

Base-Divisor Degree Formulas for Two-View Algebraic Curves

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23 October 2026

Abstract

Let two fixed projective cameras with distinct centers have integral plane image curves of degrees m, n , neither an epipolar line. We describe all integral reconstructions over an algebraically closed field of characteristic zero using the normalized components of the fiber product of the moving epipolar pencils. If a component has pencil degree h and pulled-back epipolar base divisors B_X, B_Y , its triangulated world curve has degree $h + \deg \max(B_X, B_Y)$. This retains singular branches and images passing through the epipoles. We derive the total residual degree and an explicit formula for the baseline cycle multiplicity in terms of branch base orders and pencil ramification indices. When both moving pencils have degree one, the compatible integral world curve is unique, and the possible degrees for fixed m, n are exactly $\max(m, n), \dots, m + n - 1$. The classical fiber-product method and twisted-cubic example are credited; the formulas here make their base-divisor corrections explicit.

1 Scope and context

Throughout, the ground field k is algebraically closed of characteristic zero. A curve is integral and projective unless otherwise specified. The image of a curve under a camera means the closure of the image away from the camera center. We call two image curves *weakly compatible* if some integral world curve has those image closures, and *birationally compatible* if both camera projections on such a curve are birational.

AIM's algebraic-vision list asks when two curve images are compatible for fixed cameras [1, Problem 1.25]. A separate item asks about degree and smoothness restrictions. These questions admit several interpretations. We give an exact statement for integral algebraic curves, including a formula for every possible reconstruction degree; we do not treat real visibility, positive depth, or arbitrary nonreduced input schemes.

The back-projected-cone and epipolar-cover viewpoint is classical. Kaminski, Fryers and Teicher use a fiber product and monodromy to separate, in their generic situation, the degree- d world curve from a degree- $d(d - 1)$ residual curve [2, Theorem 2]. Rydell and Sundelius study plane-curve and twisted-cubic multiview varieties, and give the two-view conic ideal [3, Theorem 4.1]. Ballico distinguishes set-theoretic and scheme-theoretic reconstruction; in particular his Example 5 describes two conic projections of a twisted cubic whose cone intersection also contains the baseline [4].

Our argument resolves the base divisors on the image normalizations, then computes the line bundle defining triangulation. It gives componentwise degree corrections, a branch formula for the

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baseline multiplicity, and a sharp degree spectrum for degree-one pencils. We make no priority claim for the fiber-product method, the usual single-camera degree formula, or the twisted-cubic example. No absolute novelty claim is implied by the targeted literature comparison. The precise results below do not settle every interpretation of the open-ended AIM question.

2 Moving epipolar pencils

Choose world and image coordinates so the cameras are

$$P[x : y : u : v] = [x : u : v], \quad Q[x : y : u : v] = [y : u : v]. \quad (1)$$

This is possible for any two rank-three cameras with distinct centers: their row spaces intersect in dimension two, giving u, v ; adjoin one coordinate from each row space. Their centers are $c_P = [0 : 1 : 0 : 0]$ and $c_Q = [1 : 0 : 0 : 0]$. The baseline is $L = \{u = v = 0\}$, and both epipoles are $[1 : 0 : 0]$. The common epipolar pencil is $T = \mathbf{P}^1$, with coordinate pair $[u : v]$.

Let $X, Y \subset \mathbf{P}^2$ have degrees m, n , neither an epipolar line. Write $\nu_X : \bar{X} \rightarrow X$, $\nu_Y : \bar{Y} \rightarrow Y$ for the smooth normalizations. Let D_X, D_Y be the common zero divisors of the pullbacks of u, v . Set

$$\epsilon_X = \deg D_X, \quad \epsilon_Y = \deg D_Y.$$

These are the respective plane-curve multiplicities at the epipoles, with value zero when an epipole is absent. On each branch the intersection order with a generic pencil line is the minimum of the orders of u, v ; summing gives this assertion.

