

Exact Positive-Focal Certificates for Fundamental-Matrix Pencils

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Abstract

For an independent rational 3-by-3 matrix pencil, we give an exact procedure that counts its real projective rank-two members compatible with two positive finite focal lengths, or reports that there are infinitely many. Principal points are known, skew is zero and aspect ratio is one. A positive-kernel formulation of the two-view Kruppa relation yields a division-free polynomial sign predicate. Classical Hermite trace forms evaluate this predicate without constructing or isolating determinant roots. Repeated roots, rank-one members, the projective point at infinity and critical calibration families are included. For identically singular pencils, derivative-root sampling also detects bounded feasible intervals. The complementary count identifies incompatible rank-two members. On a guarded generic chart, the count reduces to four Tarski queries. The construction has polynomial bit complexity for this fixed-size input; no empirical speed advantage over numerical reconstruction is asserted.

1 The counting problem

Image coordinates are centered at the known principal points. Throughout, $x_2^T F x_1 = 0$ is the epipolar convention. Put

$$K(x) = \text{diag}(\sqrt{x}, \sqrt{x}, 1), \quad x > 0.$$

A real rank-two matrix F is *admissible* if $K(y)FK(x)$ is an essential matrix for some $x, y > 0$. Thus the two focal lengths are finite, positive and independently variable. The property concerns existence of a calibrated camera pair; it does not assert positive depths for a prescribed set of scene points.

Let $F_0, F_1 \in \mathbb{Q}^{3 \times 3}$ be linearly independent and write

$$F_{[s:t]} = sF_0 + tF_1, \quad [s : t] \in \mathbb{P}^1(\mathbb{R}).$$

Independence ensures that the parameterization is injective as a map to projective matrices and never gives the zero matrix. We count projective matrices, not their possibly infinite sets of calibrations.

Theorem 1. *There is an exact algorithm using rational arithmetic and sign tests that computes each of the following counts in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$:*

1. *the number of admissible members of the pencil;*
2. *the number of rank-two members that are not admissible.*

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It does not construct or isolate algebraic roots. It applies without a genericity hypothesis, including when $\det(sF_0 + tF_1)$ vanishes identically. For this fixed-size input its bit complexity is polynomial in the bit length of F_0, F_1 .

Seven rational point correspondences whose epipolar design matrix has rank seven give such a pencil by rational nullspace computation. The theorem tests the pencil before choosing its determinant roots; it does not avoid computing the nullspace pencil itself.

Relation to the source question and prior work

AIM Algebraic vision Problem 1.5 asks for an efficient way to anticipate nonreal focal lengths before constructing a fundamental matrix [1]. Theorem 1 is an exact-arithmetic answer in a specified pencil model, not a claim that the whole practical, open-ended question is resolved.

The Kruppa relation and essential-matrix characterization are classical. Melanitis and Maragos use linear relations for camera-pair calibration from a projective reconstruction [2]. Kocur, Kyselica and Kukulova use a real-focal check after obtaining a candidate fundamental matrix and before RANSAC scoring [3]. Kosaka, Duff, Pajdla and Sugimoto give a broader algebraic framework for varying intrinsic priors; their public supplement derives the full projective Kruppa relation [4]. We compare with that public material, not with an unavailable full chapter.

Hermite trace signatures and incremental sign determination are standard real algebraic geometry [5, 6]. The generic four-query indicator below was already present in the frozen dataset’s generated research aid [7]. Our contribution here is an explicit positive-kernel formulation and uniform exceptional-case construction, with reproducible exact code. No new general sign-determination theorem or absolute priority claim is made.

2 A strictly positive kernel

For $e \in \mathbb{R}^3$, let $[e]_\times$ be the matrix of $v \mapsto e \times v$. Set $J = \text{diag}(1, 1, 0)$ and $Q = \text{diag}(0, 0, 1)$. For a rank-two F and a nonzero $e \in \ker F^T$, define

$$A = F J F^T, \quad B = F Q F^T, \quad C = [e]_\times J [e]_\times^T, \quad D = [e]_\times Q [e]_\times^T.$$

Using the six symmetric coordinates 00, 01, 02, 11, 12, 22, form

$$L(F, e) = [\text{vec}(A) \quad \text{vec}(B) \quad -\text{vec}(C) \quad -\text{vec}(D)]. \quad (1)$$

