

IPhO 2026

Theory problems and solutions · Bucaramanga, Colombia · 3 theory problems with complete solutions

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Problem 1: Three problems (A: Hydrostatic gate; B: Electron-positron pair; C: Photodissociation of ozone)

QUESTION T1 [10 pts]. Three problems

This question consists of three independent problems.

T1-A [3.0 pts]. Hydrostatic gate

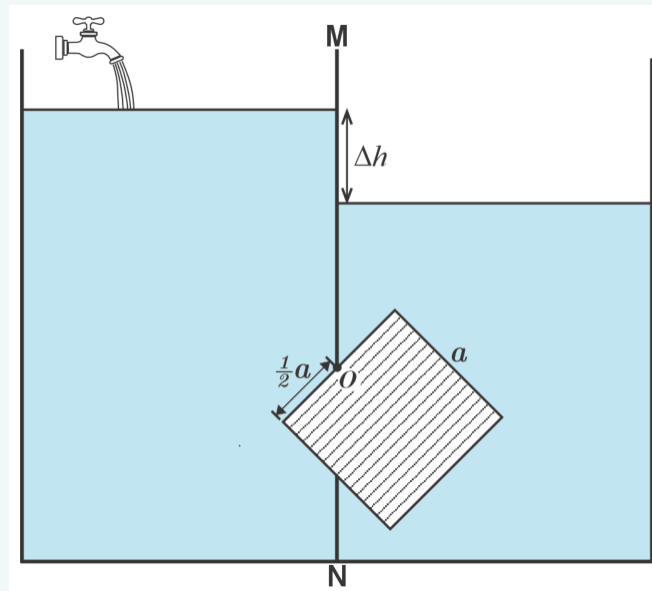


Fig. 1a.

Two water reservoirs are separated by a vertical wall MN, as shown in the figure 1a. The left reservoir can receive water from a source. It must be ensured that the maximum possible difference between the water levels is $\Delta h = 1.41 \text{ m}$. To achieve this, a slot of vertical size $a\sqrt{2}/2$ is cut in MN and sealed with a solid cubic block of side length a . The block has density $3\rho_0$, with ρ_0 the density of water. The block is fully submerged, fixed to the wall at O and can rotate without friction about the axis perpendicular to the plane of the figure that passes through point O .

T1-A1 [3.0 pts]. Calculate the value of a that ensures the maximum permissible difference in water levels is not exceeded.

T1-B [3.5 pts]. Electron-positron pair

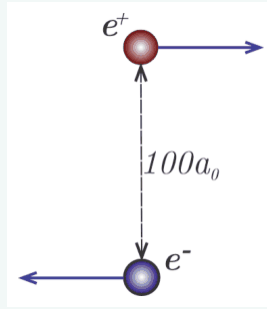


Fig. 1b.

At a certain instant of time, a positron e^+ is located at a distance of $100a_0$ from an electron e^- . The particles are moving in such a way that their velocities are antiparallel at that instant, perpendicular to their separation, as shown in figure 1b. It is known that each particle has an angular momentum of magnitude $\mu\hbar$ with respect to the system's center of mass, where μ is a dimensionless numerical factor. Assume that the only interaction between e^+ and e^- is electrostatic (both particles have the same mass m and equal magnitude of the charge but of opposite sign). Furthermore, assume that this system remains isolated and that its dynamics are classical and non-relativistic.

The Bohr radius a_0 is given by $a_0 = 4\pi\epsilon_0\hbar^2/(me^2)$.

T1-B1 [1.0 pts]. If $\mu = 4$, the system is bound, meaning that the particles move in a closed orbit around the system's center of mass. Find the maximum separation distance between e^+ and e^- in terms of a_0 .

T1-B2 [2.5 pts]. If $\mu = \frac{15}{2}$, the system is unbound, meaning that the particles are no longer in closed orbit around the system's center of mass. Let $\vec{u}_\infty$ be the velocity of e^+ relative to e^- as the separation distance between them tends to infinity. Calculate the angle between $\vec{u}_\infty$ and the initial line of motion of e^+ . Give your answer in degrees.

Hint 1: The eccentricity ε of a conic trajectory for the present case is given by

$$\varepsilon = \sqrt{1 + \frac{4L^2E}{k^2e^4m}}$$

where E and L are the total energy and the magnitude of the total angular momentum, respectively. k is Coulombs constant $k = 1/(4\pi\epsilon_0)$.

Hint 2: The equation of a conic in polar coordinates is

$$r = \frac{a}{1 - \varepsilon \cos(\theta)}$$

T1-C [3.5 pts]. Photodissociation of ozone

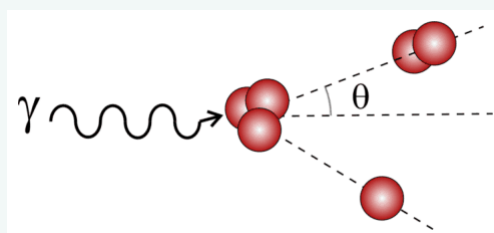


Fig. 1c.

A photon with angular frequency ω strikes an ozone molecule O_3 at rest, dissociating it (breaking it apart) into an oxygen molecule O_2 and an oxygen atom O , as shown in figure 1c. The photon is absorbed in this process. Let U_i and U_f be the ground state energies of the ozone and oxygen molecules, respectively. Assume that this system remains isolated and that its dynamics is classical (potential energies do not contribute to mass and the motion of the Oxygen atoms are non-relativistic). Use the relation $p = E/c$ for the linear momentum of the photon, where E is its energy.

T1-C1 [2.5 pts]. If the angle that the momentum of the outgoing O_2 with respect to the incident photon is θ , determine the minimum angular frequency $\omega_{\min}$ required for this dissociation to occur at this angle. Write your answer in terms of $\hbar$, c , θ , $\Delta U = U_f - U_i$, and the mass m of an oxygen atom.

T1-C2 [1.0 pts]. If $\theta = \pi/6$, calculate the value of $\hbar\omega_{\min} - \Delta U$. Give your answer in electronvolts (eV). Take $\Delta U = 1.10$ eV and $m = 16.0$ amu.

Hint:

$$(1 + x)^r = 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots$$

Physical Constants

Reference values for selected physical constants.

Constant	Notation	Value	Units
Speed of light in vacuum	c	2.99792458×10^8	$\text{m} \cdot \text{s}^{-1}$
Planck constant	h	$4.135667696 \dots \times 10^{-15}$	$\text{eV} \cdot \text{s}$
Reduced Planck constant	$\hbar = h/2\pi$	$6.582119569 \dots \times 10^{-16}$	$\text{eV} \cdot \text{s}$
Avogadro number	N_A	$6.02214076 \times 10^{23}$	mol^{-1}
Boltzmann constant	k_B	$8.617333262 \times 10^{-5}$	$\text{eV} \cdot \text{K}^{-1}$
Ideal gas constant	R	8.31446261815324	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Elementary charge	e	$1.602176634 \times 10^{-19}$	$\text{A} \cdot \text{s}$
Universal gravitational constant	G	$6.67430(15) \times 10^{-11}$	$\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Stefan Boltzmann constant	σ	$5.670374419 \times 10^{-8}$	$\text{kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$
Vacuum permeability	μ_0	$1.25663706127(20) \times 10^{-6}$	$\text{kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity	ϵ_0	$8.8541878188(14) \times 10^{-12}$	$\text{A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$
Mass of the electron	m_e	$5.1099895069(16) \times 10^5$	eV/c^2
Mass of the proton	m_p	$9.3827208943(29) \times 10^8$	eV/c^2
Atomic mass units	amu	$9.3149410372(29) \times 10^8$	eV/c^2

Official English text of the problem.

