

A Rank-One Obstruction for Odd-Primary Wood Equivalences

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Abstract

Let p be odd, $h = (p-1)p^{r-1}m$ with $r \geq 1$ and $m > 1$, and let $H = C_{p^r}$ be a subgroup of the height- h Morava stabilizer. For Honda Morava E -theory over $\mathbb{F}_{p^h}$, put $A = E^{hH}$. We prove that no finite p -local spectrum F satisfies $A \wedge F \simeq E$ as A -modules. More generally, if $A \wedge F \simeq E^{\vee q}$ with $q > 0$, then $q \geq 2$. There is no parity restriction on the cells of F . The argument combines Ravenel's computation of the first cohomology of the strict stabilizer with Galois base change and an elementary flag bound for modular representations. It explains why the known finite-wedge Wood constructions cannot have multiplicity one in these cases. The obstruction concerns external finite-spectrum realizations; it does not exclude internal filtrations of E by suspensions of A , and does not settle the general odd-primary Wood-filtration question.

1 Statement and scope

Fix an odd prime p . Let $E = E_h$ be the Morava E -theory of the height- h Honda formal group over $k = \mathbb{F}_{p^h}$, with

$$E_0 = W(k)[[u_1, \dots, u_{h-1}]], \quad \mathfrak{m} = (p, u_1, \dots, u_{h-1}), \quad E_0/\mathfrak{m} = k.$$

We use the stabilizer of the formal group over k , without the Galois automorphisms of k . Let $H = \langle g \rangle$ be a subgroup of order $P = p^r$, and set $A = E^{hH}$. Smash products and module equivalences are derived.

Theorem 1.1. *Suppose*

$$h = (p-1)p^{r-1}m, \quad r \geq 1, \quad m > 1.$$

If F is a finite p -local spectrum and there is an equivalence of A -modules

$$A \wedge F \simeq E^{\vee q}, \quad 0 < q < \infty, \tag{1}$$

then $q \geq 2$. In particular no such F gives a single copy of E . The same conclusion holds for a single suspended copy of E .

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The qualifier “as A -modules” is essential to the base-change argument. The theorem makes no assumption on cell parity or on a BP -module structure on F . It also applies when F is written as the Spanier–Whitehead dual of a finite spectrum.

The classical Wood equivalence and its odd-primary analogues motivate asking for finite spectra that convert homotopy fixed points into the underlying complex-oriented theory. Meier, Noel and Naumann construct finite even-cell spectra giving a nonzero finite wedge of E for any finite stabilizer subgroup, and obtain a single copy for $H = C_p$ at height $p - 1$ [3, Appendix B]. Their general existence theorem does not assert multiplicity one. Carrick’s chromatic-defect results give different restrictions, on the stage at which orientability is achieved [1, Section 6.2].

The related AIM question [5] is recorded in the frozen `ulamai/UnsolvedMath v1.6.0` dataset as

AIM-ALGEBRAIC_GEOMETRY-0035.

It asks about odd-primary analogues of Wood cofibre sequences, including filtrations of E with graded pieces given by suspensions of A . An internal filtration in Mod_A need not arise by smashing A with the cells of a finite spectrum. Theorem 1.1 addresses the latter, stronger realization requirement only.

2 A flag bound for pro- p representations

For a field k of characteristic p , let $U_D(k)$ be the group of upper-unitriangular $D \times D$ matrices. Let $J \subset M_D(k)$ be the ideal of strictly upper-triangular matrices inside the algebra of upper-triangular matrices. Thus $J^D = 0$, and J^2 consists of the matrices whose diagonal and first superdiagonal vanish.

Lemma 2.1. *Let G be a pro- p group and give k the discrete topology. Suppose $g \in G$ is annihilated by every continuous additive character $G \rightarrow (k, +)$. If $\rho : G \rightarrow \text{GL}(V)$ is a continuous representation on a discrete vector space of dimension $D > 0$, then*

$$(\rho(g) - I)^s = 0 \quad \text{whenever } 2s \geq D. \quad (2)$$

Proof. The stabilizers of a basis have open intersection, so ρ has finite image, necessarily a p -group. Every representation of a finite p -group over k has a full invariant flag with trivial one-dimensional quotients. To recall the elementary argument, choose a central element z of order p in a nontrivial finite p -group. The operator $\rho(z) - I$ has p th power zero, hence a nonzero kernel. This kernel is invariant and carries an action of the quotient by $\langle z \rangle$. Induction gives a nonzero fixed vector. Repeating on the quotient of V by its span gives the full flag; the trivial-group case starts the induction.