Proposition 2. *The matrix F is admissible if and only if $\ker L(F, e) \cap \mathbb{R}_{>0}^4 \neq \emptyset$. For a positive kernel vector $w = (w_0, w_1, w_2, w_3)$, the quantities*

$$x = \frac{w_0}{w_1}, \quad y = \frac{w_2}{w_3}, \quad \lambda = \frac{w_3}{w_1} \quad (2)$$

satisfy

$$F \text{diag}(x, x, 1) F^T = \lambda [e]_\times \text{diag}(y, y, 1) [e]_\times^T. \quad (3)$$

Proof. Let $K_1 = K(x)$, $K_2 = K(y)$ and $t = K_2^{-1}e$. The cross-product change-of-coordinates identity gives

$$K_2 [e]_\times K_2 = \det(K_2) [t]_\times = y [t]_\times.$$

Conjugating (3) by K_2 therefore gives

$$E E^T = \lambda y^2 [t]_\times [t]_\times^T, \quad E = K_2 F K_1. \quad (4)$$

The right side has two equal positive eigenvalues and one zero eigenvalue. Since $\text{rank } E = 2$, this is the essential-matrix criterion.

For completeness, choose $E = U \text{diag}(\sigma, \sigma, 0) V^T$ with $U, V \in SO(3)$; the signs of their null columns can be changed independently. For

$$W = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t_0 = \sigma U(0, 0, 1)^T, \quad R = U W V^T,$$

we have $R \in SO(3)$ and $[t_0]_{\times} R = E$. Thus (4) gives an actual real calibrated pair. Conversely, for an essential E , the matrix EE^T is a positive multiple of $[t]_{\times} [t]_{\times}^T$ for every nonzero $t \in \ker E^T$. Here $\ker E^T = K_2^{-1} \ker F^T$, so reversing the conjugation gives (3) with $\lambda > 0$. Finally (1) has the positive kernel vector $(x, 1, \lambda y, \lambda)$ precisely when (3) holds. $\square$

Replacing e by a nonzero scalar multiple preserves feasibility. All four symmetric matrices kill e , so $1 \leq \text{rank } L \leq 3$. The lower bound follows from $A + B = FF^T \neq 0$. The projectivized positive kernel is in bijection with the admissible pairs (x, y) : the scale λ in (3) is unique because the two matrices being compared are nonzero.

3 A division-free sign predicate

A *positive circuit* of a real matrix M is a nonnegative kernel vector with support S , positive on every coordinate of S , such that $\dim \ker M_S = 1$.

Lemma 3. *The kernel of M contains a strictly positive vector if and only if the supports of its positive circuits cover all columns.*

Proof. The sum of a covering family is strictly positive. Conversely normalize a nonnegative kernel vector v to have coordinate sum one, and let S be its support. If $\dim \ker M_S > 1$, take a kernel vector u not proportional to v and replace it by $u - (\sum u_i)v$. This is nonzero with sum zero, hence has both signs. The interval of r for which $v + ru \geq 0$ has a finite positive and a finite negative endpoint. Both endpoint vectors have smaller support, and v is their convex combination. Repeated support reduction expresses v as a convex combination of normalized positive circuits. If $v > 0$, those circuits cover every coordinate. $\square$

For L in (1), enumerate nonempty $S \subseteq \{0, 1, 2, 3\}$ with $|S| = k$ and every $(k-1)$ -subset I of its six rows. In the inherited order of S put

$$h_j = (-1)^j \det(L_{I,S} \text{ with column } j \text{ deleted}). \quad (5)$$

The empty determinant is one. The conditions

$$(h_j > 0 \text{ for all } j \text{ or } h_j < 0 \text{ for all } j), \quad L_S h = 0 \quad (6)$$

certify a positive circuit. A nonzero maximal minor gives rank at least $k-1$, and the full residual condition gives rank at most $k-1$. Every positive circuit is found by choosing independent rows. Singleton supports include zero columns. The number of choices is

$$\sum_{k=1}^4 \binom{4}{k} \binom{6}{k-1} = 120.$$

Coverage of the four columns is therefore a Boolean formula in polynomial signs.