Answers

- **T1-A1:** The largest permissible side length, giving opening at the specified level difference, is

$$a = \frac{\Delta h}{2\sqrt{2}} = 0.499 \text{ m.}$$

- **T1-B1:**

$$r_{\max} = \frac{1600}{9}a_0 \approx 178a_0.$$

- **T1-B2:**

$$\alpha = \sin^{-1}\left(\frac{2}{7}\right) = 16.6^\circ,$$

directed toward the electron's initial side of the line of motion.

- **T1-C1:** For $0 \leq \theta < \pi/2$, as in the figure, let $F = 1 + 2 \sin^2 \theta$. Then

$$\omega_{\min} = \frac{3mc^2}{\hbar F} \left(1 - \sqrt{1 - \frac{2F\Delta U}{3mc^2}} \right) = \frac{2\Delta U}{\hbar \left(1 + \sqrt{1 - \frac{2F\Delta U}{3mc^2}} \right)}.$$

For $\pi/2 \leq \theta \leq \pi$, replace F by 3; the result is an infimum approached as the outgoing O_2 momentum tends to zero.

- **T1-C2:**

$$\hbar\omega_{\min} - \Delta U \approx \frac{(\Delta U)^2}{4mc^2} = 2.03 \times 10^{-11} \text{ eV.}$$

Solution

T1-A1

We calculate the torque about O at the onset of opening. The hinge force has no moment about O , and the contact force at the lower edge of the slot vanishes at this threshold.

Write the slot height as

$$H = \frac{a}{\sqrt{2}}.$$

The marked distance $a/2$ places O at the midpoint of the upper-left side of the square cross section. Consequently, the cube's center is horizontally to the right of O by

$$d = \frac{a}{2\sqrt{2}}.$$

Decompose the water pressure into two contributions. First, imagine that both reservoirs have the right-hand water level. This hydrostatic pressure gives an upward buoyant force

$$B = \rho_0 g a^3$$

through the center of the cube. Its weight is

$$W = 3\rho_0 g a^3.$$

Their combined clockwise torque is therefore

$$\tau_{\text{closing}} = (W - B)d = 2\rho_0 g a^3 \frac{a}{2\sqrt{2}} = \frac{\rho_0 g a^4}{\sqrt{2}}.$$

Raising the left water level by Δh adds the uniform pressure

$$\Delta p = \rho_0 g \Delta h$$

to the cube's surface on the left of the wall.

The resultant force and torque of this additional pressure are equivalent to those on a vertical rectangle spanning the slot. To see this, close the protruding surface with that rectangle: uniform pressure over the resulting closed surface has zero net force and torque.

Thus the additional pressure produces a horizontal force

$$F_{\Delta p} = \Delta p a H$$

acting halfway down the slot. Its counterclockwise, opening torque is

$$\tau_{\text{opening}} = \Delta p a H \frac{H}{2} = \frac{\rho_0 g \Delta h a^3}{4}.$$

At the opening threshold,

$$\frac{\rho_0 g \Delta h a^3}{4} = \frac{\rho_0 g a^4}{\sqrt{2}},$$

so

$$\Delta h = 2\sqrt{2} a.$$

Setting this threshold equal to the permitted level difference gives

$$a = \frac{1.41}{2\sqrt{2}} \text{ m} = 0.498510 \dots \text{ m} \approx 0.499 \text{ m}.$$

Smaller values of a would make the gate open at a smaller level difference.

T1-B1

Let r be the separation of the particles and define

$$\kappa = k e^2 = \frac{\hbar^2}{m a_0}.$$

We work in the center-of-mass frame. The reduced mass is $m/2$, and the two angular momenta have the same direction, so their total magnitude is

$$L = 2\mu\hbar.$$

Conservation of mechanical energy gives

$$E = \frac{m}{4} \dot{r}^2 + \frac{L^2}{m r^2} - \frac{\kappa}{r}.$$

Initially the relative velocity is perpendicular to the separation, hence $\dot{r} = 0$ at $r_0 = 100a_0$. Therefore,

$$E = \frac{4\mu^2\hbar^2}{m(100a_0)^2} - \frac{\kappa}{100a_0} = \frac{\kappa}{a_0} \left(\frac{\mu^2}{2500} - \frac{1}{100} \right).$$

For $\mu = 4$,

$$E = -\frac{9}{2500} \frac{\kappa}{a_0} < 0.$$

At either extreme of the separation, $\dot{r} = 0$. Writing $x = r/a_0$, the turning-point equation is

$$-\frac{9}{2500} = \frac{64}{x^2} - \frac{1}{x}.$$

Equivalently,

$$9x^2 - 2500x + 160000 = (x - 100)(9x - 1600) = 0.$$

The two turning points are therefore

$$r_{\min} = 100a_0, \quad r_{\max} = \frac{1600}{9}a_0.$$

Thus the given initial configuration is the point of minimum separation, and

$$r_{\max} = \frac{1600}{9}a_0 \approx 178a_0.$$

T1-B2

For $\mu = 15/2$, the same expressions give

$$L = 15\hbar, \quad E = \frac{\kappa}{80a_0} > 0.$$

Using the supplied eccentricity formula,

$$\varepsilon = \sqrt{1 + \frac{4L^2E}{m\kappa^2}} = \sqrt{1 + \frac{4(15\hbar)^2}{m\kappa^2} \frac{\kappa}{80a_0}}.$$

Since $ma_0\kappa = \hbar^2$,

$$\varepsilon = \sqrt{1 + \frac{45}{4}} = \frac{7}{2}.$$

Choose a polar angle φ measured from the periapsis direction, increasing in the direction of motion.

With this choice, the conic equation is

$$r = \frac{p}{1 + \varepsilon \cos \varphi}, \quad p = \frac{L^2}{(m/2)\kappa} = 450a_0.$$

The periapsis separation is

$$r_p = \frac{p}{1 + \varepsilon} = \frac{450a_0}{1 + 7/2} = 100a_0,$$

so the initial position is indeed at periapsis.

On the outgoing branch, $r \rightarrow \infty$ when

$$1 + \varepsilon \cos \varphi_\infty = 0,$$

giving

$$\cos \varphi_{\infty} = -\frac{1}{\varepsilon} = -\frac{2}{7}.$$

The transverse relative speed is $L/[(m/2)r]$, which tends to zero at infinity. Hence the final relative velocity points along the limiting radial direction. The initial velocity is perpendicular to the periapsis direction, so the requested angle is

$$\alpha = \varphi_{\infty} - \frac{\pi}{2}.$$

It follows that

$$\sin \alpha = \frac{1}{\varepsilon} = \frac{2}{7},$$

and therefore

$$\alpha = \sin^{-1}\left(\frac{2}{7}\right) = 16.6^{\circ}.$$

In the figure, the outgoing relative velocity points below the positron's initial horizontal direction.

