

Four- and Five-View Thresholds for Calibration with Known Focal Scales

Alper Ferudun*

October 15, 2026

Abstract

Suppose projective camera matrices are available and the two diagonal focal scales are known in every view, while skew and principal point may vary independently between views. We give explicit sharp generic thresholds for admissible rank-three metric upgrades over the complex numbers: four views give a finite fiber of degree at least two, whereas five compatible views give a unique upgrade. The four-view ambiguity persists on a real open set, with two positive-semidefinite upgrades, affinely independent camera centers, and scene points in front of all cameras in both reconstructions. The proof uses classical dual-quadric calibration, an irreducible incidence variety, two exact regular rational solutions, and a separator for every false upgrade. In the fifth view the product of the two focal scales alone is sufficient. We do not compute the four-view degree or settle other interpretations of the open-ended AIM question on partial calibration.

1 Model and statement

A projective camera is a rank-three matrix $P \in \mathbb{C}^{3 \times 4}$ modulo nonzero scalar multiplication. Fix nonzero numbers f_i, g_i for each view. The intrinsic matrices allowed in that view are

$$K_i = \begin{pmatrix} f_i & s_i & u_i \\ 0 & g_i & v_i \\ 0 & 0 & 1 \end{pmatrix}, \quad s_i, u_i, v_i \in \mathbb{C}. \quad (1)$$

In particular, the residual intrinsics are not required to be constant between views. Projective camera matrices are assumed to have already been reconstructed; this is not a minimal image-correspondence problem.

Write $\mathcal{Q}_3$ for the rank-three locus in $\mathbb{P}(\text{Sym}^2 \mathbb{C}^4)$. For a camera tuple, an *admissible upgrade* is a point $[Q] \in \mathcal{Q}_3$ such that $W_i = P_i Q P_i^T$ is proportional to $K_i K_i^T$ for a matrix (1). We require $(W_i)_{33} \neq 0$. The rank-three and nonzero-entry restrictions are part of the definition; points introduced on the boundary by projective closure are not upgrades.

Let $\mathcal{C}_m$ be the space of m rank-three projective cameras, and let $\mathcal{I}_m \subset \mathcal{Q}_3 \times \mathcal{C}_m$ be the incidence of admissible upgrades and cameras. “Generic” means on a nonempty Zariski open subset of the specified irreducible variety over $\mathbb{C}$.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
Unrefereed preprint. CC BY 4.0.

Theorem 1.1. *For the model (1) the following hold.*

1. *For $m = 1, 2, 3$, generic cameras in $\mathcal{C}_m$ have an admissible upgrade fiber of dimension $8 - 2m$. Without genericity, every nonempty component of an admissible fiber has dimension at least $8 - 2m$.*
2. *For four generic cameras in $\mathcal{C}_4$, the admissible fiber is finite and nonempty. Its generic complex degree is at least two.*
3. *For a generic point of $\mathcal{I}_5$, the five cameras have exactly one admissible upgrade. Thus genericity here is among compatible cameras with a true upgrade, not among arbitrary five-camera tuples.*
4. *There is a nonempty real open set of four-camera tuples with at least two distinct positive-semidefinite rank-three upgrades. With positive known focal scales, both reconstructions have positive focal scales, affinely independent centers, and a common front-facing scene region.*

Consequently four is the sharp generic finiteness threshold and five is the sharp generic complex uniqueness threshold. The real assertion is an open ambiguity set; it does not assert a constant number of real solutions on all generic real inputs.

The AIM Algebraic vision question on partial calibration [1, Reconstruction, Problem 1.2] is recorded in the frozen `ulamai/UnsolvedMath v1.6.0` dataset as `AIM-ALGEBRAIC_GEOMETRY-0184`. Its wording does not fix the model (1). Two axis scales in each camera must not be conflated with the focal lengths of two cameras whose other intrinsics are known. Theorem 1.1 addresses the stated model, not every reading of that question or its known-center branch.

Relation to earlier calibration methods

The dual absolute quadric and intrinsic-subgroup normalization are classical; see Kahl, Triggs and Åström [3, Sections 2–4]. Their detailed critical-motion cases concern other intrinsic families [3, Section 5]. Heyden and Åström establish global ambiguity restrictions for flexible calibration and discuss finite-view budgets [2, Theorem 4.1 and p. 353]. They explicitly distinguish the arbitrary-camera choices in the group argument from a finite sequence. These are antecedents of the present method. Dal Cin et al. [4, Sections 1.1 and 1.3] instead impose constant intrinsics across views, unlike (1).

This note supplies a finite-incidence proof and a regular physical two-upgrade certificate for the stated focal-scale model. It makes no absolute priority claim. The exact four-view degree, a classification of critical motions, and robust numerical recovery remain outside its scope.

