)$, with nonzero coefficient $\det(q_p)$. Evaluation $h \mapsto h(p)$ is a nonzero linear functional on the space of degree-eight forms. Consequently a proper affine hyperplane of parameters is excluded at each of the sixty points; outside their union all are nodes.

It remains to exclude singularities off H . On the chart $w = 1$, the restrictions of degree-eight homogeneous forms h are all polynomials of degree at most eight in four variables. At each fixed $q \in \mathbb{A}^4$, the value and four first derivatives of h can be prescribed independently, already using constant and linear polynomials. Write N for the dimension of the affine space of parameters h . The five equations saying that $f + g + h$ vanishes and has zero differential at q therefore impose five independent affine conditions on h . The incidence variety of such pairs (h, q) has dimension $N - 5 + 4 = N - 1$. The closure of its image in parameter space has dimension at most $N - 1$ and is proper. Avoiding it and the finitely many nodality hyperplanes gives the required h . $\square$

The hypersurface in Proposition 5 is reduced and irreducible. Indeed, a repeated factor would give a positive-dimensional singular locus. Two distinct positive-degree hypersurface components in $\mathbb{P}^4$ meet in dimension at least two, and their intersection is singular on their union. Either possibility contradicts the finite singular locus. Thus the construction is a nodal threefold, not merely an abstract incidence configuration.

6 Projection, the two counterexamples and factoriality

Let L be a general line in $\mathbb{P}^4$. It meets H at a general point P . A linear map defining projection from L restricts on H to a surjective map with one-dimensional vector-space kernel corresponding to P . Its restriction is therefore the general plane projection from P , up to target coordinates.

For a general P this projection is defined and injective on Z : exclude the points of Z and their finitely many secant lines. By the known H_4 property in Section 2, the projected points lie on a sextic B . Hence

$$\#(\pi(\text{Sing } X) \cap B) = 60 > 6(10 - 1) = 54.$$

This disproves (1). At the same time, $60 \leq 80 = (10 - 1)^2 - 1$ and (9) imply all the hypotheses (2) with $n = 10$, so this is also a counterexample to [2, Conjecture 1.6].

Finally, [3, Theorem 1.4] proves factoriality of nodal degree- n threefold hypersurfaces with fewer than $(n-1)^2$ nodes, so X is factorial. One can also use [2, Theorem 1.1 and the following paragraph]: factoriality is equivalent to independent conditions at the nodes in degree $2n - 5$. Lemma 3 implies independence in degree $15 = 2 \cdot 10 - 5$ by multiplying interpolants by ℓ^9 , where ℓ is nonzero on Z . Forms on H extend to $\mathbb{P}^4$. The degree n in the workshop's statement of this criterion is a source error; the criterion used here is in degree $2n - 5$. This completes the proof of Theorem 1.

7 Exact calculation files

The accompanying `reproducibility/verify.py` constructs (4) from its monomial-symmetric definition and checks the three Hessian determinants over $\mathbb{Q}(\sqrt{5})$, the three Gröbner bases over $\mathbb{Q}$, and the sixty-by-sixty minor (8) over $\mathbb{F}_{101}$. It uses Python and SymPy and contains no randomized search. Normal and optimized Python runs give identical mathematical reports. The program verifies these finite algebraic certificates; it is not a proof-assistant verification of the geometric argument or of the cited projection theorem.

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finite checks and the written proof are available for scrutiny; no independent review, formal certification or absolute priority is claimed.

References

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