T1-C1

Let the photon energy and momentum magnitude be

$$E_{\gamma} = \hbar\omega, \quad P = \frac{E_{\gamma}}{c}.$$

Take its direction as the x -axis. Let the outgoing oxygen molecule have momentum $q\hat{\mathbf{n}}$, where

$$q \geq 0, \quad \hat{\mathbf{n}} \cdot \hat{\mathbf{x}} = \cos \theta.$$

Under the stated mass convention, the oxygen molecule has mass $2m$, and the atom has mass m . Momentum conservation gives

$$\mathbf{p}_O = P\hat{\mathbf{x}} - q\hat{\mathbf{n}}.$$

At the minimum photon energy, no additional internal excitation is needed. Energy conservation is then

$$E_{\gamma} - \Delta U = \frac{q^2}{4m} + \frac{P^2 + q^2 - 2Pq \cos \theta}{2m}.$$

Completing the square,

$$E_{\gamma} - \Delta U = \frac{3}{4m} \left(q - \frac{2P \cos \theta}{3} \right)^2 + \frac{P^2}{6m} (3 - 2 \cos^2 \theta).$$

For the forward-emission geometry, $0 \leq \theta < \pi/2$, the kinetic energy is minimized at the physically allowed momentum

$$q_* = \frac{2P \cos \theta}{3} > 0.$$

Define

$$F = 3 - 2 \cos^2 \theta = 1 + 2 \sin^2 \theta.$$

The minimum final kinetic energy for a given photon energy is therefore

$$K_{\min} = \frac{FP^2}{6m} = \frac{FE_{\gamma}^2}{6mc^2}.$$

At the dissociation threshold,

$$E_{\gamma,\min} - \Delta U = \frac{FE_{\gamma,\min}^2}{6mc^2}.$$

Solving this quadratic and taking the lower root gives

$$\omega_{\min} = \frac{3mc^2}{\hbar(1 + 2\sin^2 \theta)} \left[1 - \sqrt{1 - \frac{2(1 + 2\sin^2 \theta)\Delta U}{3mc^2}} \right].$$

The other root has energy of order mc^2 and lies outside the assumed non-relativistic regime.

An equivalent form, useful when $\Delta U \ll mc^2$, is

$$\omega_{\min} = \frac{2\Delta U}{\hbar \left[1 + \sqrt{1 - \frac{2(1 + 2\sin^2 \theta)\Delta U}{3mc^2}} \right]}.$$

Its expansion is

$$\hbar\omega_{\min} = \Delta U + \frac{(1 + 2\sin^2 \theta)(\Delta U)^2}{6mc^2} + O\left(\frac{(\Delta U)^3}{m^2c^4}\right).$$

For completeness, the statement does not explicitly restrict the angular range. If $\pi/2 \leq \theta \leq \pi$, the unconstrained minimizer q_* is nonpositive. Restricting to $q \geq 0$ instead gives

$$\inf K = \frac{P^2}{2m}.$$

Thus the threshold formula uses $F = 3$ for these angles. It is only an infimum for a specified outgoing direction: at $q = 0$ that direction is undefined, while arbitrarily small positive q permits energies arbitrarily close to the threshold.

T1-C2

For $\theta = \pi/6$,

$$F = 1 + 2\sin^2 \frac{\pi}{6} = \frac{3}{2}.$$

Consequently,

$$\hbar\omega_{\min} = 2mc^2 \left(1 - \sqrt{1 - \frac{\Delta U}{mc^2}} \right).$$

Using the supplied atomic mass unit,

$$mc^2 = 16.0 (9.3149410372 \times 10^8) \text{ eV} = 1.49039 \times 10^{10} \text{ eV}.$$

The expansion parameter is

$$x = \frac{\Delta U}{mc^2} \approx 7.38 \times 10^{-11}.$$

Using

$$\sqrt{1 - x} = 1 - \frac{x}{2} - \frac{x^2}{8} + O(x^3),$$

we obtain

$$\hbar\omega_{\min} = \Delta U + \frac{(\Delta U)^2}{4mc^2} + O\left(\frac{(\Delta U)^3}{m^2c^4}\right).$$

Thus

$$\hbar\omega_{\min} - \Delta U \approx \frac{(1.10 \text{ eV})^2}{4(1.49039 \times 10^{10} \text{ eV})},$$

which gives

$$\hbar\omega_{\min} - \Delta U = 2.03 \times 10^{-11} \text{ eV}$$

to three significant figures.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Cooking with sunlight?

QUESTION T2 [10 pts]. Cooking with sunlight?

The redistribution of luminous flux into regions of high and low intensity due to reflection or refraction can lead to the formation of *caustics*: curves formed by the intersection of infinitesimally close light rays. As an example, Figure 2a shows the caustic formed by rays incident paraxially on a half-cylindrical mirror. Figure 2b shows a photograph of the caustic formed by the reflection of light on the bottom part of a coffee cup. The brightest and central point of the caustic is called the *cusp*; at this point, the light intensity is maximal.

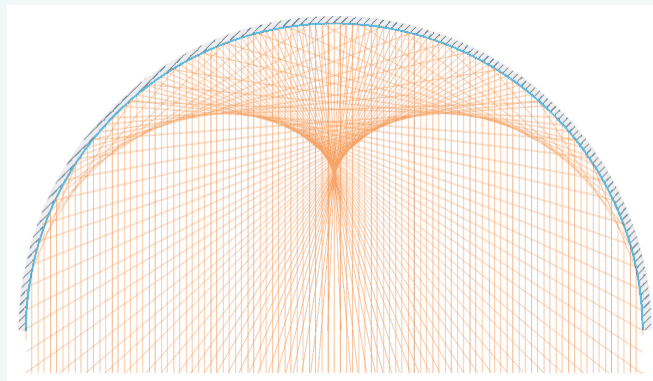


Fig. 2a.



Fig. 2b.

Caustics have a variety of applications in science, ranging from high-precision optics and quantum optics to astrophysics. In this question, you will study the caustic formation by light reflection on a half-cylinder mirror of radius R , as shown in figure 2c. You will also investigate other aspects of this mirror, such as its application in *solar cookers*.

Throughout the problem, assume that the incident rays come from a distant external light source and are parallel to the mirror's optical axis. By symmetry, it is sufficient to study reflection in one plane. Figure 2d shows such a plane, the mirror, and the coordinate system used throughout the problem.

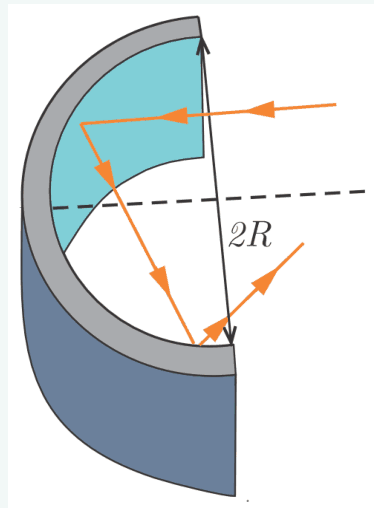


Fig. 2c.

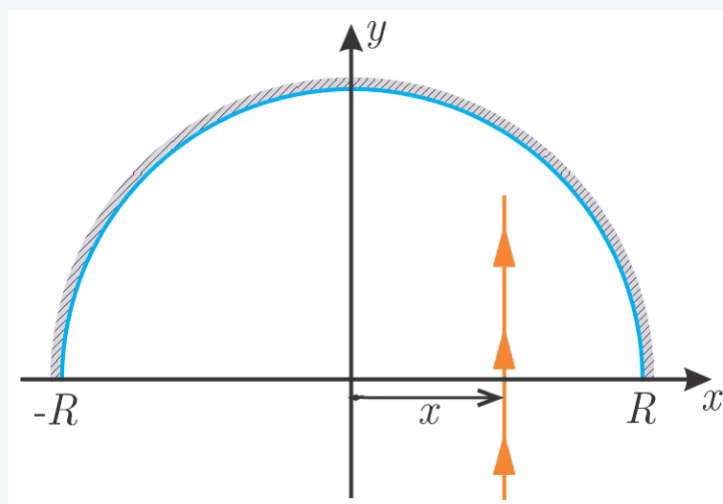


Fig. 2d.