2 Equations and irreducible incidence

Multiplying P_i on the left by $\text{diag}(1/f_i, 1/g_i, 1)$ reduces to $f_i = g_i = 1$. This is a fixed invertible change of image coordinates. For a symmetric 3×3 matrix W , put

$$F(W) = W_{22}W_{33} - W_{23}^2 - W_{33}^2, \quad G(W) = \det W - W_{33}^3. \quad (2)$$

Lemma 2.1. *On $W_{33} \neq 0$, admissibility is equivalent to $F(W) = G(W) = 0$. The admissible image-quadric family is a smooth threefold of codimension two in projective symmetric 3×3 matrices.*

Proof. Set $V = W/W_{33}$. Take $u = V_{13}$, $v = V_{23}$, and $s = V_{12} - uv$. The first equation gives $V_{22} = 1 + v^2$. Also

$$\det V = (V_{11} - u^2)(V_{22} - v^2) - (V_{12} - uv)^2.$$

The second equation therefore gives $V_{11} = 1 + s^2 + u^2$. These identities say exactly that $V = KK^T$ for the unique unit upper triangular matrix K with entries s, u, v . Conversely this factorization satisfies both equations. The parameters s, u, v give an affine three-space with a regular inverse in the chart $V_{33} = 1$. $\square$

The rank-three locus $\mathcal{Q}_3$ is the smooth eight-dimensional open part of the determinant hypersurface in $\mathbb{P}^9$. Each view imposes two equations there. The height theorem already gives the unconditional componentwise lower bound in Theorem 1.1(1). Sufficiency at four views needs more than this count.

Lemma 2.2. *The incidence $\mathcal{I}_m$ is irreducible of dimension $8 + 9m$.*

Proof. First fix $Q_0 = \text{diag}(1, 1, 1, 0)$ and write $P = [M \mid t]$. Then $W = MM^T$, and admissibility makes M invertible. Since $(\det M)^2 = W_{33}^3$, the nonzero scalar

$$\lambda = \frac{\det M}{W_{33}} \quad \text{satisfies} \quad \lambda^2 = W_{33}, \quad \lambda^3 = \det M.$$

It scales linearly when the representative of P is rescaled. Thus P/λ is a well-defined normalized representative. By Lemma 2.1, $M/\lambda = KR$ with K unit upper triangular and $R \in \text{SO}_3(\mathbb{C})$. Conversely every such K, R and any last column give an admissible camera. The fiber is isomorphic to

$$U_3 \times \text{SO}_3(\mathbb{C}) \times \mathbb{A}^3,$$

an irreducible variety of dimension nine, where U_3 is the group of unit upper triangular matrices.

The congruence action of GL_4 is transitive on $\mathcal{Q}_3$. The morphism from the irreducible variety

$$\text{GL}_4 \times (U_3 \times \text{SO}_3 \times \mathbb{A}^3)^m$$

given by $Q = HQ_0H^T$ and $P_i = [K_iR_i \mid t_i]H^{-1}$ is surjective onto $\mathcal{I}_m$. Thus the incidence is irreducible. All fibers over $\mathcal{Q}_3$ have dimension $9m$, proving the claim. $\square$

3 A regular four-view ambiguity

Set

$$R = \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}, \quad \begin{aligned} D_1 &= \text{diag}(1, 1, 1), & D_2 &= \text{diag}(1, -1, -1), \\ D_3 &= \text{diag}(-1, 1, -1), & D_4 &= \text{diag}(-1, -1, 1). \end{aligned} \quad (3)$$

Take $P_i = [RD_i \mid t_i]$, where

$$t_1 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}, \quad t_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}, \quad t_4 = \begin{pmatrix} -1 \\ -1 \\ 7 \end{pmatrix}. \quad (4)$$

The matrices RD_i are rotations. Two upgrades are

$$Q_0 = \text{diag}(1, 1, 1, 0), \quad Q_1 = \text{diag}(16, 4, 1, 0). \quad (5)$$

For Q_0 every image quadric is I . For Q_1 every image quadric is

$$4K_*K_*^T, \quad K_* = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

Both satisfy (2) and are positive-semidefinite of rank three.

Use the affine chart

$$Q = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & 1 & h \\ d & g & h & k \end{pmatrix}, \quad (a, b, c, d, e, f, g, h, k), \quad (7)$$

and order the nine equations as $\det Q$ followed by F_i, G_i for $i = 1, \dots, 4$. At the two solutions the Jacobian determinants are

$$\det J(Q_0) = \frac{18907136}{6561}, \quad \det J(Q_1) = \frac{7555843906076672}{2187}. \quad (8)$$

Appendix A gives integer matrices certifying these values, with a dependency-free exact-arithmetic checker in the source archive. In particular both solutions are regular.