Corollary 4. *Every rational admissible F has a calibration with positive rational squared focal lengths.*

Proof. Choose rational $e \in \ker F^T$. The cofactor vectors in (5) are rational; orient and sum a covering family. The resulting positive rational kernel vector gives rational x, y by (2). $\square$

Now use the affine coordinate $F(z) = F_1 + zF_0$. Each cross product of two columns is a candidate left-null vector when $\det F(z) = 0$. Use all three charts, requiring the chosen cross product to be nonzero. Also require determinant zero and at least one nonzero 2-by-2 minor. On each chart apply (6) and support coverage. Their disjunction defines a Boolean polynomial-sign predicate $\Psi(z)$, true exactly at admissible finite parameters.

The first two columns of L have degree at most two, and the last two at most four. Cofactors have degree at most ten; each residual term uses distinct columns of L and has degree at most twelve. An unsimplified predicate uses at most $10 + 3(3 + 336 + 720) = 3187$ polynomial references, all of degree at most twelve. Many vanish identically, repeat or become constant. This bound is only a uniform complexity bound.

4 Hermite queries without roots

Let $q \in \mathbb{Q}[z]$ be squarefree of degree $d > 0$, and work in $V = \mathbb{Q}[z]/(q)$. For $g \in \mathbb{Q}[z]$ define the Hermite matrix

$$H_g(i, j) = \text{Tr}(m_{gz^i+j}), \quad 0 \leq i, j < d, \quad (7)$$

where m_a is multiplication by a in V . The classical Tarski query is

$$\text{TaQ}_q(g) = \sum_{\substack{q(\alpha)=0 \\ \alpha \in \mathbb{R}}} \text{sign } g(\alpha) = \text{signature } H_g. \quad (8)$$

To see the identity, extend V to $\mathbb{R}$. Its squarefree decomposition has one real field factor for each real root and one complex factor for each nonreal conjugate pair. A real factor contributes $\text{sign } g(\alpha)$ to the trace form. On a complex factor with $g = a + ib$, the real trace form in the basis $1, i$ is

$$\begin{pmatrix} 2a & -2b \\ -2b & -2a \end{pmatrix}.$$

It has signature zero, also when $g = 0$. This decomposition proves the identity; the algorithm does not compute its roots.

Compute (7) from the rational companion matrix. Exact symmetric elimination gives its inertia. A nonzero diagonal pivot contributes its sign and a Schur complement. If all diagonal entries vanish but an off-diagonal entry b is nonzero, pivot on $\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$: it contributes one positive and one negative direction. Only a zero remainder contributes solely nullity. No numerical eigenvalues are needed.

4.1 Incremental sign determination

For completeness, the following elementary form of classical sign determination [5, 6] suffices. Maintain the $k \leq d$ realized sign cells of the polynomials processed so far, their root counts, and k products b_j of those polynomials with exponents zero, one or two, such that the sign-evaluation matrix U is invertible. Initially $b_1 = 1$ and $U = (1)$, unless $\text{TaQ}_q(1) = 0$.

For a new polynomial h query $\text{TaQ}_q(b_j h^e)$ for $e = 0, 1, 2$. Let T_e be the query column and $v_e = U^{-1}T_e$. Within old cell i , the numbers of roots with new signs zero, positive and negative are respectively

$$(v_0 - v_2)_i, \quad (v_2 + v_1)_i/2, \quad (v_2 - v_1)_i/2. \quad (9)$$

Retain cells with positive counts. The evaluation matrix of the $3k$ products $b_j h^e$ has full column rank on these new cells: U separates old cells, and $1, t, t^2$ separate any subset of $\{-1, 0, 1\}$ within one old cell. Choose independent rows as the next basis. Reduction modulo q preserves all values at its roots. Thus at most $3d$ queries per processed polynomial suffice. Evaluating any Boolean sign predicate on the resulting cells counts its satisfying roots exactly.

5 All pencils and the complementary count

Proof of Theorem 1. Test F_0 separately by Proposition 2 and Lemma 3; it is the single projective point $[1 : 0]$ omitted by $F(z) = F_1 + zF_0$. Put $p(z) = \det F(z)$.

