Smooth Hypersurfaces Beyond the Chevalley-Warning Range over Prime Fields

Alper Ferudun*

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Abstract

For every prime p, integer $n \geq 2$, and degree $d \geq n+1$, we construct a geometrically smooth hypersurface $X \subset \mathbb{P}^n_{\mathbb{F}_p}$ of degree d such that $\#X(\mathbb{F}_p) \not\equiv 1 \pmod p$. The construction stays over the specified prime field in every characteristic and degree. A finite-field moment polynomial supplies nonzero coefficients with the required point count. In odd characteristic not dividing the degree, a sharper individual-degree bound permits simultaneous avoidance of the singular parameter. When the characteristic divides the degree, positive exponent compositions make a triangular family smooth for all nonzero coefficients, apart from one explicitly handled boundary case. A separate uniform formula treats characteristic two in odd degree. The coefficient selection is a finite deterministic procedure; no efficient complexity bound is asserted.

1 Introduction

The Chevalley–Warning congruence implies $\#X(\mathbb{F}_p) \equiv 1 \pmod p$ for a hypersurface of positive degree $d \leq n$ in $\mathbb{P}^n_{\mathbb{F}_p}$. Beyond that range, the issue considered here is to violate the congruence while preserving smoothness over the specified prime field.

Problem 9.5 of the AIM list *Rational subvarieties in positive characteristic* asks for a degree-d hypersurface over $\mathbb{F}_p$, for each $d \geq n+1$, whose point count is not 1 modulo p or which is not rationally connected [2]. The displayed AIM wording does not explicitly impose smoothness. A closely matching original question from 2015 does impose smoothness and explicitly asks for every triple (d, p, n) in this range [11]. We address this stronger numerical formulation.

Theorem 1.1. *Let p be any prime, $n \geq 2$, and $d \geq n+1$. There is a geometrically smooth, geometrically integral degree-d hypersurface $X \subset \mathbb{P}^n_{\mathbb{F}_p}$ satisfying*

$$\#X(\mathbb{F}_p) \not\equiv 1 \pmod p.$$

Its defining equation can be obtained by a finite deterministic coefficient-selection procedure over $\mathbb{F}_p$.

The last assertion means a finite search in a specified sparse family whose success is proved below. It does not mean a single closed formula for all coefficients or a polynomial-time algorithm. The count property and smoothness always hold for the *same* selected equation.

Several ingredients and overlapping families are established. Finite-field coefficient formulas for these congruences are classical; at the degree-$(n+1)$ boundary, see [8, Proposition 5.15 and

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.

Exercise 5.17]. The discussion of [11] includes Fermat examples when $n+1$ divides $p-1$, as well as pencil constructions. Poonen's Bertini theorem with prescribed local conditions gives smooth point-free examples in every sufficiently large degree for fixed p, n [9, Theorem 1.2]. That statement alone does not supply the remaining degrees.

Diagonal hypersurfaces with a monomial deformation and their point counts were studied by Koblitz [6, Section 3] and, without the earlier restriction $d \mid (q-1)$, by McCarthy [7, Theorem 2.1]. The diagonal smoothness condition used below is a specialization of Kloosterman's criterion [4, Lemma 3.7]. The undeformed triangular chain in dividing characteristic is Gabber's construction as recorded by Katz and Sarnak [3, Section 11.4.6 of the author-hosted version].

Generic Hasse–Witt results [1] and Delsarte deformations [5] provide further related frameworks. A nonempty generic open subset need not contain an $\mathbb{F}_p$-point. Here the simultaneous prime-field choice follows from a sharper $p-3$ individual-degree bound, positive exponent compositions for the deformed chain, and a separate binary formula. The contribution is their uniform combination for every required degree and characteristic. We make no priority claim for the finite-field moment method, the known diagonal smoothness criterion, or the underlying diagonal and chain family shapes.

Riedl and Woolf prove that a smooth complete intersection over $\mathbb{F}_q$ whose point count is not 1 modulo q is not geometrically uniruled [10, Corollary 3.10]. Thus the hypersurfaces of theorem 1.1 also have that property. This is an application of their result, not a new proof of the geometric implication. The elementary numerical construction below does not depend on it.

