

Generic Multiplicities of Hilbert Schemes on Thickened Lines

Alper Ferudun*

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Abstract

Let $T_{n,\beta} = \text{Hilb}^n(\text{Spec } \mathbb{C}[x, y]/(y^\beta))$. For the component indexed by a partition $\lambda \vdash n$ with $\lambda_1 \leq \beta$, we compute its ambient generic multiplicity as $\prod_i \binom{\beta}{\lambda_i}$. The proof uses disjoint-support factorization and a cyclic chart in which the defining condition is divisibility of t^β by a monic polynomial. The remainder equations form a weighted complete intersection. In particular, a chart of $T_{2,3}$ has generic local ring $K[u]/(u^3)$, whereas imposing a repeated-root discriminant produces $K[u]/(u^2)$; this separates ambient multiplicity from the multiplicity of a support stratum. We compare the result with the known multinomial multiplicity of the characteristic-polynomial nilpotent fiber, and give a direct description of all reduced Jordan strata and the distinction between their closures and orbit-rank preimages.

1 Results and scheme structures

Write $H_n = \text{Hilb}^n(\mathbb{A}^2)$, over $\mathbb{C}$, and let $\mathcal{E}$ be its tautological rank- n algebra. Denote multiplication by y on $\mathcal{E}$ by N . There are several natural ways to impose nilpotence on N , and they need not define the same scheme. We use

$$H_y = \{I \in H_n : N_I \text{ is nilpotent}\}_{\text{red}}, \quad (1)$$

$$Z_n = \{I \in H_n : \det(t - N) = t^n\}, \quad (2)$$

$$T_{n,\beta} = \text{Hilb}^n(\text{Spec } \mathbb{C}[x, y]/(y^\beta)) = \{I \in H_n : N^\beta = 0\}. \quad (3)$$

In (2) we impose all n characteristic coefficients. In (3) we impose the universal relation $y^\beta = 0$ in the quotient algebra. In particular, (3) is an equality of moduli functors, including over nonreduced bases.

For a partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, let S_λ be the reduced locally closed locus in H_y where N has Jordan type λ , and let $C_\lambda = \overline{S_\lambda}$ with reduced structure. For a scheme T and an irreducible component C of its reduction, we define

$$\text{mult}_C T = \text{length } \mathcal{O}_{T, \eta_C},$$

where η_C is the generic point of C . This is the length of the *ambient* generic local ring, not the length after restricting to another support condition.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Theorem 1 (Thickened-line multiplicities). *Every S_λ is smooth, irreducible, and n -dimensional. The components of $(T_{n,\beta})_{\text{red}}$ are exactly C_λ for $\lambda_1 \leq \beta$. Their generic multiplicities are*

$$\text{mult}_{C_\lambda} T_{n,\beta} = \prod_{i=1}^{\ell} \binom{\beta}{\lambda_i}. \quad (4)$$

The component classification is known; in particular it is the thickened-line case of Luan's classification [1, Theorem 1.1]. We include a cyclic-pair proof to fix the Jordan indexing and the generic charts used for (4). Our calculation retains all transverse monic-polynomial coefficients before imposing $y^\beta = 0$.

For comparison, the characteristic-polynomial fiber has

$$\text{mult}_{C_\lambda} Z_n = \frac{n!}{\prod_i \lambda_i!}. \quad (5)$$

This multinomial and its factorization algebra are already present in Minets–Živanović [2, Corollaries 4.14–4.15], in the locally equivalent smooth-curve setting and with a transposed partition convention. We give a short direct calculation in Section 4, rather than claim a new spectral-fiber formula.

The ambient interpretation is important when comparing with the stratum calculation in [1, Section 6]. The formula there gives $\beta - m + 1$ on a one-part support stratum and is subsequently described as a component multiplicity. Section 5 exhibits an ambient local ring of length three for $m = 2, \beta = 3$, while the imposed repeated-root slice has length two. Thus the slice calculation does not compute the ambient length in (4).

2 Jordan strata and generic support

We first describe the reduced geometry in (1). A frame of $\mathcal{E}$ identifies a point of H_n with a commuting pair (X, N) and a jointly cyclic vector $v \in \mathbb{C}^n$, modulo simultaneous change of basis. The corresponding ideal is

$$I = \{p(x, y) \in \mathbb{C}[x, y] : p(X, N)v = 0\}.$$

Conversely the quotient algebra gives the pair and the vector $1 \bmod I$. This description also holds in families after choosing a frame of the quotient algebra.

