

IPhO 2025

Theory problems and solutions · Paris, France · 3 theory problems with complete solutions

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Problem 1: Hydrogen and galaxies

This problem aims to study the peculiar physics of galaxies, such as their dynamics and structure. In particular, we explain how to measure the mass distribution of our galaxy from the inside. For this we will focus on hydrogen, its main constituent.

Throughout this problem we will only use $\hbar$, defined as $\hbar = h/2\pi$.

Part A - Introduction

Bohr model

We assume that the hydrogen atom consists of a non-relativistic electron, with mass m_e , orbiting a fixed proton. Throughout this part, we assume its motion is on a circular orbit.

A.1 — 0.2pt

Determine the electron's velocity v in a circular orbit of radius r .

In the Bohr model, we assume the magnitude of the electron's angular momentum L is quantized, $L = n\hbar$ where $n > 0$ is an integer. We define $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx 7.27 \times 10^{-3}$.

A.2 — 0.5pt

Show that the radius of each orbit is given by $r_n = n^2 r_1$, where r_1 is called the Bohr radius.

Express r_1 in terms of α , m_e , c and $\hbar$ and calculate its numerical value with 3 digits. Express v_1 , the velocity on the orbit of radius r_1 , in terms of α and c .

A.3 — 0.5pt

Determine the electron's mechanical energy E_n on an orbit of radius r_n in terms of e , ϵ_0 , r_1 and n . Determine E_1 in the ground state in terms of α , m_e and c . Compute its numerical value in eV.

Hydrogen fine and hyperfine structures

The rare spontaneous inversion of the electron's spin causes a photon to be emitted on average once per 10 million years per hydrogen atom. This emission serves as a hydrogen tracer in the universe and is thus fundamental in astrophysics. We will study the transition responsible for this emission in two steps.

First, consider the interaction between the electron spin and the relative motion of the electron and the proton. Working in the electron's frame of reference, the proton orbits the electron at a distance r_1 . This produces a magnetic field $\vec{B}_1$.

A.4 — 0.5pt

Determine the magnitude B_1 of $\vec{B}_1$ at the position of the electron in terms of μ_0 , e , α , c and r_1 .

Second, the electron spin creates a magnetic moment $\vec{\mathcal{M}}_s$. Its magnitude is roughly $\mathcal{M}_s = \frac{e}{m_e} \hbar$. The *fine* (F) structure is related to the energy difference ΔE_F between an electron with a magnetic moment $\vec{\mathcal{M}}_s$ parallel to $\vec{B}_1$ and that of an electron with $\vec{\mathcal{M}}_s$ anti-parallel to $\vec{B}_1$. Similarly, the *hyperfine* (HF) structure is related to the energy difference ΔE_{HF} , due to the interaction between parallel and anti-parallel magnetic moments of the electron and the proton. It is known to be approximately $\Delta E_{HF} \simeq 3.72 \frac{m_e}{m_p} \Delta E_F$ where m_p is the proton mass.

A.5 — 0.5pt

Express ΔE_F as a function of α and E_1 .

Express the wavelength λ_{HF} of a photon emitted during a transition between the two states of the hyperfine structure and give its numerical value with two digits.

Part B - Rotation curves of galaxies

Data

- Kiloparsec: $1 \text{ kpc} = 3.09 \times 10^{19} \text{ m}$
- Solar mass : $1 M_\odot = 1.99 \times 10^{30} \text{ kg}$

We consider a spherical galaxy centered around a fixed point O . At any point P , let $\rho = \rho(P)$ be the volumetric mass density and $\varphi = \varphi(P)$ the associated gravitational potential (i.e. potential energy per unit mass). Both ρ and φ depend only on $r = \|\vec{OP}\|$. The motion of a mass m located at P , due to the field φ , is restricted to a plane containing O .

B.1 — 0.2pt

In the case of a circular orbit, determine the velocity v_c of an object on a circular orbit passing through P in terms of r and $\frac{d\varphi}{dr}$.

Fig. 1(A) is a picture of the spiral galaxy NGC 6946 in the visible band (from the 0.8m Schulman Telescope at the Mount Lemmon Sky Center in Arizona). The little ellipses in Fig. 1(B) show experimental measurements of v_c for this galaxy. The central region ($r < 1 \text{ kpc}$) is named the bulge. In this region, the mass distribution is roughly homogeneous. The red curve is a prediction for v_c if the system were homogeneous in the bulge and keplerian ($\varphi(r) = -\beta/r$ with $\beta > 0$) outside it, i.e. considering that the total mass of the galaxy is concentrated in the bulge.