Conventions

Put $m = n+1$, so throughout the proof $m \geq 3$ and $d \geq m$. Write $x = (x_0, \ldots, x_{m-1})$. A monomial has *full support* if every coordinate occurs with positive exponent. All smoothness calculations are over an algebraic closure of $\mathbb{F}_p$. When $p \mid d$, a common zero of the partial derivatives need not lie on the hypersurface; the equation $F = 0$ will always be retained.

2 A coefficient-selection lemma over the prime field

Let $v = (v_0, \ldots, v_{m-1})$ have positive integer entries of sum d, and put $Q = x^v$. Let $H_0, \ldots, H_{r-1}$ be degree-d monomials, each missing at least one variable. For indeterminates $A = (A_0, \ldots, A_{r-1})$, set

$$F_A = Q + \sum_{j=0}^{r-1} A_j H_j, \tag{1}$$

$$C(A) = (-1)^m \sum_{x \in \mathbb{F}_p^m} F_A(x)^{p-1} \quad \text{in } \mathbb{F}_p[A_0, \ldots, A_{r-1}]. \tag{2}$$

In equation (2), the coordinates are summed over the field but the parameters remain indeterminates.

Lemma 2.1. *The polynomial C has constant coefficient* 1 *and* $\deg_{A_j} C \leq p-2$ *for every j. Consequently, some $A \in (\mathbb{F}_p^\times)^r$ satisfies $C(A) \neq 0$.*

Proof. A term of degree $p-1$ in A_j can arise only from H_j^{p-1}. Since H_j misses a variable, summing in that coordinate gives the factor $p = 0$ in $\mathbb{F}_p$. The individual-degree bound follows.

For every positive integer a, $\sum_{u\in\mathbb{F}_p} u^{a(p-1)} = -1$. Hence the constant coefficient is

$$(-1)^m \sum_{x\in\mathbb{F}_p^m} Q(x)^{p-1} = (-1)^m \prod_{i=0}^{m-1} \left(\sum_{u\in\mathbb{F}_p} u^{v_i(p-1)} \right) = 1.$$

Evaluation on $(\mathbb{F}_p^\times)^r$ is injective on polynomials whose individual degrees are less than $p-1$. Indeed, if such a polynomial vanishes on that grid, fix all but its last variable. The resulting polynomial has $p-1$ roots and degree less than $p-1$, so each coefficient vanishes on the smaller grid. Induction on r, starting with the univariate root bound, proves the assertion. Apply this to the nonzero polynomial C. When $p=2$, all individual degrees are zero and C is the constant polynomial 1. ☐

Lemma 2.2. *For every actual parameter tuple $A \in \mathbb{F}_p^r$, the hypersurface defined by F_A has*

$$N_{\text{proj}} = 1 + (-1)^m C(A) \pmod p. \tag{3}$$

Proof. Let N_{aff} count affine zeros, including the origin. Homogeneity gives $N_{\text{aff}} = 1 + (p-1)N_{\text{proj}}$. The indicator of a zero of F_A is $1 - F_A^{p-1}$. Summing over $\mathbb{F}_p^m$ therefore gives

$$N_{\text{aff}} = -\sum_{x\in\mathbb{F}_p^m} F_A(x)^{p-1} \pmod p.$$

Combining these equalities yields equation (3). ☐

It remains to make a count-good tuple from lemma 2.1 geometrically smooth. Generic smoothness alone is not used for this selection.

3 Odd characteristic not dividing the degree

Assume p is odd and $p \nmid d$. Set $t = d - m + 1 \geq 1$ and use

$$Q = x_0^t x_1 \cdots x_{m-1}, \qquad F_A = Q + \sum_{i=0}^{m-1} A_i x_i^d, \qquad A_i \in \mathbb{F}_p^\times. \tag{4}$$

Lemma 3.1. *For the family in equation* (4)*,* $\deg_{A_{m-1}} C \leq p-3$.

