

Presentation Independence of Ambient Adjoint Cartier Algebras

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Abstract

Let $R = S/I$ be an integral algebra of finite type over a perfect field of characteristic $p > 0$, with S polynomial and I of height c . We give a direct proof that the restriction to R of the degree- e Cartier maps with coefficients in $I^{c(p^e-1)}$ is independent of the polynomial presentation, in every degree. The key calculation extracts one coefficient after adjoining a dummy variable; graph presentations then compare arbitrary embeddings. The resulting Cartier algebra, its ideal-pair twists, and their test ideals localize and glue without a $\mathbb{Q}$ -Gorenstein assumption. We also explain how the established adjoint-ideal and restriction theorems identify these test ideals with reductions of Mather–Jacobian multiplier ideals for each fixed rational pair on sufficiently general closed fibres. The ambient construction and reduction comparison are credited to prior work; this note makes no absolute-priority claim.

1 The construction and its scope

The question attributed to Ishii in the AIM workshop on test and multiplier ideals asks for a test ideal related to the Mather discrepancy multiplier ideal without a $\mathbb{Q}$ -Gorenstein hypothesis [1, Problem 2.15]. The terminology in Ein–Ishii–Mustață [4] includes the Jacobian correction; we use the name *Mather–Jacobian*, abbreviated MJ, to distinguish it from a construction using only the Mather discrepancy.

The ambient adjoint test-ideal construction is already present in Takagi [2]. Smolkin [3, Example 4.8, Proposition 4.10, Theorem 5.2] treats its Cartier algebra, restriction, and comparison with adjoint ideals. Our purpose is to record explicitly that the restricted algebra itself is independent of a polynomial presentation at each fixed characteristic. The proof is an elementary stabilization argument. The characteristic-zero comparison below is a consequence of the cited results, not a new comparison theorem.

Fix a perfect field k of characteristic $p > 0$. A degree- e Cartier map on a k -algebra B is an element of $\text{Hom}_B(F_*^e B, B)$, equivalently an additive map $\phi : B \rightarrow B$ satisfying $\phi(b^{p^e} x) = b\phi(x)$. We use the latter notation when composing maps. Right multiplication $\phi \cdot a$ means input multiplication: $(\phi \cdot a)(x) = \phi(ax)$.

Let R be a finitely generated integral k -algebra. Choose a surjection $S = k[x_1, \dots, x_n] \rightarrow R$ with prime kernel I and $c = \text{ht } I$. For $e \geq 1$ and $q = p^e$, set

$$\mathcal{D}_e(S, I) = \sum_{a \in I^{c(q-1)}} \text{Hom}_S(F_*^e S, S) \cdot a, \quad \mathcal{C}_e(S, I) = \mathcal{D}_e(S, I)|_R. \quad (1)$$

Set $\mathcal{D}_0 = S$ and $\mathcal{C}_0 = R$. Powers of exponent zero mean the whole ring, also when $I = 0$. We first justify the restriction in (1).

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Theorem 1.1. *The algebra $\bigoplus_{e \geq 0} \mathcal{C}_e(S, I)$ is independent of the polynomial presentation, as an actual graded subalgebra of $\bigoplus_{e \geq 0} \text{Hom}_R(F_*^e R, R)$. Denote it by $\mathcal{C}(R)$. For every nonzero ideal $\mathfrak{a} \subseteq R$ and real $t \geq 0$, the pieces*

$$\mathcal{C}(R, \mathfrak{a}^t)_e = \mathcal{C}_e(R) \cdot \mathfrak{a}^{\lceil t(p^e - 1) \rceil} \quad (2)$$

form a nondegenerate Cartier algebra, independent of the choice of ambient lifts of $\mathfrak{a}$. These algebras and their test ideals commute with localization. Consequently they define sheaves on integral finite-type k -schemes.

Here the test ideal of a nondegenerate Cartier algebra on a domain is the smallest nonzero ideal stable under all its maps. Its existence is the standard test-ideal theorem recalled in [3, §2]. No normality or $\mathbb{Q}$ -Gorenstein assumption on R is imposed.

2 Compatibility and stabilization

Lemma 2.1. *Every map in $\mathcal{D}_e(S, I)$ preserves I . The graded pieces $\mathcal{D}_e(S, I)$ are closed under Cartier composition, and their restriction to R is nondegenerate.*

Proof. These assertions specialize [3, Example 4.8]; we include the elementary argument. To test whether an element of S belongs to the prime I , it suffices to localize at I . The regular local ring S_I has dimension c and maximal ideal $\mathfrak{m}$ generated by c parameters. Pigeonhole counting gives

$$\mathfrak{m}^{c(q-1)+1} \subseteq \mathfrak{m}^{[q]},$$

where $\mathfrak{m}^{[q]}$ is generated by the q -th powers of its elements. Every degree- e Cartier map sends $\mathfrak{m}^{[q]}$ into $\mathfrak{m}$. Thus $\phi(aI) \subseteq I$ for $a \in I^{c(q-1)}$.