Removing the common zero divisors defines finite morphisms

$$\rho_X : \bar{X} \rightarrow T, \quad \rho_Y : \bar{Y} \rightarrow T.$$

They are nonconstant, since a constant pencil would force an integral image curve to be an epipolar line. Their degrees are

$$r = m - \epsilon_X > 0, \quad s = n - \epsilon_Y > 0. \quad (2)$$

Indeed the ordered pencil sections identify

$$\nu_X^* \mathcal{O}_{\mathbf{P}^2}(1)(-D_X) \simeq \rho_X^* \mathcal{O}_T(1),$$

and similarly for Y . We always use the identifications preserving the ordered sections, not arbitrary line-bundle isomorphisms.

Let Z_j be the normalizations of the irreducible components of $\bar{X} \times_T \bar{Y}$, with projections p_j, q_j . Put

$$\begin{aligned} a_j &= \deg p_j, & b_j &= \deg q_j, & \tau_j &= \rho_X p_j = \rho_Y q_j, & h_j &= \deg \tau_j, \\ B_{X,j} &= p_j^* D_X, & B_{Y,j} &= q_j^* D_Y, & E_j &= \max(B_{X,j}, B_{Y,j}). \end{aligned}$$

Minimum, maximum and positive part of divisors are coefficientwise.

3 The component degree formula

Theorem 1. *Every Z_j triangulates to an integral world curve C_j , and $Z_j \rightarrow C_j$ is its normalization. These are precisely all integral world curves with image closures X, Y . The generic camera degrees are a_j, b_j , and*

$$\deg C_j = h_j + \deg E_j \quad (3)$$

$$= ma_j + nb_j - h_j - \deg \min(B_{X,j}, B_{Y,j}). \quad (4)$$

In particular weak algebraic compatibility holds.

Proof. The pencil covers are finite flat, since finite morphisms of smooth integral curves are flat. Their fiber product is finite flat over both factors and T . Its generic algebra is reduced by separability; a nilpotent would be generically zero and therefore torsion over the base, contrary to flatness. All components dominate both factors. The fiber product need not be normal, which is why its components are normalized explicitly.

Fix j and suppress its index. On Z , write $M_0 = \tau^* \mathcal{O}_T(1)$. The pulled-back image line bundles, with their ordered pencil sections, are $M_0(B_X)$ and $M_0(B_Y)$. In $M = M_0(E)$, take four sections: the first image's x section included from $M_0(B_X)$; the second image's y section included from $M_0(B_Y)$; and the two pencil sections included from M_0 .

These sections have no common zero. Outside E , at least one pencil section is nonzero. At a point with positive coefficient in E , one of B_X, B_Y attains that coefficient. Its inclusion has zero vanishing order there. The first coordinate of the corresponding image is nonzero because the other two coordinates vanish at its epipole. Thus the four sections define a morphism $\phi : Z \rightarrow \mathbf{P}^3$ on the whole smooth curve.

Away from the epipolar base fibers, the two camera projections give the specified image pair. Such a pair triangulates uniquely off the baseline. In the chart $v \neq 0$, the world point is $[x/v : y/v : u/v : 1]$. Since Z is birational to its fiber-product component, ϕ is birational onto its image C . It is nonconstant because τ is nonconstant. Properness and the curve dimension imply that it is the normalization map. Consequently $\phi^* \mathcal{O}_{\mathbf{P}^3}(1) = M$ and $\deg C = \deg M = h + \deg E$.

Moreover $ma = h + \deg B_X$ and $nb = h + \deg B_Y$. The coefficientwise identity $\max + \min = B_X + B_Y$ proves (4).

Conversely, the normalization of any integral world curve with image closures X, Y maps to both image normalizations over T . Its joint image is a fiber-product component. The joint camera map is generically injective off the baseline, so its world curve is exactly the triangulation already constructed. The same observation separates distinct components and proves the claimed camera degrees. $\square$

Corollary 2. *Let $g = \gcd(r, s)$. There are positive integers k_j with*

$$\sum_j k_j = g, \quad a_j = \frac{s}{g} k_j, \quad b_j = \frac{r}{g} k_j, \quad h_j = \text{lcm}(r, s) k_j.$$

There are at most g integral reconstructions, and exactly one if r, s are coprime.

