

A Pivot-Independent Toric Model for Framed Carlsson–Weinshall Duality

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Abstract

An ordered projective anchor simplex and n further marks in its complement determine a configuration torus modulo diagonal translation. Choosing a pivot mark identifies this quotient with a product of tori, which has a product-permutohedral compactification. We construct a pivot-independent normal projective model dominating all these compactifications. Its fan is the arrangement of rectangle characters, or equivalently the normal fan of an explicit lattice zonotope. Inversion extends regularly; in dimension three it gives a biregular Carlsson–Weinshall duality on the closures of the reduced-measurement graphs. For ambient dimension $d \geq 1$ and $n \geq 2$ marks, the common model is smooth exactly when $d = 1$ or $n = 2$. Otherwise it contains a four-dimensional cone with five extreme rays and is not $\mathbb{Q}$ -factorial. The result concerns an anchor-framed torus chart and its minimal common toric model, not a stable compactification of unframed camera moduli.

1 Scope and the framed quotient

The AIM workshop list on algebraic vision asks for a moduli-space interpretation of Carlsson–Weinshall (CW) duality and suggests a framed multiview variety [1, Problem 5.3]. We give one concrete construction on an anchor-framed open locus. The classical duality is not a new claim: Trager–Hebert–Ponce give a coordinate-free formulation [8, Lemma 2.2 and Propositions 2.3–2.5], and Connelly–Duff–Loucks–Tavitas describe reduced image formation as a rational quotient [2, Theorem 12].

The issue considered here is the dependence of a boundary model on a normalizing mark. Taking the minimal normal toric model that dominates every such model removes that dependence. The principal additional calculation is a sharp smoothness criterion for the resulting family. The ingredients from toric geometry are standard. We make no claim that the resulting boundary classifies stable degenerate cameras, or that it answers every interpretation of the workshop question.

Work over an algebraically closed field k of characteristic zero. Let $d \geq 1$, and let $I = \{0, \dots, d\}$ index ordered anchors $Z = (z_i)_{i \in I}$ in $\mathbb{P}^d$ whose representatives form a basis of k^{d+1} . Write

$$T_Z = \mathbb{P}^d \setminus \bigcup_{i \in I} \text{span}\{z_j : j \neq i\}.$$

Let L be an ordered label set of size $n \geq 2$, and take marks $y_l \in T_Z$, $l \in L$. Marks may coincide. Let $U_{d,L}$ be this configuration space, including the anchors, with the diagonal action of PGL_{d+1} . Sending the anchors to the coordinate vertices leaves the diagonal stabilizer

$$T = (\mathbb{G}_m)^{d+1} / \mathbb{G}_m.$$

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Changing the lifts of the anchors translates all marks by the same element of T .

Proposition 1.1. *The action on $U_{d,L}$ is free, and its geometric quotient is*

$$S_{d,L} = T^L / T_{\text{diag}}, \quad \dim S_{d,L} = d(n-1).$$

For any $p \in L$, the ratios $x_l = y_l/y_p$ give an isomorphism $S_{d,L} \simeq T^{L \setminus \{p\}}$. The torus $S_{d,L}$ and its inversion do not require the choice of p .

Proof. The anchors together with y_p form a projective frame. A unique projectivity sends this frame to the coordinate vertices and $[1 : \cdots : 1]$. The normalizing map is regular on the frame locus: its formulas invert only nonzero frame determinants and coordinates. It gives an isomorphism $U_{d,L} \simeq \text{PGL}_{d+1} \times T^{L \setminus \{p\}}$, with the group acting on the first factor. This proves freeness and the geometric-quotient assertion. Equivalently, dividing T^L by diagonal translation gives the same slice. If the pivot changes from p to s , the transition on the torus is

$$x'_l = x_l/x_s \quad (l \neq p, s), \quad x'_p = 1/x_s. \quad (1)$$

Finally, $(ty_l)^{-1} = t^{-1}y_l^{-1}$, so inversion is well-defined on the quotient and independent of anchor lifts. $\square$

For $d = 3$, partition $L = C \sqcup Q$ into nonempty sets of camera centers and nonanchor scene points. In these center coordinates, CW is inversion followed by exchange of the two colors. This differs from the factor-swap formula in reduced camera-parameter coordinates: a center c has reduced parameter c^{-1} . No unique Cremona transformation of the original physical space is asserted before taking the quotient.