T2-A [1.5 pts]. Multiple reflections

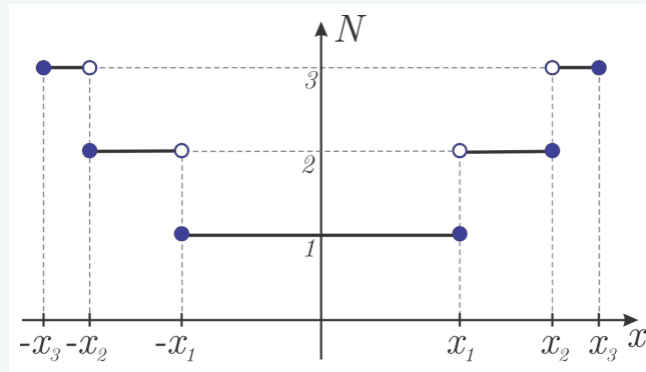


Fig.2e.

Figure 2e shows the relationship between the x -coordinate of an incident ray and the number N of reflections it undergoes, up to $N = 3$. As x varies over the interval $(-R, R)$, the variable N varies over all natural numbers greater than or equal to 1. The maximum distance from the optical axis that allows an incident ray to be reflected by the mirror at most N times is denoted x_N .

T2-A1 [1.5 pts]. Find the general expression for x_N in the infinite sequence $x_1, x_2, x_3, \dots$ that limits the coordinates of the rays with at most N reflections. Express your answer in terms of R and N .

T2-B [4.0 pts]. Solar Cooker

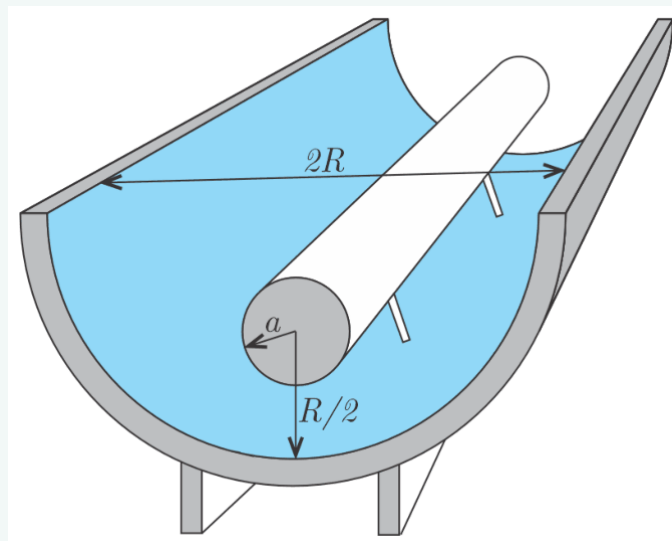


Fig.2f.

Consider a system consisting of a half-hollow-cylinder mirror and a cylindrical container. The cylinder is fully absorbing: any ray of light hitting it will be absorbed. The mirror radius is R . The container radius a . The axes of the mirror and the container are parallel. The center of the container is $R/2$ from the mirror on the system's symmetry plane, as shown in figure 2f. The mirror harnesses solar radiation to heat food inside the container. Assume that the sunlight is of constant and uniform intensity (power per unit area), and that the light rays are parallel to the optical axis of the mirror. Furthermore, assume that a is such that any ray that is absorbed by the

cylinder reflects from the mirror at most once. Let $\theta_{\max}$ be the maximum angle of incidence on the mirror (measured with respect to the normal drawn at the point of incidence) of any reflected ray striking the container.

T2-B1 [2.0 pts]. The container radius is $a = \alpha \sin(\theta_{\max}) + \beta \sin(2\theta_{\max})$. Write α and β in terms of R .

If you do not find α and β , use the expression given for the rest of the problem.

T2-B2 [1.5 pts]. Let P_0 be the power that would be received by the cylinder if the mirror was not present. Write P/P_0 , the ratio of the actual received power P to P_0 , in terms of $\theta_{\max}$.

T2-B3 [0.5 pts]. If $R = 1.0$ m, write the value of a such that $P = 5P_0$. Give your answer in cm.

T2-C [4.5 pts]. Caustics and Cusp

T2-C1 [0.5 pts]. Consider a ray A that strikes the mirror at an angle θ , as shown in Figure 2g. Upon reflection, the equation corresponding to this ray in the coordinate system defined in Figure 2g. is $y = m_A x + b_A$. Write m_A and b_A in terms of θ and R

Hint: the expected form of $m_A = K_1 \cot(K_2 \theta)$ and $b_A = R \frac{K_3}{\cos(K_4 \theta)}$, with K_i numerical constants.

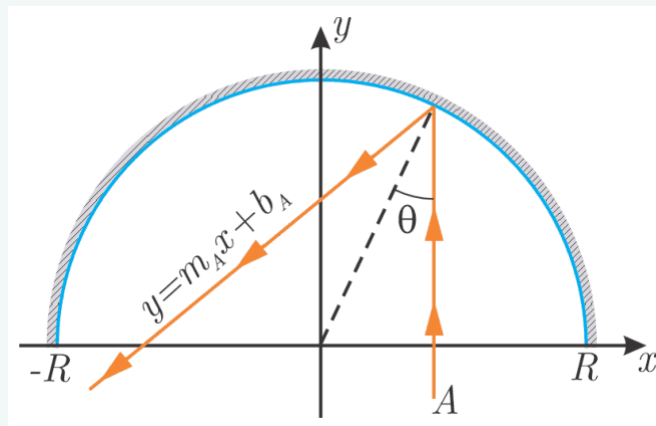


Fig. 2g.

T2-C2 [2.0 pts]. Now consider a ray B , parallel to A , that strikes the surface at an angle $\theta + \Delta\theta$, with $\Delta\theta \ll \theta$. Upon reflection, the equation corresponding to this ray is $y = m_B x + b_B$. Write m_B and b_B in terms of θ , $\Delta\theta$, and R , approximating to first order in $\Delta\theta$.

T2-C3 [1.0 pts]. Find the coordinates of the point of intersection (X_c, Y_c) between the reflected rays of A and B . Write your answer in terms of R and θ .

T2-C4 [1.0 pts]. For $\theta \ll 1$, we have $Y_c = v|X_c|^{p/q} + u$. Write v and u in terms of R and find the values of the integers p and q (equivalently, give the rational number p/q).

Official English text of the problem.

Answers

- T2-A1:

$$x_N = R \cos\left(\frac{\pi}{2N+1}\right), \quad N = 1, 2, 3, \dots$$

- T2-B1:

$$\alpha = R, \quad \beta = -\frac{R}{2}.$$

- T2-B2:

$$\frac{P}{P_0} = \frac{1}{1 - \cos \theta_{\max}}.$$

- T2-B3:

$$a = 12 \text{ cm}.$$

- T2-C1:

$$m_A = \cot(2\theta), \quad b_A = \frac{R}{2 \cos \theta}.$$

- T2-C2: To first order in $\Delta\theta$,

$$m_B \simeq \cot(2\theta) - \frac{2\Delta\theta}{\sin^2(2\theta)}, \quad b_B \simeq \frac{R}{2 \cos \theta} + \frac{R \sin \theta}{2 \cos^2 \theta} \Delta\theta.$$

- T2-C3: In the neighboring-ray limit $\Delta\theta \rightarrow 0$,

$$X_c = R \sin^3 \theta, \quad Y_c = \frac{R}{2} \cos \theta (1 + 2 \sin^2 \theta).$$

- T2-C4:

$$v = \frac{3}{4} R^{1/3}, \quad u = \frac{R}{2}, \quad p = 2, \quad q = 3.$$

Solution

T2-A1

By symmetry, consider an incident ray with $x \geq 0$. Let ϕ be the polar angle of its first point of incidence, measured from the positive x -axis. Then

$$x = R \cos \phi, \quad 0 < \phi \leq \frac{\pi}{2}.$$

Since the incident ray travels vertically, its angle of incidence relative to the normal is

$$\theta = \frac{\pi}{2} - \phi.$$

The law of reflection makes the reflected ray form the same angle θ with the radial normal. A chord connecting two successive points of reflection therefore subtends the central angle

$$\pi - 2\theta = 2\phi.$$

The triangle formed by that chord and the two radii is isosceles, so the angle of incidence remains the same at successive reflections.