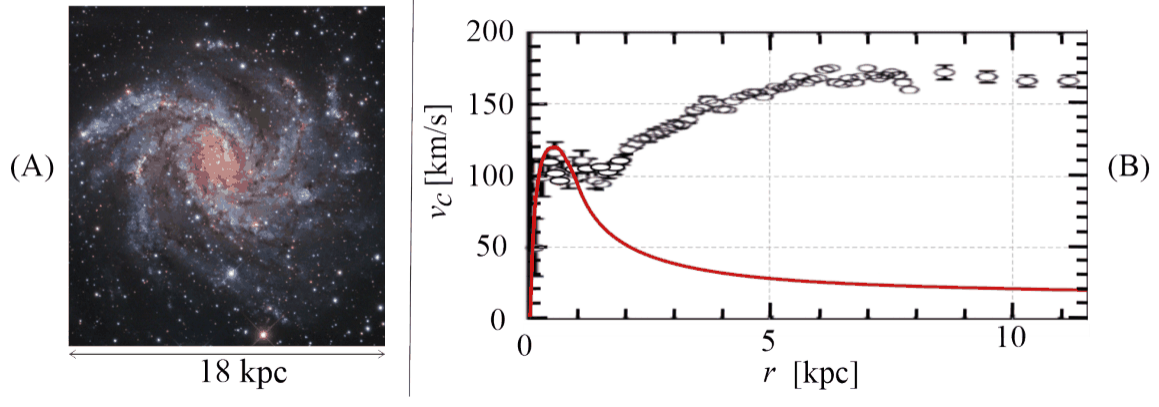


Fig. 1: NGC 6946 galaxy: Picture (A) and rotation curve (B).

B.2 — 0.5pt

Deduce the mass M_b of the bulge of NGC 6946 from the red rotation curve in Fig. 1(B), in solar mass units.

Comparing the keplerian model and the experimental data makes astronomers confident that part of the mass is invisible in the picture. They thus suppose that the galaxy's actual mass density is given by

$$\rho_m(r) = \frac{C_m}{r_m^2 + r^2} \quad (1)$$

where $C_m > 0$ and $r_m > 0$ are constants.

B.3 — 1.8pt

Show that the velocity profile $v_{c,m}(r)$, corresponding to the mass density in Eq. 1, can be written

$$v_{c,m}(r) = \sqrt{k_1 - \frac{k_2 \cdot \arctan\left(\frac{r}{r_m}\right)}{r}}.$$

Express k_1 and k_2 in terms of C_m , r_m and G .

(Hints: $\int_0^r \frac{x^2}{a^2 + x^2} dx = r - a \arctan(r/a)$, and: $\arctan(x) \simeq x - x^3/3$ for $x \ll 1$.)

Simplify $v_{c,m}(r)$ when $r \ll r_m$ and when $r \gg r_m$.

Show that if $r \gg r_m$, the mass $M_m(r)$ embedded in a sphere of radius r with the mass density given by Eq. 1 simplifies and depends only on C_m and r .

Estimate the mass of the galaxy NGC 6946 actually present in the picture in Fig. 1(A).

Part C - Mass distribution in our galaxy

For a spiral galaxy, the model for Eq. 1 is modified and one usually considers the gravitational potential is given by

$$\varphi_G(r, z) = \varphi_0 \ln\left(\frac{r}{r_0}\right) \exp\left[-\left(\frac{z}{z_0}\right)^2\right],$$

where z is the distance to the galactic plane (defined by $z = 0$), and $r < r_0$ is now the axial radius and $\varphi_0 > 0$ a constant to be determined. r_0 and z_0 are constant values.

C.1 — 0.5pt

Find the equation of motion on z for the vertical motion of a point mass m in such a potential, assuming r is constant. Show that, if $r < r_0$ the galactic plane is a stable equilibrium state by giving the angular frequency ω_0 of small oscillations around it.

From here on, we set $z = 0$.

C.2 — 0.6pt

Identify the regime, either $r \gg r_m$ or $r \ll r_m$, in which the model of Eq. 1 recovers a potential of the form $\varphi_G(r, 0)$ with a suitable definition of φ_0 .

Under this condition $v_c(r)$ no longer depends on r . Express it in terms of φ_0 .

Therefore, outside the bulge the velocity modulus v_c does not depend on the distance to the galactic center. We will use this fact, as astronomers do, to measure the galaxy's mass distribution from the inside.