2 The common toric model

Let M be the character lattice of $S_{d,L}$ and $N = \text{Hom}(M, \mathbb{Z})$. One has the concrete description

$$M = \left\{ (m_{li}) \in \mathbb{Z}^{L \times I} : \sum_i m_{li} = 0 \text{ for every } l, \quad \sum_l m_{li} = 0 \text{ for every } i \right\}. \quad (2)$$

For $l \neq h$ and $i \neq j$ define the rectangle character

$$\alpha_{lh;ij} = (e_l - e_h) \otimes (e_i - e_j), \quad \chi^{\alpha_{lh;ij}}(y) = \frac{y_{li}y_{hj}}{y_{lj}y_{hi}}.$$

For fixed $p \in L$, the characters $\alpha_{lp;id}$, $l \neq p$, $i < d$, form an integral basis of M . Indeed, the entries outside row p and column d determine a matrix in (2) uniquely. Set

$$Z_{d,L} = \sum_{l < h, i < j} [-\alpha_{lh;ij}, \alpha_{lh;ij}], \quad \Sigma_{d,L} = \mathcal{N}(Z_{d,L}). \quad (3)$$

Here $\mathcal{N}$ denotes the normal fan; changing pair orientations does not change the zonotope. Its support function is $\sum_{l < h, i < j} |\langle \alpha_{lh;ij}, u \rangle|$, so its normal fan is precisely the central arrangement of rectangle hyperplanes

$$\langle \alpha_{lh;ij}, u \rangle = 0. \quad (4)$$

For example, this follows by taking maximizing faces of each segment in (3). Since the rectangle characters span M , the zonotope is full-dimensional and the fan is complete and pointed.

Let $X_{d,L}$ be the normal toric variety of $\Sigma_{d,L}$. For each pivot p , write

$$P_p = (X_{A_d})^{L \setminus \{p\}},$$

viewed as a compactification of the *same* torus $S_{d,L}$ via Proposition 1.1. The permutohedral variety X_{A_d} has the braid fan in $\mathbb{Z}^{d+1}/\mathbb{Z}(1, \dots, 1)$; see [5, Definitions 3.2–3.4].

Theorem 2.1. *The variety $X_{d,L}$ is a pivot-independent normal projective compactification of $S_{d,L}$. It has proper birational toric morphisms $\pi_p : X_{d,L} \rightarrow P_p$ restricting to the identity on $S_{d,L}$. Any normal toric compactification of $S_{d,L}$ with identity-on-torus morphisms to every P_p factors uniquely through $X_{d,L}$. Furthermore, $X_{d,L}$ is the normalization of the closure of the diagonal $S_{d,L}$ in $\prod_{p \in L} P_p$.*

Proof. In pivot- p coordinates, the product braid fan has walls

$$(u_{li} - u_{pi}) - (u_{lj} - u_{pj}) = 0, \quad l \neq p, \quad i \neq j.$$

These are exactly the rectangle walls whose label pair contains p . Their union over all p is (4). Thus $\Sigma_{d,L}$ is the common refinement of all pivot fans. The toric morphism criterion gives π_p . It is proper since the fans are complete, and birational since it is the identity on the torus. A fan refining every pivot fan refines their common refinement, proving the factorization. Equality on the dense torus proves uniqueness. Projectivity follows from the full-dimensional lattice polytope (3).

For the normalized-closure assertion, take affine charts corresponding to cones σ_p in the pivot fans. The diagonal closure in their product has the coordinate semigroup generated by the $\sigma_p^\vee \cap M$. Normalization saturates this semigroup in M . Its rational cone is $\sum_p \sigma_p^\vee$, dual to $\bigcap_p \sigma_p$. These are the common-refinement cones. The descriptions agree on affine overlaps, proving the assertion. $\square$

Corollary 2.2. *Permutations of labels, permutations of anchors, and inversion extend to automorphisms of $X_{d,L}$. In particular, for $d = 3$ CW extends biregularly between the compactifications with colorings (C, Q) and (Q, C) .*

Proof. Each operation permutes the segments of (3). Inversion acts by $-\text{id}$ on the lattice. All therefore preserve the fan. $\square$

3 When is the common model smooth?

Theorem 3.1. *For $d \geq 1$ and $n = |L| \geq 2$, the variety $X_{d,L}$ is smooth if and only if $d = 1$ or $n = 2$. In these cases it is permutohedral. If $d \geq 2$ and $n \geq 3$, it is not $\mathbb{Q}$ -factorial.*

Proof. When $n = 2$, the difference between the two rows identifies the cocharacter lattice with $\mathbb{Z}^{d+1}/\mathbb{Z}\mathbf{1}$ and the fan with the A_d braid fan. When $d = 1$, taking the difference between the two columns gives $\mathbb{Z}^n/\mathbb{Z}\mathbf{1}$ and the A_{n-1} braid fan. These are integral lattice identifications. A maximal braid cone is generated by the classes of successive partial sums of a permuted coordinate basis; they form a lattice basis. Thus both cases are smooth.