Proof. The degree-$(p-1)$ term is zero by lemma 2.1. A degree-$(p-2)$ term in A_{m-1} uses $x_{m-1}^{d(p-2)}$ and exactly one further summand of F_A. If that summand is another pure power, at most two variables occur; since $m \geq 3$, a missing-coordinate sum vanishes. If the summand is Q, any coordinate indexed by $1, \dots, m-2$ has exponent one. Its sum over $\mathbb{F}_p$ is zero because p is odd. Thus that term vanishes as well. ☐

The singular-parameter calculation in the next proof recovers the relevant case of [4, Lemma 3.7]. We include it to make the coefficient-selection argument self-contained. Normalizing diagonal coefficients by taking d-th roots would be legitimate geometrically, but is not assumed possible over $\mathbb{F}_p$.

Proposition 3.2. *Some nonzero coefficient tuple in equation* (4) *defines a geometrically smooth hypersurface with point count not* 1 *modulo* p*.*

Proof. First consider a common zero of the partial derivatives on a coordinate boundary. If at least two coordinates are zero, all derivatives of Q vanish. The derivative at any nonzero coordinate is then $dA_i x_i^{d-1} \neq 0$, a contradiction. If exactly x_j is zero and $j \neq 0$, the x_0 derivative is nonzero. If $j = 0$, the derivative at any other coordinate is nonzero; its derivative of Q still contains x_0. Thus no such common zero exists.

On the coordinate torus, put $v = (t, 1, \dots, 1)$. Multiplying the derivative equations by their coordinates gives

$$dA_i x_i^d = -v_i Q. \tag{5}$$

If $p \mid t$, the equation with $i = 0$ is impossible, so every nonzero parameter tuple is smooth. Otherwise, raise the i-th equation to the positive integer v_i, multiply, and cancel Q^d. This gives the necessary condition

$$d^d A_0^t A_1 \cdots A_{m-1} = (-1)^d t^t. \tag{6}$$

For fixed $A_0, \dots, A_{m-2}$, at most one nonzero A_{m-1} satisfies this condition. Necessity suffices; no converse is asserted.

Choose a tuple with $C(A) \neq 0$ using lemma 2.1, and fix its first $m-1$ entries. The resulting nonzero polynomial in A_{m-1} has degree at most $p-3$ by lemma 3.1. At least two elements of $\mathbb{F}_p^\times$ therefore keep C nonzero. At most one satisfies equation (6). Choose a different one. The resulting equation is smooth and satisfies equation (3) with $C(A) \neq 0$. If $p \mid t$, the first count-good tuple already suffices. □

4 Degrees divisible by the characteristic

Assume $p \mid d$. For a positive vector v of sum d, put

$$F_A = \sum_{i=0}^{m-2} A_i x_i^{d-1} x_{i+1} + A_{m-1} x_{m-1}^d + Q, \qquad Q = x^v, \quad A_i \in \mathbb{F}_p^\times. \tag{7}$$

Each monomial other than Q misses a coordinate because $m \geq 3$. Thus lemmas 2.1 and 2.2 apply. After reversing the coordinates, the part without Q is the Gabber chain from [3, Section 11.4.6 of the author-hosted version]. The following sufficient conditions control the added full-support monomial. Write

$$T_i = x_i^{d-1} x_{i+1}, \qquad s_i = \sum_{j=0}^{i} v_j \ (0 \leq i \leq m-2), \qquad B = 1 + \sum_{i=0}^{m-2} s_i \quad \text{in } \mathbb{F}_p.$$

Lemma 4.1. *Every nonzero coefficient tuple in equation* (7) *is geometrically smooth if either a proper prefix s_i is divisible by p, or $B = 0$ in $\mathbb{F}_p$.*

Proof. At a common derivative zero on the torus, multiplication by the corresponding coordinate gives, since $d = 0$ in $\mathbb{F}_p$,

$$\begin{aligned} &A_0 T_0 = v_0 Q, \\ &A_i T_i = A_{i-1} T_{i-1} + v_i Q = s_i Q \qquad\qquad (1 \leq i \leq m-2). \end{aligned}$$

The final equation is $A_{m-2} T_{m-2} + v_{m-1} Q = dQ = 0$. If a proper s_i is zero, its equation is impossible on the torus. Otherwise, the value of the defining equation at such a critical point is

$$F_A = BQ + A_{m-1} x_{m-1}^d. \tag{8}$$

When $B = 0$, this value is nonzero. The critical point therefore does not lie on the hypersurface.