Consequently, on the corresponding full circle, successive contact points have polar angles

$$\phi, \quad 3\phi, \quad 5\phi, \quad \dots, \quad (2j-1)\phi.$$

Only the upper semicircle is a mirror. The ray exits through the diameter when the next full-circle contact would be on or below it.

At the limiting incident coordinate x_N , the ray undergoes N reflections and then reaches the far rim. Thus the would-be $(N+1)$ -st contact is at polar angle π :

$$(2N+1)\phi_N = \pi.$$

It follows that

$$x_N = R \cos\left(\frac{\pi}{2N+1}\right).$$

At equality, the ray exits at the rim, consistent with the endpoints shown in Fig.2e. The first values are

$$x_1 = \frac{R}{2}, \quad x_2 \approx 0.8090R, \quad x_3 \approx 0.9010R,$$

and $x_N \rightarrow R$ as $N \rightarrow \infty$.

T2-B1

Use the coordinates of Fig.2d. The container centre is

$$C = \left(0, \frac{R}{2}\right).$$

A ray with incidence angle $\theta \geq 0$ strikes the mirror at

$$M = (R \sin \theta, R \cos \theta).$$

The outward unit normal there is

$$\hat{\mathbf{n}} = (\sin \theta, \cos \theta),$$

and the incident propagation direction is $\hat{\mathbf{k}} = (0, 1)$.

The vector form of the law of reflection gives the reflected unit direction:

$$\begin{aligned} \hat{\mathbf{u}} &= \hat{\mathbf{k}} - 2(\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\ &= (0, 1) - 2 \cos \theta (\sin \theta, \cos \theta) \\ &= (-\sin 2\theta, -\cos 2\theta). \end{aligned}$$

The perpendicular distance from C to this reflected line is the magnitude of the two-dimensional cross product of $C - M$ with $\hat{\mathbf{u}}$:

$$\begin{aligned} d(\theta) &= |(C_x - M_x)\hat{u}_y - (C_y - M_y)\hat{u}_x| \\ &= \left| -R \sin \theta + \frac{R}{2} \sin 2\theta \right| \\ &= R \sin \theta (1 - \cos \theta). \end{aligned}$$

For $0 \leq \theta < \pi/2$, this distance increases with θ , since

$$d'(\theta) = R(1 - \cos \theta)(1 + 2 \cos \theta) \geq 0.$$

The limiting accepted reflected ray is tangent to the container, so $d(\theta_{\max}) = a$. Therefore

$$a = R \sin \theta_{\max} - \frac{R}{2} \sin 2\theta_{\max}.$$

Comparison with the given expression yields

$$\boxed{\alpha = R, \quad \beta = -\frac{R}{2}}.$$

T2-B2

Let I be the uniform incident intensity and L a common illuminated length along the cylinder axes. Without the mirror, the container presents projected area $2aL$, so

$$P_0 = 2aLI.$$

The greatest incoming distance from the optical axis that leads to absorption is

$$h_{\max} = R \sin \theta_{\max}.$$

Rays with $|x| \leq a$ strike the container directly. Rays with

$$a < |x| \leq h_{\max}$$

reach the mirror and are then absorbed after one reflection.

The directly absorbed rays shadow the corresponding portion of the mirror. Thus the direct and reflected contributions are

$$P_{\text{direct}} = 2aLI, \quad P_{\text{reflected}} = 2(h_{\max} - a)LI.$$

By the stated restriction, no additional absorbed power comes from rays undergoing two or more reflections. For the ideal mirror, conservation of power therefore gives

$$P = 2h_{\max}LI = 2R \sin \theta_{\max}LI.$$

Hence

$$\frac{P}{P_0} = \frac{R \sin \theta_{\max}}{a}.$$

Substituting the result of T2-B1,

$$\boxed{\frac{P}{P_0} = \frac{1}{1 - \cos \theta_{\max}}}.$$

T2-B3

The condition $P = 5P_0$ gives

$$1 - \cos \theta_{\max} = \frac{1}{5}, \quad \cos \theta_{\max} = \frac{4}{5}.$$

Thus

$$\sin \theta_{\max} = \frac{3}{5}$$

and

$$a = R \sin \theta_{\max} (1 - \cos \theta_{\max}) = R \frac{3}{5} \frac{1}{5} = \frac{3R}{25}.$$

With $R = 1.0$ m,

$$\boxed{a = 0.12 \text{ m} = 12 \text{ cm}}.$$

T2-C1

From T2-B1, the reflected ray passes through

$$M = (R \sin \theta, R \cos \theta)$$

and has direction

$$\hat{\mathbf{u}} = (-\sin 2\theta, -\cos 2\theta).$$

Its slope is therefore

$$\boxed{m_A = \frac{\hat{u}_y}{\hat{u}_x} = \cot 2\theta}.$$

Using the coordinates of M ,

$$\begin{aligned} b_A &= R \cos \theta - R \sin \theta \cot 2\theta \\ &= R \cos \theta - \frac{R \cos 2\theta}{2 \cos \theta} \\ &= \boxed{\frac{R}{2 \cos \theta}}. \end{aligned}$$

In the notation of the hint,

$$K_1 = 1, \quad K_2 = 2, \quad K_3 = \frac{1}{2}, \quad K_4 = 1.$$

The central reflected ray, at $\theta = 0$, is the vertical line $x = 0$. Results involving its slope are understood through the limit $\theta \rightarrow 0$.

T2-C2

For ray B , replace θ by $\theta + \Delta\theta$. Taylor expansion to first order gives

$$m_B = m_A + \frac{dm_A}{d\theta} \Delta\theta + O((\Delta\theta)^2).$$

Since

$$\frac{d}{d\theta} \cot 2\theta = -2 \csc^2 2\theta,$$

we obtain

$$\boxed{m_B \simeq \cot 2\theta - \frac{2\Delta\theta}{\sin^2 2\theta}}.$$

Similarly,

$$\frac{d}{d\theta} \left(\frac{R}{2 \cos \theta} \right) = \frac{R \sin \theta}{2 \cos^2 \theta},$$

so

$$b_B \simeq \frac{R}{2 \cos \theta} + \frac{R \sin \theta}{2 \cos^2 \theta} \Delta \theta.$$

These expansions retain only terms through first order in $\Delta \theta$, as requested.

T2-C3

At the intersection of the two reflected rays,

$$m_A X_c + b_A = m_B X_c + b_B.$$

Consequently,

$$(m_B - m_A) X_c + (b_B - b_A) = 0.$$

Using the first-order differences from T2-C2 and taking the neighboring-ray limit,

$$-\frac{2X_c}{\sin^2 2\theta} + \frac{R \sin \theta}{2 \cos^2 \theta} = 0.$$

Therefore

$$X_c = \frac{R \sin \theta \sin^2 2\theta}{4 \cos^2 \theta}.$$

Since $\sin 2\theta = 2 \sin \theta \cos \theta$,

$$X_c = R \sin^3 \theta.$$

Substitution into ray A's equation gives

$$\begin{aligned} Y_c &= R \sin^3 \theta \cot 2\theta + \frac{R}{2 \cos \theta} \\ &= \frac{R}{2 \cos \theta} (1 + \sin^2 \theta \cos 2\theta) \\ &= \boxed{\frac{R}{2} \cos \theta (1 + 2 \sin^2 \theta)}. \end{aligned}$$

Here the simplification uses

$$1 + \sin^2 \theta \cos 2\theta = \cos^2 \theta (1 + 2 \sin^2 \theta).$$

These parametric equations describe the caustic. The branch on the other side of the optical axis follows by symmetry: X_c changes sign while Y_c remains unchanged.