If p is nonzero and nonconstant, replace it by its squarefree part $q = p/\gcd(p, p')$ and apply Section 4 to Ψ . Each distinct real root is counted once, including repeated roots of p . Rank-one roots fail the predicate. If p is a nonzero constant, no finite parameter contributes.

Suppose $p = 0$ identically. Remove identically zero and constant polynomials from the sign list, and let H be the squarefree least common multiple of the remaining polynomials. If the list is empty, Ψ is constant and can be evaluated at $z = 0$. Otherwise Ψ is constant on every component of $\mathbb{R} \setminus H^{-1}(0)$. Test both tail sign patterns from leading coefficients and degrees. Count feasible roots of H by sign determination. Finally apply sign determination to the squarefree part of H' , retaining only roots where $H \neq 0$.

Rolle's theorem gives at least one such derivative root between every pair of consecutive real roots of H . Together with the tails these sample every complementary interval. Conversely a feasible sample where $H \neq 0$ has a whole feasible neighborhood. If any interval sample is feasible, return infinity; otherwise return the feasible H -root count plus the separate infinity test. This covers every real projective parameter.

For incompatible rank-two members use the predicate $\text{rank } F = 2$ and $\neg\Psi$ with the same procedure. If $p \neq 0$ there is a shorter total-count calculation: let

$$R(z) = \sum_{\substack{I, J \subseteq \{1, 2, 3\} \\ |I| = |J| = 2}} \det(F(z)_{I, J})^2.$$

At a real root of p , sign R is one exactly when the rank is two. Hence $\text{TaQ}_q(R)$ counts finite rank-two members; subtract the admissible count and handle infinity separately. For $p = 0$, include R in the sign list before forming H .

Finally, all dimensions, polynomial counts and degrees are bounded independently of the rational input bit length. Rational polynomial gcds, traces, sign queries and congruences in those bounded dimensions have polynomial bit complexity. In the singular branch, $\deg H \leq 12 \cdot 3187$ is a crude universal bound for the admissibility predicate; adding R changes it by a constant. The assertion is for fixed-size input, not a claim of favorable practical constants. $\square$

Fixed-dimensional real quantifier elimination already implies broad decidability and polynomial-bit bounds for this problem. The explicit positive kernel and univariate construction above make the exceptional cases and their certificates accessible without a black-box multivariate solver.

6 The guarded four-query formula

Let $p \neq 0$ and q be its nonconstant squarefree part. Suppose a cross-product chart e satisfies $\gcd(q, e_0, e_1, e_2) = 1$. Suppose also that three rows of L have cofactor vector $h = (h_0, h_1, h_2, h_3)$ with

$$Lh = 0 \pmod{q}, \quad \gcd(q, h_0 h_1 h_2 h_3) = 1. \quad (10)$$

At each root, this cofactor vector spans the one-dimensional kernel. Put $g_1 = h_0 h_1$ and $g_2 = h_2 h_3$. The ratios x, y in (2) are positive exactly when $g_1, g_2 > 0$. In that event $\lambda > 0$ automatically: the two sides of (3) before multiplication by λ are nonzero positive-semidefinite matrices. The finite admissible count is consequently

$$\frac{\text{TaQ}_q(1) + \text{TaQ}_q(g_1) + \text{TaQ}_q(g_2) + \text{TaQ}_q(g_1 g_2)}{4}. \quad (11)$$

This is the familiar two-sign indicator, not a new general formula. All guards are rational polynomial tests. If they fail, use the full predicate rather than declaring infeasibility.

7 Exact examples and reproducibility

7.1 A seven-correspondence pencil

Consider

$$F_0 = \begin{pmatrix} -2/9 & 2/3 & 2/3 \\ 0 & 1/3 & -2 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} 1/6 & 1/4 & 0 \\ -1/12 & 0 & -1/2 \\ 0 & -1/4 & 1 \end{pmatrix}.$$

The pairs $(u, v) \leftrightarrow (u', v')$ in Table 1, with homogeneous final coordinate one, give an epipolar design matrix of rank seven with nullspace spanned by F_0, F_1 . Here

u	v	u'	v'
$-2/5$	0	$9/2$	$3/2$
$4/5$	$-4/5$	18	0
0	1	5	4
$-4/5$	$4/5$	$18/5$	$12/5$
$2/5$	$4/5$	9	6
2	2	9	12
0	0	6	2