Proposition 3.1. *The projection $\mathcal{I}_4 \rightarrow \mathcal{C}_4$ is dominant and generically finite, of generic degree at least two. For $m \leq 3$ the generic fiber dimension is $8 - 2m$.*

Proof. The nonzero determinants (8) allow the implicit function theorem to solve for the nine chart coordinates as functions of the camera entries near each solution. The rank-three and admissibility conditions remain open there. Thus the projection is locally an analytic isomorphism at both points and is dominant. Its source and target are irreducible of dimension 44, by Lemma 2.2, so it is generically finite.

After restricting to a nonempty Zariski open target, a dominant generically finite morphism has finite fibers of fixed cardinality in characteristic zero. The two distinct regular branches persist on a common small analytic neighborhood of the displayed camera tuple. This neighborhood intersects that generic open set, so the generic degree is at least two. This reasoning neither counts the entire special fiber nor asserts that all its other points are isolated.

Forgetting all but the first $m \leq 3$ cameras gives dominance of $\mathcal{I}_m \rightarrow \mathcal{C}_m$. The fiber dimension theorem gives $(8 + 9m) - 11m = 8 - 2m$. $\square$

4 A fifth view separates all false upgrades

Lemma 4.1. *Fix a true upgrade Q . For every rank-three symmetric Q' not proportional to Q , the equation $G(PQ'P^T) = 0$ is not identically satisfied on the family of cameras compatible with Q .*

Proof. A world projectivity reduces to $Q = Q_0$. It suffices to use the subfamily $P = [R \mid t]$ with $R \in \mathrm{SO}_3(\mathbb{C})$. Write

$$Q' = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix}.$$

For fixed R , the determinant of $PQ'P^T$ has degree at most two in t . Indeed Cauchy–Binet expresses it as a sum of products of maximal minors of $[R \mid t]$ and fixed minors of Q' , and each camera minor has degree at most one in t .

Set $t = xe_3$ and let z be the third row of R . Then

$$(PQ'P^T)_{33} = zAz^T + 2xzb + cx^2.$$

If G vanished for every x and R , comparing the coefficient of x^6 would give $c = 0$. Comparing the coefficient of x^3 then gives $(zb)^3 = 0$ for every z with $zz^T = 1$, hence $b = 0$. Since Q' has rank three, A is invertible. The remaining identity is

$$\det A = (zAz^T)^3 \quad \text{for every } z \text{ with } zz^T = 1.$$

Every such z is a third row of a complex special orthogonal matrix. The affine quadric $zz^T = 1$ is irreducible; the regular function zAz^T has values in the finite set of cube roots of $\det A$ and is therefore constant, say λ . Evaluating at e_i and $(e_i + e_j)/\sqrt{2}$ gives $A = \lambda I$. Thus $Q' = \lambda Q_0$, contrary to the hypothesis. $\square$

Proof of Theorem 1.1(3). Restrict the first four cameras to a nonempty open set $B \subset \mathcal{C}_4$ where the projection is quasi-finite. The off-diagonal fiber product

$$\mathcal{J} = (\mathcal{I}_4|_B) \times_B (\mathcal{I}_4|_B) \setminus \{[Q] = [Q']\}$$

has dimension at most 44. Over each of its points, a fifth camera compatible with the first selected upgrade Q varies in an irreducible nine-dimensional family. By Lemma 4.1, imposing compatibility with the distinct Q' restricts it to dimension at most eight. Thus the ambiguity incidence has dimension at most 52.

The true five-view incidence over B is a nonempty open subset of the irreducible $\mathcal{I}_5$, of dimension $8 + 45 = 53$. The closure of the image of the ambiguity incidence is consequently proper. Outside that closure there is no false admissible upgrade. The true upgrade exists by definition. This argument uses the dimension of the image closure, not the incorrect assertion that a nonproper projection is closed. $\square$

Corollary 4.2. *If the first four views have both focal scales specified, then a generic compatible fifth view with only their nonzero product specified still gives a unique admissible upgrade.*

Proof. Normalize that known product to one. Its sole constraint is $G = 0$. The true fifth-camera fiber is irreducible of dimension ten: the intrinsic matrices have $K_{33} = 1$, $\det K = 1$ and nonzero diagonal, an irreducible four-dimensional family. Lemma 4.1 uses G alone. The bad incidence has dimension at most $44 + 9 = 53$, whereas the true incidence has dimension $44 + 10 = 54$. $\square$

5 Physical ambiguity and positive scene depths

Let $L = \text{diag}(4, 2, 1)$, $H = \text{diag}(4, 2, 1, 1)$ and

$$S = \frac{1}{3} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}.$$

Direct multiplication gives $RL = 2K_*S$, and $S \in \text{SO}_3(\mathbb{R})$. Since L commutes with every D_i ,

$$P_i H = 2K_*[SD_i \mid K_*^{-1}t_i/2]. \quad (9)$$

These cameras have the same positive unit focal scales as the original ones, but different skew and principal point. The world metrics are not similarity-equivalent because Q_1 and Q_0 are not proportional.