In an adapted basis, $\rho(G) \subset U_D(k)$. For each $1 \leq i < D$, the function

$$\chi_i(s) = \rho(s)_{i,i+1}$$

is an additive character: in the $(i, i+1)$ entry of a product of upper-unitriangular matrices, only the two corresponding entries contribute. By hypothesis $\chi_i(g) = 0$ for all i . Therefore $\rho(g) - I \in J^2$, and its s th power lies in J^{2s} . This proves (2). $\square$

Corollary 2.2. *Under the hypotheses of Lemma 2.1, suppose $H = \langle g \rangle$ has order $P = p^r$ and $V|_H$ is free of positive rank q over $k[H]$. Then*

$$\dim_k V = qP \geq 2P - 1, \quad q \geq 2. \quad (3)$$

Proof. The identification $k[H] \cong k[T]/(T^P)$, with $T = g - 1$, shows that $(g - 1)^{P-1}$ is nonzero on every nonzero free module. Equation (2) implies $2(P - 1) < \dim_k V$. Since the dimension is qP and $P > 1$, the result follows. $\square$

The character hypothesis is the usual Frattini condition. We use the explicit vanishing formulation, so no description of the entire Frattini subgroup is needed.

3 Torsion and the first strict-stabilizer cohomology

Let D_h be the central division algebra over $\mathbb{Q}_p$ of invariant $1/h$, with maximal order $\mathcal{O}_h$. Choose a uniformizer Π with $\Pi^h = p$, and normalize the valuation by $v(p) = 1$. The strict stabilizer is the pro- p group

$$S_h = 1 + \Pi\mathcal{O}_h.$$

The quotient of $\mathcal{O}_h^\times$ by S_h is $k^\times$, of order prime to p . Thus $H \subset S_h$.

We use the following classical input, stated in the notation needed here. The scalar extension in this statement matters: it is over $k = \mathbb{F}_{p^h}$, not an identification of the unextended Hopf algebra with the group algebra over $\mathbb{F}_p$.

Proposition 3.1 (Ravenel). *For odd p and $h > 1$, the continuous cohomology $H_{\text{cts}}^1(S_h; k)$, with trivial coefficient action, is spanned over k by*

$$t_1, t_1^p, \dots, t_1^{p^{h-1}}, \quad \zeta_h(s) = \frac{\det(s) - 1}{p} \pmod{p}. \quad (4)$$

Here $t_i(s)$ are the residual digits in the expansion

$$s = 1 + \sum_{i \geq 1} [t_i(s)] \Pi^i, \quad (5)$$

and $\det : S_h \rightarrow 1 + p\mathbb{Z}_p$ is the determinant of the standard faithful matrix representation over $W(k)$.

This is Theorem 6.3.12 of [4], together with Theorem 6.2.3, Corollary 6.2.4 and Proposition 6.2.6 there. The powers in (4) are functions on the group. For trivial coefficients, degree-one cohomology is exactly the space of continuous additive characters, since every one-coboundary is zero. The digit convention (5) is also described in the proof of Theorem B.5 of [3].

Lemma 3.2. *Under the hypotheses of Theorem 1.1, every continuous additive character $S_h \rightarrow k$ vanishes on g .*

Proof. The element g has exact order p^r , so it is a primitive p^r th root of unity in the subfield $\mathbb{Q}_p(g) \subset D_h$. The cyclotomic valuation gives

$$v(g - 1) = \frac{1}{(p - 1)p^{r-1}} = \frac{m}{h}. \quad (6)$$

For completeness, apply the valuation to $p = \prod_a (1 - g^a)$, where a runs over the units modulo p^r . All factors have the same valuation, because $(1 - g^a)/(1 - g)$ is a unit when $p \nmid a$. There are $(p - 1)p^{r-1}$ factors. This proves (6); compare [1, Corollary 6.9].