Lemma 2. *For every $\lambda \vdash n$, the stratum S_λ is smooth, irreducible, and n -dimensional. Its closure is an irreducible component of H_y , and all components arise in this way.*

Proof. Fix a Jordan matrix J_λ . Its endomorphism centralizer $Z_{\text{End}}(J_\lambda)$ is a vector space of dimension

$$c_\lambda = \sum_{i,j} \min(\lambda_i, \lambda_j).$$

Its invertible elements form a smooth open algebraic group G_λ of the same dimension. The locus

$$U_\lambda = \{(X, v) \in Z_{\text{End}}(J_\lambda) \times \mathbb{C}^n : \mathbb{C}[X, J_\lambda]v = \mathbb{C}^n\}$$

is open. It is nonempty: choose distinct scalars α_i , set $X = \alpha_i 1 + J_{\lambda_i}$ on the i th Jordan block, and let v have a cyclic starting vector in each block. Polynomial spectral projectors of X separate the blocks, after which powers of J_{λ_i} generate each one.

Thus U_λ is smooth and irreducible of dimension $c_\lambda + n$. The framed locus with N in the orbit of J_λ is $\mathrm{GL}_n \times^{G_\lambda} U_\lambda$. The change-of-basis action on cyclic triples is free: an automorphism commuting with X, N and fixing v fixes all $p(X, N)v$. The frame torsor over S_λ therefore gives smooth descent and

$$\dim S_\lambda = (n^2 - c_\lambda) + (c_\lambda + n) - n^2 = n.$$

Irreducibility descends as well. Rank equalities for the powers of N show that this locus is locally closed.

There are finitely many Jordan types, and their strata partition H_y . Their closures are irreducible of the same dimension. Two closures cannot coincide: the two disjoint, locally closed, dense strata would then contain intersecting dense open subsets of one irreducible variety. No one of these closures properly contains another of the same dimension. They are exactly the irreducible components. $\square$

Lemma 3. *A dense open subset of S_λ consists of disjoint local subschemes with ideals*

$$(x - f_i(y), y^{\lambda_i}), \quad \deg f_i < \lambda_i, \quad f_i(0) = \alpha_i, \quad (6)$$

where the α_i are pairwise distinct.

Proof. Every support point accounts for at least one Jordan block of N . Thus a point of S_λ has at most ℓ distinct support points. If equality holds, each local quotient has just one Jordan block. It is therefore generated by 1 as a $\mathbb{C}[y]$ -module, and has the presentation (6).

Conversely (6) has the required Jordan type and ℓ distinct support points. This locus is nonempty. The locus of cycles with at most $\ell - 1$ distinct support points is closed in $\mathrm{Sym}^n \mathbb{A}^2$, so its Hilbert–Chow inverse image is closed. Its complement in S_λ is consequently the asserted nonempty open subset, which is dense by Lemma 2. $\square$

The f_i in (6) have a total of n free coefficients, with distinct constant terms, up to permutations of equal-sized pieces. These will be the parameters along the component; the next section retains the transverse parameters.

3 Cyclic charts and monic divisors

Lemma 4 (Cyclic chart). *On the open chart of H_m where $1, y, \dots, y^{m-1}$ form a basis of the quotient algebra, every ideal has a unique presentation*

$$I = (q(y), x - f(y)), \quad q(t) = t^m + c_1 t^{m-1} + \dots + c_m, \quad \deg f < m. \quad (7)$$

The coefficients of q and f identify the chart with $\mathbb{A}^{2m}$, functorially over arbitrary coefficient rings.

Proof. Multiplication by y determines the unique monic relation q . Since 1 is cyclic for y , every endomorphism commuting with multiplication by y is determined by its value on 1. Multiplication by x is therefore the unique polynomial $f(y)$ of degree less than m . Conversely the quotient by (7) is free with the indicated basis over any coefficient ring. These constructions are inverse in families. $\square$

Near a point (6), the disjoint-union map from the open subset of $\prod_i H_{\lambda_i}$ with mutually disjoint supports to H_n is étale. One can check this infinitesimally: the mutually orthogonal idempotents

of a finite commutative quotient algebra lift uniquely through nilpotent thickenings, so every deformation splits uniquely into the specified pieces. The union map is of finite presentation, and formal étaleness implies étaleness. See also [2, Proposition 4.1].

At the generic point of C_λ , the coefficients of the f_i become residue-field parameters, while the reduced values of the q_i are t^{λ_i} . Generic length is unchanged by the étale passage and the resulting residue-field extension. There is no factorial correction for equal parts: ordering disjoint support is an étale cover, not a ramified collision cover.

Lemma 5 (Monic-divisor algebra). *Let $1 \leq m \leq \beta$. The scheme of monic degree- m polynomials dividing t^β is supported at t^m . With weight j on the coefficient c_j , its coordinate algebra has Hilbert series*

$$\frac{\prod_{d=\beta-m+1}^{\beta} (1 - z^d)}{\prod_{j=1}^m (1 - z^j)} \quad (8)$$

and length $\binom{\beta}{m}$.