T2-C4

Near the optical axis, use the small-angle expansions

$$\sin \theta = \theta + O(\theta^3), \quad \cos \theta = 1 - \frac{\theta^2}{2} + O(\theta^4).$$

The caustic coordinates then become

$$X_c = R\theta^3 + O(R\theta^5)$$

and

$$\begin{aligned} Y_c &= \frac{R}{2} \left(1 - \frac{\theta^2}{2} + O(\theta^4) \right) (1 + 2\theta^2 + O(\theta^4)) \\ &= \frac{R}{2} + \frac{3R}{4} \theta^2 + O(R\theta^4). \end{aligned}$$

For both symmetric branches,

$$\theta^2 \simeq \left(\frac{|X_c|}{R} \right)^{2/3}.$$

Eliminating θ therefore gives, to leading order near the cusp,

$$Y_c \simeq \frac{R}{2} + \frac{3}{4} R^{1/3} |X_c|^{2/3}.$$

Thus

$$v = \frac{3}{4} R^{1/3}, \quad u = \frac{R}{2}, \quad p = 2, \quad q = 3.$$

The cusp is at $(0, R/2)$. Its two symmetric branches open toward increasing y and share a vertical tangent there.

The solution was checked against the official solution.

Problem 3: Chasing the absolute zero

QUESTION T3 [10 pts]. Chasing the absolute zero

A refrigeration cycle must be carried out by a system with characteristics appropriate for the temperatures to be achieved. One hundred years ago, in 1926, the chemist William Giauque and, independently, the physicist Peter Debye proposed paramagnetic salts as ideal systems for lowering temperatures far below 1 K. Indeed, beginning in the following decade, temperatures as low as 0.0044 K were achieved (Wander de Haas and Gerrit Wiersma, 1935, Leiden Cryogenic Laboratories). In this question, the paramagnetic Carnot refrigeration cycle is studied.

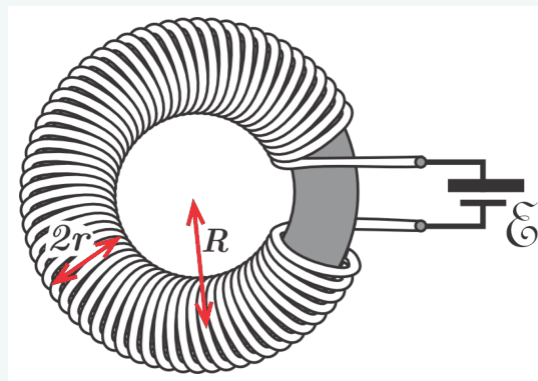


Fig. 3a.

Consider a torus with mean radius R and inner radius r made of a homogeneous, isotropic paramagnetic material, around which an insulated conducting wire is wound, with its ends connected to an external voltage source (emf), as shown in figure 3a. The resistance of the wire is so low that energy losses due to the heating of the wire can be neglected. The winding is dense, with N turns in total, and it is assumed that $r \ll R$, so that the fields $\vec{H}$ and $\vec{B}$ and the

magnetization $\vec{M}$ have approximately constant magnitudes throughout the torus. Let V and A be the volume and the cross-section area of the torus. Hereinafter, the abbreviation Pm-T is used to refer to the paramagnetic torus.

In paramagnetic materials, $\vec{M}$ is parallel to $\vec{H}$ and the following relation holds:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}.$$

Energy transfer sign convention: in this problem, Work (W) and Heat (Q) are considered positive when they flow into the system, and negative when they flow out.

T3-A [1.0 pts]. Work in the Pm-T

Hint: in magnetic materials, $\oint_C \vec{H} \cdot d\vec{\ell} = I_C$, where I_C is the net free current passing through the area bounded by the closed curve C .

T3-A1 [0.2 pts]. Let H be the magnitude of $\vec{H}$ in the torus. Write H in terms of N , A , V and the instantaneous electric current I in the wire.

T3-A2 [0.6 pts]. Let dW_{emf} be the work that the external voltage source performs to change the magnitude of $\vec{B}$ by dB . Write dW_{emf} in terms of V , H , and dB .

T3-A3 [0.2 pts]. The total work done by the voltage source, dW_{emf} , can be divided in two parts: the work that would be performed to change the magnetic field if the toroid had a vacuum core, dW_{vac} , and the work done on the paramagnetic material itself, dW . Write dW in terms of μ_0 , H , V and dM .

T3-B [3.0 pts]. Heat and temperature in the Pm-T

The following table compares two thermodynamic systems: the ideal gas and the Pm-T. Use this information and your answer to question T3-A3 to solve the following questions. This information will also be used in part C. If you did not find an answer for dW in question T3-A3, you may use $dW = \alpha V \mu_0 H dM$ where α is a number. The volume V of the paramagnetic torus is constant, so that there is no work due to pressure. Below, T is the temperature in all instances.

IDEAL GAS	PARAMAGNETIC TORUS
State functions: P , V and temperature T	State functions: H , M and temperature T
Constants: R , γ and number of moles n	Constants: K , λ and number of moles n
Equation of state: $PV = nRT$	Equation of state: $TMV = nKH$
Heat capacity at constant V : $C_V = nR/(\gamma - 1)$	Heat capacity at constant M : $C_M = n\lambda/T^2$
Internal energy U in any process satisfies $dU = C_V dT$	Internal energy U in any process satisfies $dU = C_M dT$

K and λ are constants given by the material in the torus. You may use the equations provided in the table directly without proving them.

T3-B1 [1.5 pts]. If the magnitude of $\vec{H}$ changes from H_i to H_f at a constant temperature T , an energy Q is transferred between the Pm-T and its environment. Write Q in terms of μ_0 , n , V , K , H_i , H_f , and T .

T3-B2 [1.5 pts]. If the magnitude of $\vec{H}$ changes adiabatically from H_i to H_f , the temperature of the Pm-T changes from T_i to T_f . Write $\Delta T = T_f - T_i$ in terms of μ_0 , V , n , K , λ , H_i , H_f , and T_i .

T3-C [6.0 pts]. Carnot refrigerator of the Pm-T

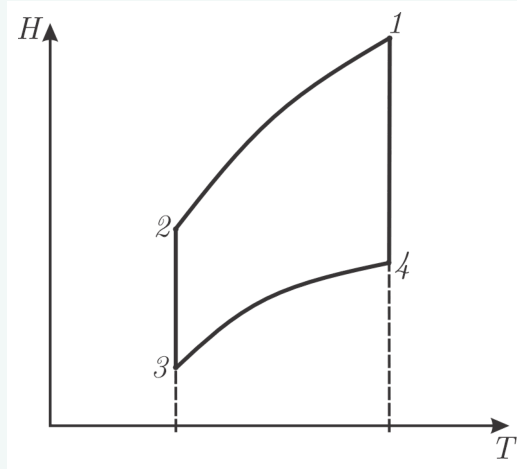


Fig.3b.

The H vs. T diagram of the Carnot refrigeration cycle $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ performed by the Pm-T is shown in figure 3b. In this cycle, let:

- T_h : temperature of the hot reservoir
- T_c : temperature of the cold reservoir
- Q_h : absolute value of the heat transferred to the hot reservoir
- Q_c : absolute value of the heat absorbed from the cold reservoir

T3-C1 [0.2 pts]. Mark T_h and T_c on the T axis of the H vs. T diagram. Also label the processes in which the transfers of Q_h and Q_c occur, respectively.