Since $v(\Pi) = 1/h$, the first nonzero digit in (5) has index m . In particular $t_1(g) = 0$ when $m > 1$. Also $\det(g)^{p^r} = 1$ in $\mathbb{Z}_p^\times$. For odd p this group has no nontrivial p -power torsion, so $\det(g) = 1$ and $\zeta_h(g) = 0$. Every spanning character in (4) therefore vanishes on g . $\square$

4 Galois base change and proof of the theorem

The natural map $A = E^{hH} \rightarrow E$ is a faithful H -Galois extension. We use [2, Proposition 3.6], whose Section 3 allows arbitrary heights and primes, despite the height $p - 1$ in the paper's title. In particular the Galois map

$$E \wedge_A E \xrightarrow{\simeq} \text{Map}(H, E), \quad b_1 \wedge b_2 \longmapsto (x \mapsto b_1 x(b_2)) \quad (7)$$

is an equivalence. The element notation describes the natural multiplication-and-action map of spectra.

Let H act on the first E factor on the left. The corresponding semilinear action on the right is

$$(s \cdot f)(x) = s(f(s^{-1}x)). \quad (8)$$

Indeed, $s(b_1)x(b_2) = s(b_1(s^{-1}x)(b_2))$. This identifies the action explicitly, including its permutation part.

Proof of Theorem 1.1. Assume (1). Base change along $A \rightarrow E$ gives

$$E \wedge F \simeq E \wedge_A (A \wedge F) \simeq (E \wedge_A E)^{\vee q} \simeq \text{Map}(H, E)^{\vee q}. \quad (9)$$

This is H -equivariant with H acting through the first base-change factor. No equivariance of (1) was assumed: it follows functorially after this base change. On the left the action is the restriction of the natural S_h -action on $E \wedge F$.

As an E -module, $E \wedge F$ is finite free and even of rank qP . Consequently

$$V = \pi_0(E \wedge F) \otimes_{E_0} k$$

is a qP -dimensional continuous k -linear representation of S_h . The coefficient action on k is trivial because S_h is the strict stabilizer over k . The residual action and its continuity are the standard Morava K -theory action recalled at the beginning of Appendix B of [3]. More explicitly, if K is the residual Morava K -theory, then finite freeness in (9) identifies V with $\pi_0(K \wedge F) = K^0(DF)$. No degeneration of a universal-coefficient spectral sequence for arbitrary F is being assumed.

Reducing (8) modulo $\mathfrak{m}$ removes the coefficient action of H , leaving its regular permutation action. Thus

$$V|_H \cong \text{Map}(H, k)^{\oplus q} \cong k[H]^{\oplus q}.$$

Lemma 3.2 and Corollary 2.2 give $q \geq 2$. If $A \wedge F \simeq \Sigma^d E$, replace F by $\Sigma^{-d} F$ to exclude a single suspended copy as well. $\square$

Example 4.1. At $p = 3$ and height $h = 4$, a subgroup C_3 corresponds to $r = 1$, $m = 2$. There is no finite spectrum F such that $E_4^{hC_3} \wedge F \simeq E_4$ as $E_4^{hC_3}$ -modules. For $H = C_9$ at height 12, the same conclusion holds with $r = 2$, $m = 2$. The theorem does not treat C_9 at height 6, where $m = 1$.

Remark 4.2. The bound $q \geq 2$ is optimal for the abstract flag argument, not asserted to be optimal for Morava theories. For $P = p^r$, take $G = U_{2P}(\mathbb{F}_p)$ and $g = I + N$, with $N_{i,i+2} = 1$ and all other entries zero. There are two Jordan blocks of size P , so g has order P and the natural module is free of rank two over $\mathbb{F}_p[[g]]$. The subgroup with zero first superdiagonal is generated by the elementary matrices $e_{i,j}(a)$ for $j - i \geq 2$, and

$$e_{i,j}(a) = [e_{i,i+1}(a), e_{i+1,j}(1)].$$

Hence $g \in [G, G]$ and every additive character vanishes on it. No realization of this example by a Morava theory or a finite spectrum is claimed.

Finite-wedge existence remains compatible with the obstruction, as does the known rank-one construction at height $p - 1$. Determining the smallest attainable q for the excluded rank-one cases is a separate question. Internal filtrations and the cases $m = 1$, $r > 1$ are not resolved here. The argument is a deduction from the cited classical cohomology and descent results; no absolute novelty or priority claim is made.

AI-assisted literature search, algebraic checks and drafting were used in preparing this unrefereed note. The accompanying finite-field regressions check the elementary matrix mechanism, not the topological input, and are not formal proof certificates. No independent review is claimed.

References

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