

Infinitesimal Lifting Does Not Imply Flatness for Green Functors

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Abstract

Let k be any nonzero commutative ring of characteristic two. We show that the ghost map from the constant C_2 -Tambara functor on k is finitely presented and has unique lifting against square-zero extensions, but is not flat for the relative Mackey box product. Its two orbit maps are the finite étale ring maps $k \rightarrow k \times k$ and $k \rightarrow k$, and both its Green and genuine Tambara differential modules vanish. The failure of flatness is witnessed by an explicit monomorphism of modules whose tensor acquires a kernel isomorphic to a top-supported copy of k . The ghost construction is known; the point here is its application to the lifting-versus-flatness question in the 2016 AIM workshop report on equivariant derived algebraic geometry. This distinguishes infinitesimal lifting from the convention that includes flatness in the definition of formal étaleness.

1 The question and the example

The report of the AIM workshop *Equivariant derived algebraic geometry* asks whether formal étaleness of a map of Green functors, defined through lifting over square-zero extensions, implies flatness [1, §4, p. 3]. We give a negative answer, even with finite presentation and with finite étale maps at each orbit.

Throughout, $G = C_2$ and k is a nonzero commutative ring with $2 = 0$. Put $C = \underline{k}$, the constant G -Tambara functor: its two orbit rings are k , restriction is the identity, transfer is multiplication by two, and the norm is squaring. For a Mackey functor or module M , write

$$M_G = M(G/G), \quad M_e = M(G/e).$$

Flatness of a C -module means exactness of $M \square_C (-)$ on all C -modules in Mackey functors.

Here *infinitesimally étale* means that for every levelwise surjection $U \rightarrow V$ of commutative C -algebras with square-zero kernel, the map

$$\mathrm{Hom}_{C\text{-alg}}(S, U) \longrightarrow \mathrm{Hom}_{C\text{-alg}}(S, V)$$

is bijective. We use the term separately in Green and Tambara algebras. This is not Hill's definition of *formally étale*, which requires both box-flatness and vanishing genuine differentials [3, Definition 5.9]. The latter convention, with finite presentation added for étaleness, is used in [6, Definition 2.13].

Define S by

$$S_G = k \times k, \quad S_e = k, \quad \mathrm{res}(a, b) = a, \quad \mathrm{tr}(a) = 0, \quad N(a) = (a^2, a^2), \quad (1)$$

with trivial Weyl action. Define $\iota : C \rightarrow S$ by the diagonal at G/G and the identity at G/e .

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Theorem 1. *The map ι is a morphism of complete C_2 -Tambara functors. In both commutative Green algebras and Tambara algebras over C , S is finitely presented and infinitesimally étale. Nevertheless, S is not box-flat over C . Both orbit maps of ι are finite étale, and*

$$\Omega_{S/C}^{1,\text{Green}} = \Omega_{S/C}^{1,\text{Tamb}} = 0.$$

The algebra S is not a new construction. It is the ghost of C , and ι is its ghost map in the sense of Chan–Mehrlé–Quigley–Spitz–Van Niel [2, Definitions 7.3 and 7.11]. Indeed, the transfer ideal of C_G is zero, its geometric fixed-point ring is k , and the ghost formulas give exactly (1). Their Remark 7.4 explains earlier Green-functor and Tambara antecedents. Our application concerns the simultaneous finite presentation, infinitesimal lifting, and explicit failure of box-flatness of this map. No priority for the ghost construction is claimed.

2 Structure and a functor of points

The formulas (1) can also be checked directly using Strickland’s description of C_2 -Tambara functors by Tambara pairs [5, Definition 7.1, Theorem 7.7]. Restriction is a unital ring map, and transfer is additive and satisfies Frobenius reciprocity. Since $2 = 0$,

$$\text{res tr}(a) = 0 = a + a, \quad \text{res } N(a) = a^2.$$

The norm is multiplicative, preserves zero and one, and satisfies

$$N(a + b) = N(a) + N(b) + \text{tr}(ab).$$

The Weyl identities are trivial. These are the Tambara-pair identities, so the formulas give a complete Tambara functor. Compatibility of ι with all operations is immediate.