At the origin the original depths are 4, 2, 3, 7; the depths in (9) are 2, 1, 3/2, 7/2. All are positive. Hence a neighborhood of the origin contains arbitrarily many points in front of all cameras in both reconstructions, with corresponding points X and $H^{-1}X$ producing the same images. The original centers $C_i = -(RD_i)^T t_i$ satisfy

$$\det \begin{pmatrix} C_1^T & 1 \\ C_2^T & 1 \\ C_3^T & 1 \\ C_4^T & 1 \end{pmatrix} = -164 \neq 0.$$

The second centers are $L^{-1}C_i$, so they too are affinely independent.

Both upgrades have three strictly positive nonzero eigenvalues. They remain positive on the two small real solution branches supplied by (8). Continuous choices of metric factorizations and rotations preserve positive scales, affine independence and the strict depth inequalities. A fixed finite scene in a sufficiently small common ball stays front-facing under these choices. This proves the real open ambiguity assertion and completes Theorem 1.1.

A Integer regularity certificates

In the coordinate order (7), the derivative of $\det Q$ is $(0, \dots, 0, 1)$ at Q_0 and $(0, \dots, 0, 64)$ at Q_1 . Delete this row and the final column from the full Jacobian. The two remaining matrices are respectively $A_0/3$ and $4A_1/3$, where

$$A_0 = \begin{pmatrix} 1 & 0 & -4 & -8 & -1 & -4 & -16 & -16 \\ 2 & -4 & -4 & -16 & -1 & -8 & -32 & -32 \\ 1 & 0 & 4 & -16 & -1 & -4 & 14 & -4 \\ 2 & 4 & 4 & -24 & -1 & -8 & 18 & 6 \\ 1 & 0 & -4 & -2 & -1 & 4 & -8 & 20 \\ 2 & 4 & -4 & -4 & -1 & 8 & -28 & 28 \\ 1 & 0 & 4 & 18 & -1 & 4 & 30 & -24 \\ 2 & -4 & 4 & 36 & -1 & 8 & 54 & -54 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & -2 & -4 & -16 & -1 & 0 & -8 & 16 \\ -1 & -16 & -16 & -88 & -4 & -32 & -128 & 64 \\ 0 & 2 & 4 & -14 & -1 & 0 & -2 & -32 \\ -1 & 16 & 16 & -60 & -4 & -32 & 72 & -192 \\ 0 & 2 & -4 & 8 & -1 & 0 & -10 & 4 \\ -1 & 16 & -16 & 50 & -4 & 32 & -112 & 16 \\ 0 & -2 & 4 & 30 & -1 & 0 & 12 & 36 \\ -1 & -16 & 16 & 162 & -4 & 32 & 216 & 144 \end{pmatrix}.$$

Exact elimination gives

$$\det A_0 = 18907136, \quad \det A_1 = 5404360704.$$

Expansion in the determinant-constraint row gives $\det J(Q_0) = \det A_0/3^8$ and $\det J(Q_1) = 64(4/3)^8 \det A_1$, which are (8).

The accompanying Python checker uses rational arithmetic from the standard library. It computes constraint derivatives by cofactors and cross-checks all 162 entries by exact polynomial interpolation. It also checks the printed matrices, intrinsic and rotation factorizations, positive origin depths and center independence. These are finite certificates for the displayed witness; the incidence and separation arguments prove the generic statements.

This unrefereed note was prepared with AI-assisted mathematical exploration, literature comparison, exact-arithmetic programming and exposition. The proof received an originating-researcher self-audit; no independent review or proof-assistant formalization is claimed.

References

- [1] American Institute of Mathematics, *Algebraic vision: Reconstruction*, Problem 1.2. <http://aimpl.org/algvision/1/>. Accessed October 15, 2026.
- [2] A. Heyden and K. Åström, Flexible Calibration: Minimal Cases for Auto-Calibration, *Proceedings of ICCV* (1999), 350–355. doi:10.1109/ICCV.1999.791241.
- [3] F. Kahl, B. Triggs and K. Åström, Critical Motions for Auto-Calibration When Some Intrinsic Parameters Can Vary, *Journal of Mathematical Imaging and Vision* **13** (2000), 131–146. doi:10.1023/A:1026524030731.
- [4] A. P. Dal Cin, T. Duff, L. Magri and T. Pajdla, Minimal Perspective Autocalibration, *Proceedings of CVPR* (2024), 5064–5073. CVF Open Access.