Proof. Divide t^β by the monic polynomial $q(t)$ in (7), and let $r_k(c)$ be the coefficient of t^k in the remainder, for $0 \leq k < m$. Divisibility is equivalent to the m equations $r_k = 0$. Giving t weight one, monic division is homogeneous, so r_k has weight $\beta - k$.

Over an algebraically closed field the only monic degree- m divisor of t^β is t^m . The remainder equations therefore have only the origin as a common zero in $\mathbb{A}_c^m$, and their ideal has height m . Since the polynomial ring is Cohen–Macaulay, these m homogeneous parameters form a regular sequence. The usual exact sequences for quotienting by a homogeneous non-zero-divisor give (8). Evaluating the resulting Hilbert polynomial at $z = 1$ gives the length:

$$\frac{\beta(\beta - 1) \cdots (\beta - m + 1)}{m!} = \binom{\beta}{m}.$$

□

Proof of Theorem 1. The assertion about S_λ is Lemma 2. Over fields, $N^\beta = 0$ is equivalent to having all Jordan blocks of size at most β . Hence the reduction of $T_{n,\beta}$ is the union of those strata. It is closed, so contains their closures. These closures are distinct and all have dimension n , giving the component assertion.

In a cyclic chart (7), $y^\beta = 0$ in the quotient is equivalent to $q(t) \mid t^\beta$ over the coefficient ring, not just over its residue field. At the generic point of C_λ , the disjoint-support charts therefore give one independent monic-divisor algebra for each part λ_i . Their tensor product over the common residue field has length equal to the product of the lengths in Lemma 5. This is (4). □

Corollary 6 (Smooth support on a nonreduced curve). *Let C be a plane curve, possibly nonreduced. Consider a component of $\text{Hilb}^n(C)$ whose generic support consists of distinct smooth points of the reduced curve, away from intersections, with local subscheme lengths m_i^j on a curve component of multiplicity β_j , with $m_i^j \leq \beta_j$. Its ambient generic multiplicity is*

$$\prod_{i,j} \binom{\beta_j}{m_i^j}.$$

Proof. At each specified smooth support point, a local equation of C is a unit times a smooth transverse parameter raised to β_j . Complete that parameter to an étale coordinate system of the ambient plane. Disjoint-support factorization identifies the component locally with a product of thickened-line components. Lemma 3 makes their generic local pieces cyclic for the transverse parameter, so the calculation reduces to Lemma 5. □

The corollary concerns the specified generic components. It does not assert a new global classification on arbitrary singular curves.

4 The characteristic-polynomial fiber

The comparison (5) follows from a different coefficient algebra. In the product of cyclic charts, the characteristic polynomial of N is $\prod_i q_i(t)$. Thus Z_n imposes the n coefficients of

$$\prod_{i=1}^{\ell} q_i(t) - t^n. \quad (9)$$

Assign weight j to the j th coefficient of each q_i . The equations in (9) have weights $1, \dots, n$. Their only common closed zero is the origin, because every root of every monic q_i must be zero. They are a regular sequence in n variables, by the same Cohen–Macaulay argument as before. The Hilbert series is

$$\frac{\prod_{j=1}^n (1 - z^j)}{\prod_i \prod_{j=1}^{\lambda_i} (1 - z^j)}.$$

Its length at $z = 1$ is $n! / \prod_i \lambda_i!$, proving (5). The closed points of Z_n are precisely H_y , so its reduced components are all the C_λ .

This coefficient algebra is also the invariant coinvariant algebra appearing in [2, Corollary 4.15], which identifies the corresponding graded algebra with the cohomology of a partial flag variety. No new claim about that identification is needed here.

Remark 7. Although H_y is set-theoretically the inverse image of cycles supported on the horizontal line, we have not identified Z_n scheme-theoretically with the inverse image of $\text{Sym}^n(\{y = 0\})$ under the full Hilbert–Chow morphism. The explicit equations (2) are essential to the calculation.

5 An ambient ring of length three

Consider $T_{2,3}$ in its y -cyclic chart:

$$I = (x - a - by, \ y^2 - uy - r).$$

Reduction modulo the quadratic relation gives

$$y^3 \equiv (u^2 + r)y + ru.$$

Consequently this chart is

$$\text{Spec } \mathbb{C}[a, b, u, r] / (u^2 + r, ru) \simeq \text{Spec } \mathbb{C}[a, b, u] / (u^3). \quad (10)$$

Its reduction is the chart of $C_{(2)}$, and its generic local ring is

$$K[u] / (u^3), \quad K = \mathbb{C}(a, b).$$

It has length three, agreeing with $\binom{3}{2}$. Length here is over the residue field K , not finite vector-space dimension over $\mathbb{C}$.