Let $e = (0, 1) \in S_G$. The following description proves finite presentation and will also control lifting.

Lemma 2. *For every commutative Green C -algebra U there is a natural bijection*

$$\text{Hom}_{C\text{-alg}}(S, U) \cong \{u \in U_G : u^2 = u, \text{ res}(u) = 0\}. \quad (2)$$

The same bijection holds in Tambara C -algebras.

Proof. A map sends e to an element of the indicated set. Conversely, write $\alpha : C \rightarrow U$ for the structure map and let u satisfy the two equations. Define

$$f_G(a, b) = \alpha_G(a) + \alpha_G(b - a)u, \quad f_e(a) = \alpha_e(a).$$

Idempotence of u makes f_G a unital ring map extending α_G on the diagonal. The restriction equation ensures compatibility with res . The transfer condition follows from

$$\text{tr}_U(\alpha_e(a)) = \alpha_G(\text{tr}_C(a)) = 0.$$

These prove the Green assertion, including uniqueness.

If U is a Tambara algebra, the only additional operation between distinct transitive orbits is its norm. On the bottom ring of S the map is the base structure map, so

$$N_U(f_e(a)) = \alpha_G(a^2) = f_G(a^2, a^2).$$

Thus norm compatibility is automatic and the same formulas prove the Tambara assertion. $\square$

Corollary 3. *In either category, S is finitely presented over C .*

Proof. Lemma 2 presents S by one generator e at G/G and two relations:

$$e^2 = e \text{ at } G/G, \quad \text{res}(e) = 0 \text{ at } G/e.$$

Equivalently, its functor of points preserves filtered colimits, which are computed levelwise: one element and finitely many equations are detected at a finite stage. For the equivalence between compactness and finite presentation of Tambara algebras, see [4, Proposition 4.3]. $\square$

This argument does not infer finite presentation just from the size or finite generation of the two orbit rings.

3 Unique infinitesimal lifts

Lemma 4. *Let $q : U \rightarrow V$ be a levelwise surjection of commutative Green C -algebras with square-zero kernel. Every C -algebra map $S \rightarrow V$ has a unique lift to U . The same statement holds for Tambara C -algebras.*

Proof. By Lemma 2, the given map is an idempotent $v \in V_G$ with $\text{res}(v) = 0$. Choose $x \in U_G$ with $q_G(x) = v$. Then $x^2 - x$ is in the square-zero kernel. In characteristic two,

$$(x^2)^2 - x^2 = (x^2 - x)^2 = 0.$$

Thus $u = x^2$ is an idempotent lift of v . Its restriction is an idempotent in the square-zero kernel of $U_e \rightarrow V_e$, and hence is zero. Lemma 2 now gives the desired lift.

If two idempotent lifts differ by j in the kernel, then commutativity and characteristic two give $(u + j)^2 = u^2 + j^2 = u$. Since $u + j$ is also idempotent, $j = 0$. Thus the lift is unique. The representing property handles all norm operations in the Tambara case, so no further lifting condition is necessary. $\square$

The same proof covers levelwise nilpotent kernels: replace x by a sufficiently large 2-power of x . In characteristic two the difference of two idempotents is itself idempotent, so a nilpotent difference must be zero. In particular, Lemma 4 applies to genuine square-zero extensions of Tambara functors as well as to the Green extensions.

4 An explicit tensor obstruction

For a k -module Q , let $D(Q)$ be the C -module with top value Q , bottom value zero, and zero restriction and transfer. Put $D = D(k)$. Define another C -module E by

$$E_G = k, \quad E_e = k, \quad \text{res}_E = 0, \quad \text{tr}_E = \text{id},$$

with trivial Weyl action. These are valid C -modules: $\text{res tr} = 1 + \sigma = 0$ and $\text{tr res} = 2 = 0$, and the scalar Frobenius identities hold. There is a monomorphism

$$j : D \longrightarrow E \tag{3}$$

which is the identity at G/G and zero at G/e .