For completeness, consider all coordinate boundary strata. If at least two coordinates are zero, every derivative of Q vanishes. The last derivative gives $x_{m-2} = 0$, and moving backwards through the other derivatives gives $x_0 = \cdots = x_{m-2} = 0$. Here $d - 2 \geq 1$ is used. The remaining possible projective point is not on $F_A = 0$, since its value is $A_{m-1}x_{m-1}^d \neq 0$.

If exactly one coordinate x_j is zero and $v_j \geq 2$, again every derivative of Q vanishes, so the same reasoning applies. It remains to suppose exactly x_j is zero and $v_j = 1$. If $j = m-1$, the first chain derivative $-A_0x_0^{d-2}x_1$ is nonzero. If $j \neq m-1$, the last derivative, whose Q-term contains x_j, forces $j = m-2$. For $m \geq 4$, the derivative indexed by $m-3$ then gives $A_{m-4}x_{m-4}^{d-1} = 0$, a second zero coordinate. For $m = 3$ and $j = 1$, every T_i and Q vanishes but $A_2x_2^d$ does not. Hence this point too is off the hypersurface. The argument covers every boundary stratum. ☐

4.1 Multiples greater than the characteristic

Proposition 4.2. *If $p \mid d$ and $d > p$, some member of equation (7) is smooth and has point count not 1 modulo p, for every $3 \leq m \leq d$.*

Proof. We have $d \geq 2p$. Choose an integer k such that

$$\max(1, m-d+p) \leq k \leq \min(p, m-1). \tag{9}$$

The interval is nonempty: its lower bound is at most p because $m \leq d$, and at most $m-1$ because $d - p \geq 1$ and $m \geq 3$. Define

$$\begin{aligned} v_0 &= p - k + 1, & v_1 &= \cdots = v_{k-1} = 1, \\ v_k &= d - p - (m-k) + 1, & v_{k+1} &= \cdots = v_{m-1} = 1. \end{aligned}$$

Empty index ranges are allowed. By equation (9), every entry is positive, their sum is d, and $s_{k-1} = p$ is a proper prefix. Lemma 4.1 makes every nonzero coefficient tuple smooth. Choose a count-good tuple by lemma 2.1 and apply lemma 2.2. ☐

4.2 Degree equal to the characteristic with fewer variables

Proposition 4.3. *If $d = p$ and $3 \leq m < p$, some member of equation (7) is smooth and has point count not 1 modulo p.*

Proof. Here $p \geq 5$. Write $v_j = 1 + e_j$, where $e_j \geq 0$ and $\sum_j e_j = L = p - m$. The value of B is

$$B = 1 + \frac{m(m-1)}{2} + \sum_{j=0}^{m-1}(m-1-j)e_j \pmod{p}. \tag{10}$$

The final sum takes every integer in $[0, (m-1)L]$. Indeed, it is a sum of L elements of $\{0, \ldots, m-1\}$, with repetition. For a prescribed sum h, use $\lfloor h/(m-1) \rfloor$ copies of $m-1$, one remainder if necessary, and then zeros. At most L terms are needed.

If $m = p-1$, the first two terms of equation (10) give 2 modulo p. The available value $h = p-2 = m-1$ makes $B = 0$. If $3 \leq m \leq p-2$, then

$$(m-1)(p-m) - 2(p-3) = (m-3)(p-m-2) \geq 0,$$

and $2(p-3) \geq p-1$. The interval of attainable values therefore contains every residue modulo p. Choose one cancelling the first two terms in equation (10). This gives a positive vector v of sum p with $B = 0$. Now apply lemmas 2.1, 2.2 and 4.1. ☐

4.3 The equal-prime boundary

Proposition 4.4. *For $d = p = m \geq 3$, choose $a \in \mathbb{F}_p^\times \setminus \{1\}$. The equation*

$$F = \sum_{i=0}^{p-2} x_i^{p-1} x_{i+1} + a x_{p-1}^p + P, \qquad P = x_0 \cdots x_{p-1}, \tag{11}$$

is smooth and has point count not 1 modulo p.