Luan’s computation [1, Section 6.1 and Corollary 6.11] first imposes a repeated-root polynomial in the x coordinate. The product of the resulting lengths is described as a component multiplicity

in [1, Theorem 6.19]. If read as the ambient generic length defined in this paper, that formula gives $3 - 2 + 1 = 2$, contrary to (10).

The exact difference between the rings is visible on the dense locus $b \neq 0$. Multiplication by $x = a + by$ has characteristic discriminant

$$b^2(u^2 + 4r).$$

Pulling back the repeated-root discriminant divisor adds $u^2 + 4r = 0$. In (10), where $r = -u^2$, this becomes $-3u^2 = 0$. The resulting slice has generic local ring

$$K[u]/(u^2),$$

of length two. Thus the restricted-slice calculation can give the stated length while discarding a nonzero transverse nilpotent of the ambient Hilbert scheme.

6 Reduced orbit preimages and infinitesimal distinctions

Write $\mu \preceq \lambda$ for dominance of partitions: $\sum_{i \leq j} \mu_i \leq \sum_{i \leq j} \lambda_i$ for every j . Let $H_{\leq \lambda}$ be the reduced closed locus in H_n defined on points by

$$\text{rank } N^k \leq \sum_i \max(\lambda_i - k, 0), \quad 1 \leq k \leq n.$$

The last inequality includes nilpotence.

Proposition 8. *One has*

$$H_{\leq \lambda} = \bigcup_{\mu \preceq \lambda} C_\mu.$$

In particular, $H_{\leq \lambda} = C_\lambda$ if and only if $\lambda = (1^n)$.

Proof. For a nilpotent matrix of type μ , the rank of its k th power is $\sum_i \max(\mu_i - k, 0)$. These ranks are bounded by the corresponding ranks for λ exactly when $\mu \preceq \lambda$. Indeed, $n - \text{rank } N^k$ is the sum of the first k parts of the conjugate partition, and conjugation reverses dominance.

Therefore $H_{\leq \lambda}$ is the union of S_μ over $\mu \preceq \lambda$. Being closed, it contains their closures, proving the equality. For $\lambda = (1^n)$ there is just one term. Otherwise the distinct component $C_{(1^n)}$ is also a term, so the union is strictly larger than C_λ . $\square$

There is a concrete witness for the strictness. A set of n distinct points on the horizontal line belongs to $H_{\leq \lambda}$ for every λ , but does not belong to C_λ when $\ell(\lambda) < n$. By Lemma 3 the dense generic locus of C_λ has only $\ell(\lambda)$ support points, and the Hilbert–Chow inverse image of cycles with at most that many support points is closed.

The reduced qualifier in Proposition 8 cannot be dropped merely by writing rank minors of powers. Cayley–Hamilton gives a closed immersion $Z_n \subset T_{n,n}$, and their reductions coincide, but their scheme structures differ already for $n = 2$. Over $R = \mathbb{C}[\epsilon]/(\epsilon^2)$ take

$$N = \epsilon \, 1_2, \quad X = \text{diag}(0, 1), \quad v = (1, 1)^T.$$

The pair commutes and is cyclic, since v, Xv are a basis. It therefore defines a flat R -point of H_2 . One has $N^2 = 0$ and $\det N = 0$, but $\text{tr } N = 2\epsilon \neq 0$. This family lies in $T_{2,2}$ and satisfies the naive power-rank equations for regular nilpotence, while it does not lie in Z_2 .

7 Context and reproducibility

The geometry above is relevant to comparisons between nilpotent-orbit stratifications and Hilbert schemes, as discussed in the AIM workshop on categorified Hecke algebras [3]. The frozen UnsolvedMath record AIM-ALGEBRAIC_GEOMETRY-0078 associates this theme with Problem 1.6 of its problem list [4]. The present results distinguish the underlying geometric schemes; they do not construct a Kazhdan–Lusztig cell equivalence or establish the proper-support convolution package in an equivariant matrix-factorization category.

The supplementary source archive includes an exact Python checker using SymPy 1.14.0. Its 4,338 finite checks cover rank and dominance identities through $n = 10$, explicit cyclic commuting pairs through $n = 8$, monic-divisor algebras with $\beta \leq 9$ and $m \leq 4$, and factorization algebras through $n = 5$. It also checks the cubic slice and the dual-number example. Normal and optimized Python runs agree. These computations are regressions for the written proofs, not an unbounded proof or proof-assistant certification.

This unrefereed manuscript was developed with AI assistance for proof development, symbolic checks, literature searches and writing. The source comparison is bounded; no absolute priority or independent review is claimed.

References

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