T3-C2 [1.5 pts]. Let M_1 , M_2 , M_3 , and M_4 be the magnitudes of $\vec{M}$ at the vertices of this cycle. Write M_1 in terms of M_2 , M_3 , and M_4 .

T3-C3 [0.8 pts]. Suppose that to cool 1.00 L of liquid helium initially at 1.00 K, we utilize a 2.0 moles Pm-T made of *potassium chromate*. For this material, $K = 1.87 \times 10^{-6} \text{ K} \cdot \text{m}^3/\text{mol}$, its density is 2730 kg/m^3 and its molar mass is 0.19 kg/mole . The Pm-T performs a Carnot refrigeration cycle with $H_1 = 411624 \text{ A/m}$, $H_2 = 311306 \text{ A/m}$, $H_3 = 204618 \text{ A/m}$ and $H_4 = 240446 \text{ A/m}$. After one operating cycle, calculate the final temperature of the liquid helium. Assume that for the liquid helium both the specific heat capacity $c = 100 \text{ J}/(\text{kg} \cdot \text{K})$ and the density $\rho = 130 \text{ kg/m}^3$ remain constant.

T3-C4 [2.0 pts]. In order to continuously lower the temperature of a body with constant heat capacity C_c , Carnot cycles are allowed to occur over time in such a way that the power P transferred to the refrigerator remains constant. It is then assumed that the temperature T_h of the hot reservoir remains constant and that in each operating cycle the temperature T_c of the cooled body decreases by dT_c , satisfying the relation $dQ_c/dQ_h = T_c/T_h$. If at $t = 0$ we have $T_c = T_0$, determine the time t that the Carnot refrigerator must operate for the cooled body to reach a temperature $T < T_0$. Express your answer in terms of C_c , T_h , P , T_0 and T .

T3-C5 [1.5 pts]. The efficiency of a refrigerator is quantified by its *coefficient of performance* $\text{COP} = Q_c/W$. Determine the overall COP of the refrigerator of question T3-C4 with all the cycles performed until time t . Write your answer in terms of T_0 and T_h and T .

Physical Constants

Reference values for selected physical constants.

Constant	Notation	Value	Units
Speed of light in vacuum	c	2.99792458×10^8	$\text{m} \cdot \text{s}^{-1}$
Planck constant	h	$4.135667696 \dots \times 10^{-15}$	$\text{eV} \cdot \text{s}$
Reduced Planck constant	$\hbar = h/2\pi$	$6.582119569 \dots \times 10^{-16}$	$\text{eV} \cdot \text{s}$
Avogadro number	N_A	$6.02214076 \times 10^{23}$	mol^{-1}
Boltzmann constant	k_B	$8.617333262 \times 10^{-5}$	$\text{eV} \cdot \text{K}^{-1}$
Ideal gas constant	R	8.31446261815324	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Elementary charge	e	$1.602176634 \times 10^{-19}$	$\text{A} \cdot \text{s}$
Universal gravitational constant	G	$6.67430(15) \times 10^{-11}$	$\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Stefan Boltzmann constant	σ	$5.670374419 \times 10^{-8}$	$\text{kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$
Vacuum permeability	μ_0	$1.25663706127(20) \times 10^{-6}$	$\text{kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity	ε_0	$8.8541878188(14) \times 10^{-12}$	$\text{A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$
Mass of the electron	m_e	$5.1099895069(16) \times 10^5$	eV/c^2
Mass of the proton	m_p	$9.3827208943(29) \times 10^8$	eV/c^2
Atomic mass units	amu	$9.3149410372(29) \times 10^8$	eV/c^2

Official English text of the problem.

Answers

- T3-A1:

$$H = \frac{NAI}{V}.$$

- T3-A2:

$$dW_{emf} = VH dB.$$

- T3-A3:

$$dW = \mu_0 VH dM, \quad \alpha = 1.$$

- T3-B1:

$$Q = \frac{\mu_0 n K}{2T} (H_i^2 - H_f^2).$$

- **T3-B2:**

$$\Delta T = T_i \left[\sqrt{\frac{\lambda + \mu_0 K H_f^2}{\lambda + \mu_0 K H_i^2}} - 1 \right].$$

- **T3-C1:** The left vertical line, through states 2, 3, is at T_c ; the right vertical line, through states 1, 4, is at T_h . Heat Q_c is absorbed during $2 \rightarrow 3$, and heat Q_h is rejected during $4 \rightarrow 1$.

- **T3-C2:**

$$M_1 = \sqrt{M_2^2 + M_4^2 - M_3^2}.$$

- **T3-C3:** Using $T_c \simeq 1.00$ K for the cold stroke,

$$T_{\text{He},f} \simeq 0.990 \text{ K}.$$

This is a small-cooling estimate that takes the cold isotherm at the initial helium temperature; it uses only the cold stroke $2 \rightarrow 3$ (see the note at the end of the T3-C3 solution).

- **T3-C4:**

$$t = \frac{C_c}{P} \left[T_h \ln \left(\frac{T_0}{T} \right) - (T_0 - T) \right].$$

- **T3-C5:**

$$\text{COP}_{\text{overall}} = \frac{T_0 - T}{T_h \ln(T_0/T) - (T_0 - T)}.$$

Solution

We use the prescribed sign convention throughout:

$$dU = dQ + dW,$$

where heat and work are positive when supplied to the paramagnetic material. The magnetic processes are treated quasistatically, with the uniform-field approximation $r \ll R$ and negligible resistive losses.

T3-A1

Choose an Amperian circle of radius R inside the torus. It links all N turns, so the enclosed free current is NI . Thus,

$$H(2\pi R) = NI.$$

Since the torus volume is $V = (2\pi R)A$,

$$H = \frac{NI}{2\pi R} = \frac{NAI}{V}.$$

T3-A2

The magnetic flux through each turn is approximately

$$\Phi = AB.$$

Faraday's law gives an induced emf $-NA dB/dt$. With negligible resistance, the applied source emf balances this:

$$\mathcal{E} = NA \frac{dB}{dt}.$$

The electrical work supplied in time dt is therefore

$$dW_{emf} = \mathcal{E} I dt = NAI dB.$$

Using $NAI = VH$ from A1,

$$\boxed{dW_{emf} = VH dB}.$$

This expression also gives the correct sign when the source receives energy during a decrease of the field.

T3-A3

Because the fields and magnetization are parallel,

$$dB = \mu_0 dH + \mu_0 dM.$$

Consequently,

$$dW_{emf} = \mu_0 VH dH + \mu_0 VH dM.$$

For a vacuum core, $B = \mu_0 H$, so the work associated with the same change of H is

$$dW_{vac} = \mu_0 VH dH = d\left(\frac{\mu_0 VH^2}{2}\right).$$

Subtracting this vacuum-field contribution leaves the work on the material:

$$\boxed{dW = dW_{emf} - dW_{vac} = \mu_0 VH dM}.$$

Thus the fallback coefficient is $\alpha = 1$.

T3-B1

The given internal-energy relation implies that an isothermal process has

$$dU = \frac{n\lambda}{T^2} dT = 0.$$

The first law then gives

$$dQ = -dW = -\mu_0 VH dM.$$

At constant T , the equation of state yields

$$M = \frac{nKH}{VT}, \quad dM = \frac{nK}{VT} dH.$$

Hence

$$Q = -\frac{\mu_0 nK}{T} \int_{H_i}^{H_f} H dH,$$

and therefore

$$\boxed{Q = \frac{\mu_0 nK}{2T} (H_i^2 - H_f^2)}.$$

The volume cancels. Isothermal demagnetization, $H_f < H_i$, absorbs heat; isothermal magnetization releases heat.