Table 1: Seven exact point correspondences.

$$\det(F_1 + zF_0) = -\frac{z(84z^2 + 43z + 3)}{54}.$$

The roots $0, -1/12, -3/7$ give squared-focal pairs

$$(4, 9), \quad \left(\frac{2520}{1223}, \frac{648}{31}\right), \quad \left(-\frac{84}{53}, -\frac{3801}{1177}\right),$$

respectively; only the first two are positive. Infinity is not rank two. The four queries in (11) are $(3, 1, 1, 3)$, so the admissible and incompatible counts are two and one. The deliberately unoptimized full predicate also returns two, using 1902 trace queries rather than four. These are query counts, not measured runtime comparisons.

7.2 Singular and projective boundary cases

The pencil $\text{diag}(z, 1, 0)$ is admissible exactly at $z = \pm 1$; its point at infinity has rank one. The pencil $\begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is admissible only at zero, whereas replacing its middle diagonal entry by two gives no admissible member.

For

$$F(z) = \begin{pmatrix} z & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the nonzero diagonal entries of EE^T are $y(z^2x + 1)$ and yx . Thus feasibility is exactly $|z| < 1$, with $x = (1 - z^2)^{-1}$ and arbitrary $y > 0$. Neither tail is feasible; derivative sampling is essential here.

The matrix $\begin{pmatrix} z & 1 & 0 \\ 0 & z & 0 \\ 0 & 0 & z-1 \end{pmatrix}$ has determinant $z^2(z - 1)$ but only the repeated root $z = 0$ is feasible, with $xy = 1$, and is counted once. For $\text{diag}(z, 1, z + 1)$ both finite roots and infinity are feasible. For $\begin{pmatrix} z & -1 & 0 \\ 1 & z & 0 \\ 0 & 0 & 1 \end{pmatrix}$ there are no finite real determinant roots, but infinity is feasible.

The accompanying Sympy 1.14.0 code uses exact rational arithmetic. Its 836 regression checks comprise 489 positive-kernel tests, 159 root-free tests, 177 boundary/design checks and 11 complementary counts. They include rational camera-generated matrices, a separate Fourier–Motzkin feasibility implementation, inertia under congruence, irreducible root-polynomial sign patterns and projective coordinate changes. Normal and optimized runs from a separate source copy agree after excluding runtime and optimization metadata. These finite tests support the implementation; the general proof is the argument above.

The note is unrefereed. AI assistance was used in developing the arguments, checking calculations and preparing the manuscript; the author is responsible for its claims. The result does not certify scene cheirality, arbitrary intrinsic models or an empirical RANSAC improvement.

References

- [1] American Institute of Mathematics, *Algebraic vision: Reconstruction*, Problem 1.5, <http://aimpl.org/algvision/1/>. Accessed 15 October 2026.
- [2] N. Melanitis and P. Maragos, *A linear method for camera pair self-calibration and multi-view reconstruction with geometrically verified correspondences*, arXiv:1906.12075v1 (2019), <https://arxiv.org/abs/1906.12075v1>.
- [3] V. Kocur, D. Kyselica and Z. Kukelova, *Robust self-calibration of focal lengths from the fundamental matrix*, Proceedings of CVPR (2024), 5220–5229, [CVF open-access paper](#).
- [4] N. Kosaka, T. Duff, T. Pajdla and A. Sugimoto, *General self-calibration with varying intrinsics*, Computer Vision–ECCV 2026, LNCS 17068, 167–183 (2026). Public abstract and supplementary material, Sections A–B, https://doi.org/10.1007/978-3-032-37152-2_10.
- [5] S. Basu, *Algorithms in real algebraic geometry: A survey*, Section 2.4.1, https://www.math.purdue.edu/~sbasu/raag_survey2011_final.pdf.
- [6] D. Perrucci, *Linear solving for sign determination*, arXiv:0911.5707v1 (2009), <https://arxiv.org/abs/0911.5707v1>.
- [7] *UnsolvedMath*, version 1.6.0, record AIM-ALGEBRAIC_GEOMETRY-0190, generated research aid, frozen commit c6e7f41b4eb6, [dataset snapshot](#).