Proof. The prime is odd, so $a = 2$ is a valid choice. The boundary analysis in lemma 4.1 does not require its prefix or B hypotheses and applies unchanged to equation (11).

On the torus, the derivative equations give $T_i = (i+1)P$. If also $F = 0$, then $\sum_{j=1}^{p-1} j = 0$ in $\mathbb{F}_p$ gives $P = -a x_{p-1}^p$. On the other hand, multiplying all T_i and using Wilson's theorem yields

$$-P^{p-1} = \prod_{i=0}^{p-2} T_i = \frac{P^p}{x_0 x_{p-1}^{p-1}}.$$

Consequently $x_0 = a x_{p-1}$. Normalize $x_{p-1} = 1$, so $x_0 = a \in \mathbb{F}_p^\times$ and $P = -a$. The recursion

$$x_{i+1} = \frac{-(i+1)a}{x_i^{p-1}} \qquad (0 \leq i \leq p-2)$$

now puts every coordinate in $\mathbb{F}_p^\times$. *After* this has been established, Fermat's little theorem sets every denominator equal to 1. The final step gives $x_{p-1} = a$, contradicting the normalization and $a \neq 1$. Thus no singular point exists.

For the count, the total degree of F^{p-1} is $p(p-1)$ in p variables. A monomial survives all coordinate sums only if its exponent in every variable equals $p-1$. Its coefficient is therefore that of P^{p-1}. Suppose T_0 occurs e_0 times and P occurs b times in a term contributing to this coefficient. The x_0 exponent equation is $(p-1)e_0 + b = p-1$. It permits only $(e_0, b) = (0, p-1)$ or $(1, 0)$. The second option would force $1 + (p-1)e_1 = p-1$ at x_1, impossible for integral e_1 and $p \geq 3$. The first option uses only P and has coefficient 1. Thus $C(F) = 1$ in the normalization equation (2); equation (3) proves the required congruence failure. ☐

5 Characteristic two in odd degree

The remaining case has $p = 2$ and d odd. Put $t = d - m + 1 \geq 1$ and define

$$Q = x_0^t x_1 \cdots x_{m-1}, \qquad F = \sum_{i=0}^{m-1} x_i^d + x_0^{d-1} x_1 + Q. \tag{12}$$

Proposition 5.1. *For every odd $d \geq m \geq 3$, the hypersurface equation* (12) *is geometrically smooth and has an even number of $\mathbb{F}_2$-points.*

Proof. Every monomial except Q misses a coordinate. Since $p - 1 = 1$, summing F over $\mathbb{F}_2^m$ leaves only Q, with sum 1. The affine-to-projective calculation of lemma 2.2 gives $N_{\mathrm{proj}} = 0 \pmod 2$.

As d is odd and $d-1$ is even, the partial derivatives are

$$F_{x_0} = x_0^{d-1} + t x_0^{t-1} \prod_{j=1}^{m-1} x_j, \tag{13}$$

$$F_{x_1} = x_1^{d-1} + Q/x_1 + x_0^{d-1}, \tag{14}$$

$$F_{x_i} = x_i^{d-1} + Q/x_i \qquad (2 \leq i \leq m-1). \tag{15}$$

Here Q/x_i denotes a polynomial monomial, so the notation is also meaningful when $x_i = 0$.

If t is even, equation (13) forces $x_0 = 0$. Then $t \geq 2$, so all other Q-derivatives vanish; the remaining equations force all other coordinates to be zero. There is no projective common derivative zero.

Suppose t is odd. If at least two coordinates vanish, all Q-derivatives vanish, and the equations again kill every coordinate. If exactly x_j vanishes with $j \neq 0$, the first derivative is nonzero. If exactly x_0 vanishes, the first derivative is nonzero when $t = 1$; for $t \geq 3$, the second derivative is nonzero. Thus a common derivative zero must lie on the torus.