If ϕ has degree e , ψ has degree f , and $r = p^f$, then

$$(\phi \cdot a) \circ (\psi \cdot b) = (\phi \circ \psi) \cdot (a^r b). \quad (3)$$

Indeed $a\psi(bx) = \psi(a^r bx)$. The coefficient exponent satisfies

$$rc(q-1) + c(r-1) = c(qr-1),$$

so composition stays in the required piece. Degree-zero actions and finite sums cause no difficulty.

For nondegeneracy, take regular parameters $u_1, \dots, u_c$ in S_I . The element $v = (u_1 \cdots u_c)^{q-1}$ is not in $\mathfrak{m}^{[q]}$. Since $F_*^e S_I$ is free, some linear functional has value outside $\mathfrak{m}$ on $F_*^e v$. Clearing denominators gives a global map in $\mathcal{D}_e(S, I)$ with nonzero restriction to R . When $c = 0$, one has $I = 0$, and the claims follow directly from the full Cartier algebra of the polynomial ring. $\square$

Lemma 2.2. *Let $T = S[z]$ and $J = IT + (z)$. Under $T/J = R$ one has $\mathcal{C}_e(T, J) = \mathcal{C}_e(S, I)$ for every $e \geq 0$.*

Proof. For $e \geq 1$, the standard polynomial trace Φ_e^S is given by

$$\Phi_e^S(\lambda x^\alpha) = \begin{cases} \lambda^{1/q} x^{(\alpha - (q-1)\mathbf{1})/q}, & \alpha_i \equiv q-1 \pmod{q} \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Perfectness of k makes the coefficient root unambiguous. This map generates $\text{Hom}_S(F_*^e S, S)$ as a right S -module. The trace on T is the product trace in x and z .

Put $M = (c + 1)(q - 1)$. Every element of $\mathcal{D}_e(T, J)$ is of the form $\Phi_e^T \cdot h$ with $h \in J^M$. Write $h = \sum_j h_j z^j$. For $b \in S$, evaluating the output at $z = 0$ yields

$$\Phi_e^T(hb) \Big|_{z=0} = \Phi_e^S(h_{q-1}b). \quad (4)$$

Terms with exponent not congruent to $q - 1$ are killed. An exponent $q - 1 + \ell q$ with $\ell > 0$ gives a factor z^ℓ , which vanishes. Moreover

$$J^M = \sum_{j=0}^M z^j I^{M-j} T \implies h_{q-1} \in \sum_{j=0}^{q-1} I^{M-j} = I^{c(q-1)}. \quad (5)$$

Restricting (4) modulo I proves one inclusion. Conversely, for every $a \in I^{c(q-1)}$, the coefficient $h = z^{q-1}a$ lies in J^M and recovers $\Phi_e^S \cdot a$ on R . This proves the reverse inclusion. Degree zero is immediate. $\square$

3 Changing presentation and localizing

Proof of Theorem 1.1. Suppose $S = k[x_1, \dots, x_n]$ and $S' = k[y_1, \dots, y_m]$ both surject onto R , with kernels I and I' . The common ring $T = k[x, y]$ has kernel K under the combined map to R . Choose $f_j(x)$ lifting the image of y_j . Then

$$K = (I, y_1 - f_1(x), \dots, y_m - f_m(x)).$$

A triangular change of variables identifies this quotient presentation with $(S[y], (I, y))$ and induces the identity on R . Ideal powers, the full Cartier Hom modules, and restriction transport under that isomorphism. Repeated application of Lemma 2.2 therefore identifies $\mathcal{C}(T, K)$ with $\mathcal{C}(S, I)$. Exchanging x and y identifies the same algebra with $\mathcal{C}(S', I')$. These are equalities inside the Cartier algebra of the specified ring R ; we have not assumed that chosen trace generators are invariant under a coordinate change.

Next, let $\tilde{\mathfrak{a}} \subseteq S$ map onto $\mathfrak{a}$, and put $m_e = \lceil t(q - 1) \rceil$. Its m_e -th power maps onto $\mathfrak{a}^{m_e}$. If two coefficient lifts differ by an element of I , their difference, followed by any I -compatible map, restricts to zero. Hence restriction of $\mathcal{D}_e(S, I) \cdot \tilde{\mathfrak{a}}^{m_e}$ is exactly (2). Twisted composition is closed because

$$r \lceil t(q - 1) \rceil + \lceil t(r - 1) \rceil \geq \lceil t(qr - 1) \rceil. \quad (6)$$

At the generic point, the nonzero ideal $\mathfrak{a}$ becomes the unit ideal, so Lemma 2.1 gives nondegeneracy and thus existence of the test ideal.

Let $0 \neq s \in R$ and choose a lift $\tilde{s} \in S$. Frobenius is finite, so localization of its Hom module agrees with the Hom module of the localization. Ideal powers, images, and the restriction maps localize as well. It remains to compare the resulting localized ambient construction with a polynomial presentation of R_s . Use

$$R_s = S[u]/(I, \tilde{s}u - 1).$$

After inverting $\tilde{s}$, the coordinate $v = u - \tilde{s}^{-1}$ identifies the new kernel with (I, v) . The coefficient proof of Lemma 2.2 works over $S_{\tilde{s}}$ with the localized trace and gives the desired equality. Localizing further on the quotient makes no change, since s is already a unit there. Presentation independence now identifies this with $\mathcal{C}(R_s)$.