For the obstruction, first take $d = 2, n = 3$. Normalize the pivot row and final anchor column to zero. Write the other rows as $(a, b, 0)$ and $(c, e, 0)$. The nine wall forms are

$$a, b, a - b, c, e, c - e, a - c, b - e, a - b - c + e.$$

The chamber containing $(-3, -2, -2, -3)$ has closure

$$a \leq b \leq 0, \quad e \leq c \leq 0, \quad a \leq c, \quad e \leq b. \quad (5)$$

The remaining sign $a - b - c + e \leq 0$ follows from the first two chains. With $(A, B, C, D) = (-a, -b, -c, -e)$, the cone is

$$K = \{B, C \geq 0; A, D \geq B, C\}.$$

Its primitive extreme rays are

$$\begin{aligned} r_A &= (1, 0, 0, 0), & r_D &= (0, 0, 0, 1), & r_0 &= (1, 1, 1, 1), \\ r_B &= (1, 1, 0, 1), & r_C &= (1, 0, 1, 1). \end{aligned}$$

To see completeness of this list, if $B \geq C$ decompose a point of K as

$$Cr_0 + (B - C)r_B + (A - B)r_A + (D - B)r_D. \quad (6)$$

All coefficients are nonnegative. For $C \geq B$, interchange B, C and r_B, r_C . Each ray is extreme: r_A and r_D are cut out by setting the other three coordinates to zero; r_B by $C = 0, A = B, D = B$; r_C by $B = 0, A = C, D = C$; and r_0 by $A = B = C = D$. In each case these equations cut out a one-dimensional intersection of supporting faces. The point $(3, 2, 2, 3)$ satisfies all six inequalities strictly, so K is four-dimensional. Five extreme rays in dimension four make it nonsimplicial.

For arbitrary $d \geq 2, n \geq 3$, keep just two entries in each of two nonpivot rows, setting every other normalized entry to zero. This defines a four-dimensional arrangement flat F : each zero condition is a rectangle hyperplane using the pivot row and last anchor column. Every rectangle form restricts to zero or, up to sign, one of the nine forms above, and all nine occur. Hence (5), embedded in F , is a cone of the ambient fan, not merely a slice of a cone. Its relative interior has the prescribed nine nonzero signs and zeros for the forms vanishing on F . It is again four-dimensional with five rays. A normal toric variety is $\mathbb{Q}$ -factorial exactly when all its cones are simplicial [3, Section 4.2]. In the present cone the Cartier criterion already gives the obstruction: five arbitrary rational coefficients on its rays cannot all be the values of a linear form on its four-dimensional span. This proves the remaining cases. $\square$

Remark 3.2. The theorem concerns the *minimal normal toric* common model. It does not preclude a smooth common refinement obtained by further subdivision, nor a different smooth modular compactification. For camera dimension $d = 3$, nonsimpliciality already occurs with three nonanchor marks.

4 Compactified reduced measurements

Return to $d = 3$ and $L = C \sqcup Q$. In the anchor-normalized retinal coordinates, the measurement of a scene point q from center c is

$$\mu(c, q) = \left[\frac{q_0}{c_0} - \frac{q_3}{c_3} : \frac{q_1}{c_1} - \frac{q_3}{c_3} : \frac{q_2}{c_2} - \frac{q_3}{c_3} \right]. \quad (7)$$

The associated camera matrix is

$$B(c^{-1}) = \begin{bmatrix} 1/c_0 & 0 & 0 & -1/c_3 \\ 0 & 1/c_1 & 0 & -1/c_3 \\ 0 & 0 & 1/c_2 & -1/c_3 \end{bmatrix},$$

whose kernel is c . Formula (7) is the usual reduced form of CW [8, 2], with the center distinguished from its reciprocal camera parameter.

Diagonal translation cancels in every ratio. Rescaling homogeneous representatives multiplies the whole vector by a common nonzero scalar. Thus (7) is well-defined on the quotient wherever $c \neq q$. The vector vanishes precisely when all q_i/c_i agree, which is exactly $c = q$. Let $S_{C,Q}^{\text{nc}}$ be the dense open locus with no camera–scene collisions and set

$$\mu_{C,Q} : S_{C,Q}^{\text{nc}} \longrightarrow (\mathbb{P}^2)^{C \times Q}.$$

Define the compactified incidence variety as the reduced graph closure

$$\Gamma_{C,Q} = \overline{\text{Graph}(\mu_{C,Q})} \subset X_{3,L} \times (\mathbb{P}^2)^{C \times Q}. \quad (8)$$

It is integral and projective, and its projection to $X_{3,L}$ is birational and an isomorphism over $S_{C,Q}^{\text{nc}}$. Normality is not assumed. In particular, taking a graph closure does not claim that the measurement itself extends to a morphism on all of $X_{3,L}$.