Lemma 5. *For every C -module M there is a natural isomorphism*

$$D \square_C M \cong D(M_G / \text{tr}_M(M_e)). \quad (4)$$

Proof. Use the universal property in terms of C -balanced Mackey pairings into a C -module W . Since $D_e = 0$, the bottom component of a pairing is zero. The top component is determined by a k -linear map $f : M_G \rightarrow W_G$, using $D_G = k$. Restriction compatibility forces $\text{res}_W f = 0$. Frobenius reciprocity forces

$$f(\text{tr}_M y) = \text{tr}_W(\text{bottom pairing of } \text{res}_D(1) \text{ and } y) = 0.$$

The other transfer relation is zero because $D_e = 0$. These are all the relations, and conversely they define a balanced pairing. Such maps are exactly C -module maps $D(M_G / \text{tr}_M(M_e)) \rightarrow W$. The representing objects are therefore naturally isomorphic. $\square$

In particular,

$$D \square_C D \cong D, \quad D \square_C E = 0. \quad (5)$$

As a C -module, S splits as $C \oplus D$: on the top ring use $(a, b) = (a, a) + (0, b - a)$, and on the bottom use the identity. The box product is additive and has unit C , so

$$S \square_C D \cong D \oplus D, \quad S \square_C E \cong E.$$

Under these identifications $S \square_C j$ sends (d_1, d_2) to $j(d_1)$. Its kernel is the second copy of D , which is nonzero because $k \neq 0$. Thus the box product with S does not preserve the monomorphism (3), proving nonflatness.

The splitting used here is a module splitting, not a product decomposition into Tambara functors with a nonzero top-supported factor. No such Tambara factor is needed.

5 Differentials and the precise conclusion

The orbit maps of ι are the identity of k and the diagonal $k \rightarrow k \times k$. They are finite étale ordinary ring maps. Also, any C -relative Green derivation $d : S \rightarrow M$ is zero at the free orbit, since $S_e = k$. At the fixed orbit it kills the base ring and satisfies

$$d(e) = d(e^2) = 2e d(e) = 0.$$

Thus it vanishes everywhere. The universal Green differential module is zero; the genuine Tambara differential module, a quotient of it, is also zero. This completes the proof of Theorem 1.

The example separates three conditions that should not be conflated: orbitwise finite étaleness, unique infinitesimal lifting, and box-flatness. It refutes the lifting-implies-flatness assertion from the AIM report, even for a finitely presented algebra admitting all Tambara norms. It does not refute a definition that explicitly includes flatness, nor answer the report's separate question about the diagonal or its spectral analogue. The broader problem of developing equivariant derived algebraic geometry is not claimed to be resolved.

The accompanying standard-library Python checker performs 224655 exact supplementary tests. These include the displayed Tambara-pair identities over $\mathbb{F}_2[z]/(z^n)$ for $1 \leq n \leq 4$, idempotent lifts for $n \leq 6$, and a separate Frobenius-relation calculation of the nonzero tensor kernel. Normal and optimized executions agree. These finite checks supplement the general proofs; they are not a formalization.

This work was developed with AI assistance for mathematical exploration, literature comparison, drafting, and exact computations. The preprint is self-audited and unrefereed; no independent review, proof-assistant certification, or absolute priority is claimed.

References

- [1] American Institute of Mathematics, *Equivariant derived algebraic geometry*, workshop report, 2016, §4, pp. 3–4. <https://aimath.org/pastworkshops/equideralggeomrep.pdf>.
- [2] D. Chan, D. Mehrle, J. D. Quigley, B. Spitz, and D. Van Niel, *On the Tambara affine line*, *Advances in Mathematics* 493 (2026), 110908. <https://arxiv.org/abs/2410.23052>.
- [3] M. A. Hill, *On the André–Quillen homology of Tambara functors*, *Journal of Algebra* 489 (2017), 115–137. <https://arxiv.org/abs/1701.06219>.
- [4] J. Schuchardt, B. Spitz, and N. Wisdom, *Algebraically closed fields in equivariant algebra*, preprint, 2025. <https://arxiv.org/abs/2505.05539>.
- [5] N. Strickland, *Tambara functors*, preprint, 2012. <https://arxiv.org/abs/1205.2516>.
- [6] N. Wisdom, *Affine étale group schemes over Tambara fields*, preprint, version 2, 2025. <https://arxiv.org/abs/2508.09365v2>.