The same argument applies to twists. Test ideals localize by the standard theorem [3, Lemma 2.11(c)]. Principal affine opens form a basis, and the equalities are inside the intrinsic Cartier Hom sheaves. Thus the algebras, twists, and test ideals glue. $\square$

4 The known comparison with MJ multiplier ideals

We record the precise comparison furnished by existing adjunction results. For a log resolution $\pi : Y \rightarrow X$ factoring through the Jacobian blowup, write $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-Z)$ and $\text{Jac}_X \mathcal{O}_Y = \mathcal{O}_Y(-J_{Y/X})$. The MJ multiplier ideal is

$$\mathcal{J}_{\text{MJ}}(X, \mathfrak{a}^t) = \pi_* \mathcal{O}_Y(\widehat{K}_{Y/X} - J_{Y/X} - \lfloor tZ \rfloor).$$

In particular the correction $-J_{Y/X}$ is part of the definition.

Corollary 4.1 (Comparison from prior work). *Let X be an integral variety over an algebraically closed field of characteristic zero, let $\mathfrak{a} \neq 0$ be an ideal sheaf, and fix $t \in \mathbb{Q}_{\geq 0}$. After spreading out to a suitable finitely generated integral $\mathbb{Z}$ -algebra, there is a dense open subset of the base such that every closed fibre X_μ over it satisfies*

$$\mathcal{J}_{\text{MJ}}(X, \mathfrak{a}^t)_\mu = \tau(X_\mu, \mathcal{C}(X_\mu), \mathfrak{a}_\mu^t). \quad (7)$$

The descent data and open subset may depend on the fixed pair and t .

Proof. Work on an affine chart embedded in a smooth affine space A with prime ideal I and codimension $c \geq 1$, adding a dummy variable if necessary. Choose a factorizing resolution $\rho : \overline{A} \rightarrow A$ that also resolves an ambient lift $\tilde{\mathfrak{a}}$ and the pullback of the Jacobian ideal. Write

$$I\mathcal{O}_{\overline{A}} = I_{\overline{X}}\mathcal{O}_{\overline{A}}(-E), \quad \tilde{\mathfrak{a}}\mathcal{O}_{\overline{A}} = \mathcal{O}_{\overline{A}}(-Z), \quad D = K_{\overline{A}/A} - cE - \lfloor tZ \rfloor.$$

Lemma 2.12 of [4] identifies the restriction of $K_{\overline{A}/A} - cE$ with $\widehat{K}_{\overline{X}/X} - J_{\overline{X}/X}$. The vanishing and exact sequence in the proof of its Proposition 2.13 make restriction onto $\rho_{X*}\mathcal{O}_{\overline{X}}(D|_{\overline{X}})$ surjective. Consequently

$$(\rho_*\mathcal{O}_{\overline{A}}(D))\mathcal{O}_X = \mathcal{J}_{\text{MJ}}(X, \mathfrak{a}^t). \quad (8)$$

The ambient ideal here is the adjoint ideal of $(A, I^c\tilde{\mathfrak{a}}^t)$ along X used in [5, 3]. To see the agreement of formulas, blow up the smooth strict transform $\overline{X}$. For its exceptional divisor F , the relative canonical term contributes $(c-1)F$, the coefficient cX contributes $-cF$, and the adjoint correction contributes F . They cancel, leaving the pullback of D ; pushing down gives $\mathcal{O}_{\overline{A}}(D)$.

Apply [3, Theorem 5.2] to the regular ambient ring with formal product $I^c\tilde{\mathfrak{a}}^t$. At I , these two ideals have orders 1 and 0, so their weighted orders add to c , exactly as required by its Setting 5.1. Proposition 4.10 of the same paper identifies the restriction of its adjoint test ideal with the test ideal of the restricted Cartier algebra (2). Combining this with (8) proves (7) after shrinking the descent base so that the finitely many relevant images commute with reduction. Geometric integrality of X ensures integral fibres after a further shrinking. A finite affine cover and Theorem 1.1 give the global assertion. $\square$

The fixed-characteristic algebra of Theorem 1.1 is defined at every prime. Corollary 4.1 does not identify it with every resolution-defined ideal at all small primes, or give a single open set valid for all real exponents. It concerns the MJ interpretation of the AIM question, not the pure-Mather variant. Arbitrary imperfect bases and general non-finite-type F -finite rings are outside the theorem's scope. We do not count the original open-ended AIM record as wholly resolved by this note.

The accompanying source archive contains the proof audit and an exact sparse-polynomial checker for finite trace, graph-change, composition, and ceiling examples. These are regression checks, not a proof of the general theorem. The ambient construction and comparison results

belong to the cited literature; novelty of the elementary independence observation is not certified. This preprint is unrefereed. AI-assisted tools supported research, computation, proof development, and manuscript preparation; the author is responsible for the claims and final text. No independent human review or proof-assistant verification is asserted.

References

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