Proof. We have $ra_j = sb_j = h_j$. The generic reduced fibers give $\sum_j a_j = s$ and $\sum_j b_j = r$. Dividing by g gives the result. $\square$

The numbers h_j are pencil degrees, not generally world degrees. Only when both epipoles are avoided do we recover $\deg C_j = \text{lcm}(m, n) k_j$. In general Theorem 1 gives

$$\max(ma_j, nb_j) \leq \deg C_j \leq ma_j + nb_j - h_j. \quad (5)$$

Corollary 3. *The base divisors of P, Q on Z_j are respectively $(B_{Y,j} - B_{X,j})_+$ and $(B_{X,j} - B_{Y,j})_+$. Hence*

$$\deg C_j = ma_j + \deg(B_{Y,j} - B_{X,j})_+ = nb_j + \deg(B_{X,j} - B_{Y,j})_+.$$

Proof. All three P -sections acquire the common factor $E_j - B_{X,j}$. After its removal they are the basepoint-free first image sections. Now $E_j - B_{X,j} = (B_{Y,j} - B_{X,j})_+$; exchange the factors for Q . $\square$

Corollary 4. *Birational compatibility holds if and only if the two normalized image curves are isomorphic as covers of the fixed pencil T .*

Proof. An isomorphism produces a graph component with $a_j = b_j = 1$. Conversely such a component projects by finite degree-one morphisms to both smooth image normalizations, hence by isomorphisms. $\square$

This criterion compares covers, not merely their branch sets.

4 Residual degree and baseline multiplicity

Let $f(x, u, v), g(y, u, v)$ define X, Y . For each $p \in \text{supp } D_X$ write $d_p = \text{ord}_p D_X$ and $e_p = e_p(\rho_X)$; for $q \in \text{supp } D_Y$ use d_q, e_q . Define

$$S = \sum_{\substack{p \in \text{supp } D_X, q \in \text{supp } D_Y \\ \rho_X(p) = \rho_Y(q)}} \min(d_p e_q, d_q e_p). \quad (6)$$

Theorem 5. *The nonbaseline components of $V(f, g)$ are precisely the C_j , each with generic multiplicity one, and*

$$\sum_j \deg C_j = mn - \epsilon_X \epsilon_Y - S, \quad \mu_L = \epsilon_X \epsilon_Y + S. \quad (7)$$

Here μ_L is the coefficient of L in the complete-intersection one-cycle, with value zero if L is absent. Saturating (f, g) by (u, v) defines the reduced union of the C_j .

Proof. The two cones are distinct irreducible surfaces. If they coincided, their equation would be independent of both x, y , and hence, by algebraic closure and irreducibility, would define an epipolar plane, contrary to the hypotheses. Thus their intersection is a complete intersection of degree mn , without embedded associated components.

A component contracted by one camera is a ray through its center. Unless it is the baseline, the other camera maps this ray to an epipolar line. Such a line cannot be contained in our other integral image curve. Therefore every nonbaseline component dominates both images and is one of the C_j . Over an open subset of T avoiding singular-image fibers, epipoles and branch values, the cone intersection is the etale fiber product in a triangulation chart. Hence every nonbaseline component has generic multiplicity one. Together with the absence of embedded components, this also proves the saturation assertion.

It remains to calculate the overlap of base divisors. At a matched pair p, q , absorb units into uniformizers; this is possible in characteristic zero over k . The completed local fiber product has equation

$$k[[z, w]]/(z^{e_p} - w^{e_q}).$$

Put $h = \gcd(e_p, e_q)$. There are h normalized branches. On each, the projection indices are e_q/h and e_p/h . The total minimum-base contribution is

$$h \min(d_p e_q/h, d_q e_p/h) = \min(d_p e_q, d_q e_p).$$

Every overlap point arises this way, so $\sum_j \deg \min(B_{X,j}, B_{Y,j}) = S$.

Sum (4) and use $\sum_j a_j = s$, $\sum_j b_j = r$, $\sum_j h_j = rs$. Since

$$ms + nr - rs = mn - (m - r)(n - s) = mn - \epsilon_X \epsilon_Y,$$

we obtain the first formula in (7). Bezout gives the second, since the remaining component is the degree-one baseline. $\square$

The theorem computes the baseline cycle coefficient; it does not describe the entire primary ideal supported on that line.