Theorem 4.1. *Let τ transpose the output labels $C \times Q \rightarrow Q \times C$. Inversion on $X_{3,L}$ times τ restricts to an isomorphism $\Gamma_{C,Q} \simeq \Gamma_{Q,C}$. It also induces an isomorphism of their normalizations.*

Proof. Direct substitution gives $\mu(q^{-1}, c^{-1}) = \mu(c, q)$, and inversion with the color exchange preserves the no-collision locus. By Corollary 2.2, the ambient map is an isomorphism sending one dense graph onto the other. It therefore sends their reduced closures isomorphically. The assertion about normalizations follows by functoriality under isomorphisms. $\square$

5 Boundary comparison and limitations

Even in dimension one, simply identifying the open quotients does not identify their pivot boundaries. For $d = 1, n = 3$, (1) reads $(x, y) \mapsto (1/x, y/x)$ on $(\mathbb{P}^1)^2$. The paths $(x, y) = (t, \lambda t)$, $\lambda \neq 0$, all tend to $(0, 0)$, but the transformed paths tend to (∞, λ) . Thus the transition does not extend there; similarly it fails at (∞, ∞) . The common model is the hexagonal permutohedral surface, the blowup of those two torus-fixed points.

Lieblisch–Van Meter study calibrated and uncalibrated camera moduli using Hilbert schemes [7]; their Section 5 raises boundary and CW questions. Loucks–Tavitas studies a different boundary issue for concentric camera pairs [6, Chapter 3]. Those spaces retain different data from our anchor-framed configuration quotient. Our construction does not supply the stability condition sought there, remove the anchors, or interpret every toric boundary point as a physical camera configuration. A modular description of the incidence boundary and its singularities remains to be developed.

Another related construction is the toric Chow quotient of a power of a toric variety by diagonal translation, studied by Di Rocco–Schaffler [4]. Their universal-family example already has nonsimplicial four-dimensional cones. We do not claim that singularity in toric configuration compactifications is new. Our definition instead uses the common refinement of all pivot *product-permutohedral* fans; no identification with their Chow quotient or universal family is assumed. Theorem 3.1 is a criterion for this specified all-pivot fan.

The quotient and standard CW formulas, the permutohedral extension of inversion, and the general common-refinement construction are credited above. The explicit rectangle fan, its simultaneous-pivot property, and Theorem 3.1 give the scoped result of this note. The source comparison was targeted, not an exhaustive priority certification.

An accompanying standard-library Python script checks the rectangle lattice bases, the all-pivot wall sets, the five extreme rays, and the measurement identities using exact arithmetic. The

general proofs are the written arguments above, not extrapolations from finite tests. This unrefereed preprint was developed with AI assistance in derivation, literature comparison, computation, and drafting. It is self-audited; no independent peer review or proof-assistant formalization is claimed.

References

- [1] American Institute of Mathematics, *Algebraic Vision*, workshop problem list, Problem 5.3, <http://aimpl.org/algvision/5/>.
- [2] E. Connelly, T. Duff, and J. Loucks-Tavitas, *Algebra and geometry of camera resectioning*, Math. Comp. **94** (2025), 2613–2643. doi:10.1090/mcom/4030. Preprint: [arXiv:2309.04028](https://arxiv.org/abs/2309.04028).
- [3] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric Varieties*, Graduate Studies in Mathematics **124**, American Mathematical Society, 2011. doi:10.1090/gsm/124.
- [4] S. Di Rocco and L. Schaffler, *Families of pointed toric varieties and degenerations*, Math. Z. **301** (2022), 4119–4139. doi:10.1007/s00209-022-03047-y.
- [5] J. Huh, *Tropical geometry of matroids*, Current Developments in Mathematics 2016, International Press, 2018, 1–46. <https://web.math.princeton.edu/~huh/TropicalMatroids.pdf>.
- [6] J. Loucks-Tavitas, *Hey! You Got Your Algebraic Geometry in My Computer Vision!*, Ph.D. dissertation, University of Washington, 2024. <https://hdl.handle.net/1773/52099>.
- [7] M. Lieblich and L. Van Meter, *Two Hilbert schemes in computer vision*, SIAM J. Appl. Algebra Geom. **4** (2020), 297–321. doi:10.1137/18M1200117.
- [8] M. Trager, M. Hebert, and J. Ponce, *Coordinate-free Carlsson–Weinshall duality and relative multi-view geometry*, Proceedings of CVPR 2019, 225–233. [Open-access proceedings](#).