T3-B2

An adiabatic process has $dQ = 0$, so

$$\frac{n\lambda}{T^2} dT = \mu_0 V H dM.$$

Now both H and T vary. Differentiating the equation of state gives

$$dM = \frac{nK}{V} \left(\frac{dH}{T} - \frac{H}{T^2} dT \right).$$

Substitution into the first law gives

$$\frac{n\lambda}{T^2} dT = \mu_0 n K \left(\frac{H}{T} dH - \frac{H^2}{T^2} dT \right).$$

Thus

$$(\lambda + \mu_0 K H^2) dT = \mu_0 K H T dH,$$

or

$$\frac{dT}{T} = \frac{\mu_0 K H}{\lambda + \mu_0 K H^2} dH.$$

Integrating between the initial and final states,

$$\ln \frac{T_f}{T_i} = \frac{1}{2} \ln \frac{\lambda + \mu_0 K H_f^2}{\lambda + \mu_0 K H_i^2}.$$

Therefore,

$$T_f = T_i \sqrt{\frac{\lambda + \mu_0 K H_f^2}{\lambda + \mu_0 K H_i^2}},$$

and

$$\Delta T = T_i \left[\sqrt{\frac{\lambda + \mu_0 K H_f^2}{\lambda + \mu_0 K H_i^2}} - 1 \right].$$

For positive K and heat capacity, decreasing H decreases T : this is adiabatic demagnetization.

An equivalent form, useful below, is the adiabatic invariant

$$\frac{\lambda + \mu_0 K H^2}{T^2} = \text{constant along an adiabat}.$$

T3-C1

The horizontal coordinate is temperature. Therefore:

- Label the left dashed projection, beneath states 2 and 3, T_c .
- Label the right dashed projection, beneath states 1 and 4, T_h .

On $2 \rightarrow 3$, the field decreases at constant T_c . By B1, the torus absorbs heat, so place Q_c , directed into the torus, beside this downward vertical stroke.

On $4 \rightarrow 1$, the field increases at constant T_h . The torus releases heat, so place Q_h , directed out of the torus, beside this upward vertical stroke.

The remaining strokes are $1 \rightarrow 2$, adiabatic cooling, and $3 \rightarrow 4$, adiabatic heating.

T3-C2

Substitute

$$H = \frac{VTM}{nK}$$

into the adiabatic invariant from B2:

$$\frac{\lambda}{T^2} + \frac{\mu_0 V^2}{n^2 K} M^2 = \text{constant}.$$

For the adiabat $1 \rightarrow 2$,

$$\frac{\mu_0 V^2}{n^2 K} (M_1^2 - M_2^2) = \lambda \left(\frac{1}{T_c^2} - \frac{1}{T_h^2} \right).$$

For the adiabat $3 \rightarrow 4$,

$$\frac{\mu_0 V^2}{n^2 K} (M_4^2 - M_3^2) = \lambda \left(\frac{1}{T_c^2} - \frac{1}{T_h^2} \right).$$

Their right-hand sides are identical, so

$$M_1^2 - M_2^2 = M_4^2 - M_3^2.$$

Since the M_j are magnitudes, the positive root is required:

$$M_1 = \sqrt{M_2^2 + M_4^2 - M_3^2}.$$

T3-C3

For the one-cycle estimate, take the cold-stroke temperature as approximately the initial helium temperature:

$$T_c \simeq 1.00 \text{ K}.$$

The heat absorbed during $2 \rightarrow 3$ is

$$Q_c = \frac{\mu_0 n K}{2 T_c} (H_2^2 - H_3^2).$$

Using the supplied values,

$$Q_c = \frac{(1.25663706127 \times 10^{-6})(2.0)(1.87 \times 10^{-6})}{2(1.00)} (311306^2 - 204618^2) \text{ J},$$

which gives

$$Q_c = 0.1293459 \dots \text{ J} \simeq 0.13 \text{ J}.$$

The helium mass and heat capacity are

$$m_{\text{He}} = \rho V_{\text{He}} = (130)(1.00 \times 10^{-3}) = 0.130 \text{ kg},$$

$$C_{\text{He}} = m_{\text{He}}c = (0.130)(100) = 13.0 \text{ J K}^{-1}.$$

Energy conservation for the helium gives

$$C_{\text{He}}(T_{\text{He},f} - 1.00 \text{ K}) = -Q_c.$$

Thus

$$T_{\text{He},f} = 1.00 \text{ K} - \frac{0.1293459 \dots \text{ J}}{13.0 \text{ J K}^{-1}} = 0.9900503 \dots \text{ K},$$

or

$$\boxed{T_{\text{He},f} \simeq 0.990 \text{ K}}.$$

The estimated temperature decrease is approximately 1%, supporting the small-change approximation for this numerical estimate. The material volume cancels from Q_c , so its density and molar mass are unnecessary here. For an exact cycle the four given fields are not fully consistent: with $M = nKH/(VT)$ on the two isotherms, the relation of T3-C2 requires

$T_h/T_c = \sqrt{(H_1^2 - H_4^2)/(H_2^2 - H_3^2)} \approx 1.42$, and the adiabats $1 \rightarrow 2$ and $3 \rightarrow 4$ then hold only with $\lambda \approx -0.062 \text{ J K mol}^{-1}$, that is, a negative heat capacity $C_M = n\lambda/T^2$. The estimate above uses only the cold stroke and does not depend on λ .

T3-C4

Let $dT_c < 0$ denote the actual temperature change of the cooled body. The positive heat extracted from it is

$$dQ_c = -C_c dT_c.$$

The specified Carnot relation gives

$$dQ_h = \frac{T_h}{T_c} dQ_c.$$

Since the refrigerator returns to its original state after each cycle, its net internal-energy change vanishes. The required work is therefore

$$dW = dQ_h - dQ_c = \left(\frac{T_h}{T_c} - 1 \right) dQ_c.$$

With constant input power,

$$P dt = dW = -C_c \left(\frac{T_h}{T_c} - 1 \right) dT_c.$$

Integrating from $T_c = T_0$ at $t = 0$ to $T_c = T$,

$$Pt = -C_c \int_{T_0}^T \left(\frac{T_h}{T_c} - 1 \right) dT_c.$$

Hence

$$Pt = C_c \left[T_h \ln \left(\frac{T_0}{T} \right) + T - T_0 \right],$$

and

$$t = \frac{C_c}{P} \left[T_h \ln \left(\frac{T_0}{T} \right) - (T_0 - T) \right].$$

For $0 < T < T_0 < T_h$, this time is positive. Moreover,

$$T \rightarrow 0^+ \implies t \rightarrow \infty.$$

Within the stated model, reaching absolute zero requires an infinite operating time.

T3-C5

The total heat extracted from the cooled body is

$$Q_{c,\text{total}} = C_c(T_0 - T).$$

The total work supplied is

$$W_{\text{total}} = Pt = C_c \left[T_h \ln \left(\frac{T_0}{T} \right) - (T_0 - T) \right].$$

Therefore,

$$\text{COP}_{\text{overall}} = \frac{Q_{c,\text{total}}}{W_{\text{total}}} = \frac{T_0 - T}{T_h \ln(T_0/T) - (T_0 - T)}.$$

As a check, for an infinitesimal temperature decrease,

$$\lim_{T \rightarrow T_0^-} \text{COP}_{\text{overall}} = \frac{T_0}{T_h - T_0},$$

which is the usual Carnot refrigerator COP at cold temperature T_0 .

The solution was checked against the official solution.