5 A sharp degree spectrum

Theorem 6. *Suppose $r = s = 1$. There is a unique compatible integral world curve. For fixed image degrees m, n , as the image curves vary subject to these hypotheses, its possible degrees are exactly*

$$\max(m, n), \max(m, n) + 1, \dots, m + n - 1.$$

Proof. Both normalized image curves identify with T , so their fiber product is T and uniqueness follows. The bounds follow from (5).

For each $0 \leq k \leq \min(m-1, n-1)$, choose coordinates $[s : t]$ on T and set

$$A = s^{m-1}, \quad B = s^k t^{n-1-k}, \quad D = \text{lcm}(A, B) = s^{m-1} t^{n-1-k}.$$

Use image parametrizations

$$[t^m : As : At], \quad [s^n + t^n : Bs : Bt]. \quad (8)$$

They have no base points: their first coordinates do not vanish at zeros of A, B , respectively, and otherwise their last two coordinates cannot vanish together. Their pencil is $[s : t]$; thus both maps are birational onto their integral images. Their image degrees are m, n , and the epipolar base divisors are those of A, B .

The world parametrization is

$$[(D/A)t^m : (D/B)(s^n + t^n) : Ds : Dt]. \quad (9)$$

It is basepoint free by the same maximum-divisor argument and is birational because its pencil is $[s : t]$. All four coordinates have degree $\deg D + 1 = m + n - 1 - k$. These are exactly the required integers. $\square$

The parameters s, t in this proof are homogeneous coordinates; they are not the earlier cover degrees.

6 Examples and scope checks

The conics

$$xv - u^2 = 0, \quad yu - v^2 = 0$$

have $r = s = 1$, with disjoint epipolar base divisors. Their saturated cone intersection is the twisted cubic $[s^3 : t^3 : s^2t : st^2]$, of degree three, whereas the moving pencil degree is one. Its ideal is

$$(xv - u^2, yu - v^2, xy - uv).$$

This is the familiar baseline-removal example [4, Example 5], not a new counterexample to an open problem. Changing the second conic to $yv - u^2 = 0$ makes the base divisors coincide. The residual curve is then a plane conic, and $\mu_L = 2$.

For cuspidal images $xv^2 - u^3 = 0$, $yv^2 - u^3 = 0$, the base orders are two and the pencil ramification indices are one. Formula (7) gives $\mu_L = 6$ and residual degree three. For $x^2u - v^3 = 0$,

$y^2u - v^3 = 0$, the base orders are one and the ramification indices are two: $\mu_L = 3$ and the residual consists of two plane cubics. With $x^2u - v^3 = 0$, $y^3u - v^4 = 0$, the moving degrees are coprime, two and three. The residual is unique, has degree nine, and $\mu_L = 3$.

More generally, take coprime positive pairs $(a, b), (c, d)$ and

$$f = x^a u^b - v^{a+b}, \quad g = y^c u^d - v^{c+d}.$$

At the epipoles the base orders are b, d and the pencil ramification indices are a, c . Their common pencil value gives

$$\mu_L = bd + \min(bc, da), \quad \sum_j \deg C_j = ac + \max(ad, bc).$$

The supplied checker verifies saturated ideals and Hilbert-function degree differences for all 81 pairs with $a + b, c + d \leq 5$. It also checks six named examples, all 285 constructions (9) with $1 \leq m, n \leq 9$, and 9,216 local-order identities. These are exact rational or integer regression checks, not substitutes for the geometric proofs.

If exactly one image is an epipolar line, weak compatibility is impossible: the common epipolar map would be both constant and nonconstant. If both images are epipolar lines, compatibility holds exactly when they specify the same epipolar plane. A line in that plane avoiding both centers gives a birational reconstruction. The finite-component and degree formulas above are not asserted in this constant-pencil case.

If the camera centers coincide, the two cameras differ by an image homography. Compatibility is then equivalent to that homography carrying one curve to the other; a plane avoiding the common center gives a birational reconstruction. Reducible or nonreduced inputs, real visibility and a smoothness classification are outside the theorems stated here.

The author used AI assistance for mathematical exploration, drafting, literature searches and the exact-checking implementation, and is responsible for the statements and proofs. This preprint is unrefereed; neither independent review nor proof-assistant verification is claimed.

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