Normalize $x_0 = 1$. Equation (13) gives $Q = 1$, and equation (15) gives $x_i^d = 1$ for every $i \geq 2$. Since $\prod_{i=1}^{m-1} x_i = Q = 1$, it follows that $x_1^d = 1$. Multiplying equation (14) by x_1 now gives

$$x_1^d + Q + x_1 = x_1 \neq 0,$$

a contradiction. The argument takes place over the algebraic closure and never assumes its torus coordinates are rational. □

6 Completion, effective selection, and scope

Proof of theorem 1.1. Set $m = n + 1$. If p is odd and $p \nmid d$, use proposition 3.2. If $p \mid d$, then $d \geq p$; the disjoint possibilities $d > p$, $d = p > m$, and $d = p = m$ are covered by propositions 4.2 to 4.4. The last possibility has $p \geq 3$. The remaining case $p = 2$, d odd, is proposition 5.1. Every equation is homogeneous of degree d and contains a nonzero pure-power term distinct from all other displayed monomials, so it is a nonzero degree-d form.

Geometric smoothness also gives geometric integrality here. Indeed, two distinct positive-degree hypersurface components in $\mathbb{P}^{m-1}$ meet when $m \geq 3$. At an intersection point the defining equation and all its partial derivatives vanish. A repeated component is likewise incompatible with smoothness. The algebraic-closure hypersurface is therefore irreducible and reduced.

All choices are finite and deterministic. In the triangular cases, use the displayed positive composition and enumerate $(\mathbb{F}_p^\times)^m$ until $C \neq 0$. In the odd diagonal case, first find a tuple with $C \neq 0$; fix its first $m - 1$ coordinates and choose a final coefficient as in proposition 3.2. The equal-prime and binary cases already have fixed displayed coefficients. These procedures succeed by the proofs above. □

Remark 6.1 (Zero-dimensional edge case)*.* For $n = 1$ and $d \geq 2$, a monic irreducible polynomial of degree d over $\mathbb{F}_p$, homogenized in two variables, defines a smooth finite-etale degree-d hypersurface of $\mathbb{P}^1_{\mathbb{F}_p}$ with no rational point. This is the classical finite-field construction, not a new geometric-integrality assertion: after algebraic closure it consists of d points. We do not add an $n = 0$ positive-degree hypersurface claim.

Remark 6.2 (Literal norm examples)*.* Without smoothness, the numerical AIM alternative already has classical point-free norm-form witnesses in every required degree, as acknowledged in [11]. They are not the contribution of this paper. In positive dimension the present construction keeps smoothness and obtains the numerical failure over the original field. The disjunctive AIM target is met by its numerical alternative; no replacement of rational connectedness by separable rational connectedness is needed.

Remark 6.3 (Ground field and optimality of the claim)*.* The theorem concerns the prime field $\mathbb{F}_p$ and a congruence modulo p. It does not assert that one chosen hypersurface violates the analogous congruence over every finite extension, or that the displayed families are the only possibilities. The method does not prescribe an exact point count, guarantee absence of points, or optimize the coefficient-search complexity.

Checks and reproducibility

The proof above is analytic. A supplementary standard-library implementation checks 76 coefficient examples, including 75 exact projective enumerations, and 894 bounded composition cases. A separately authored implementation checks 10,293 prefix compositions, 240 residue-dynamic-programming compositions, and 144 formal coefficient families. Neither a rational-point scan nor a nonzero entry of a higher-dimensional Frobenius matrix is used to infer geometric smoothness.

For eight selected equations, exact symbolic computations with SymPy 1.14.0 obtain the unit ideal from the defining equation and all homogeneous partial derivatives on each of their 26 projective charts. Those computations verify geometric smoothness of the selected examples only. They are not part of the proof of the unbounded theorem. The supplementary verification record identifies the exact code, outputs, versions, and retained unsuccessful attempts.

AI systems assisted with mathematical exploration, literature retrieval, separate model-based checking, and manuscript preparation. The reported reviews are not human peer review or formal proof-assistant verification. This manuscript is an unrefereed preprint; no absolute priority claim is made.

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