All galactic objects considered here for astronomical observations, such as stars or nebulae, are primarily composed of hydrogen. Outside the bulge, we assume that they rotate on circular orbits around the galactic center C . S is the sun's position and E that of a given galactic object emitting in the hydrogen spectrum. In the galactic plane, we consider a line of sight SE corresponding to the orientation of an observation, on the unit vector $\hat{u}_v$ (see Fig. 2).

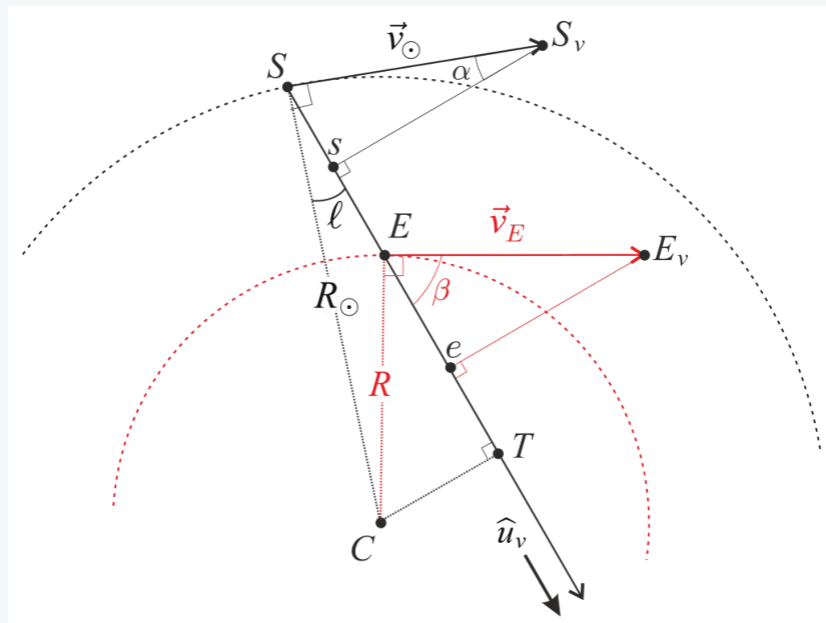


Fig. 2: Geometry of the measurement

Let ℓ be the galactic longitude, measuring the angle between SC and the SE . The sun's velocity on its circular orbit of radius $R_\odot = 8.00$ kpc is denoted $\vec{v}_\odot$. A galactic object in E orbits on another circle of radius R at velocity $\vec{v}_E$. Using a Doppler effect on the previously studied 21cm line, one can obtain the relative radial velocity $v_{rE/S}$ of the emitter E with respect to the sun S : it is the projection of $\vec{v}_E - \vec{v}_\odot$ on the line of sight.

C.3 — 0.7pt

Determine $v_{rE/S}$ in terms of ℓ , R , $R_\odot$ and $v_\odot$. Then, express R in terms of $R_\odot$, $v_\odot$, ℓ and $v_{rE/S}$.

Using a radio telescope, we make observations in the plane of our galaxy toward a longitude $\ell = 30^\circ$. The frequency band used contains the 21cm line, whose frequency is $f_0 = 1.42 \text{ GHz}$. The results are reported in Fig. 3.

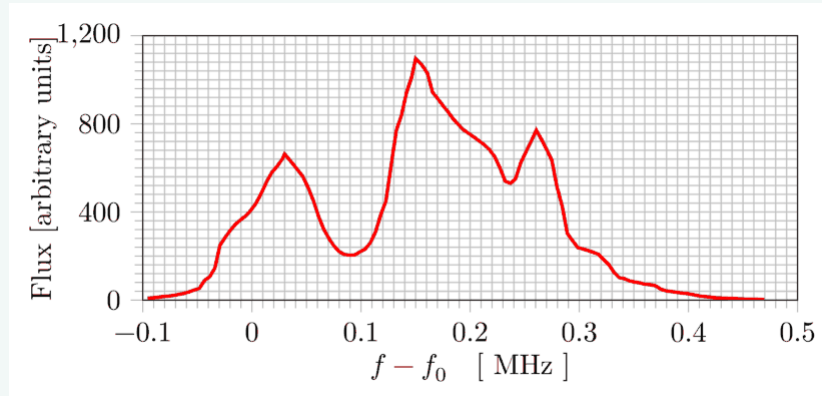


Fig. 3: Electromagnetic signal as a function of the frequency shift, measured in the radio frequency band at $\ell = 30^\circ$ using EU-HOU RadioAstronomy

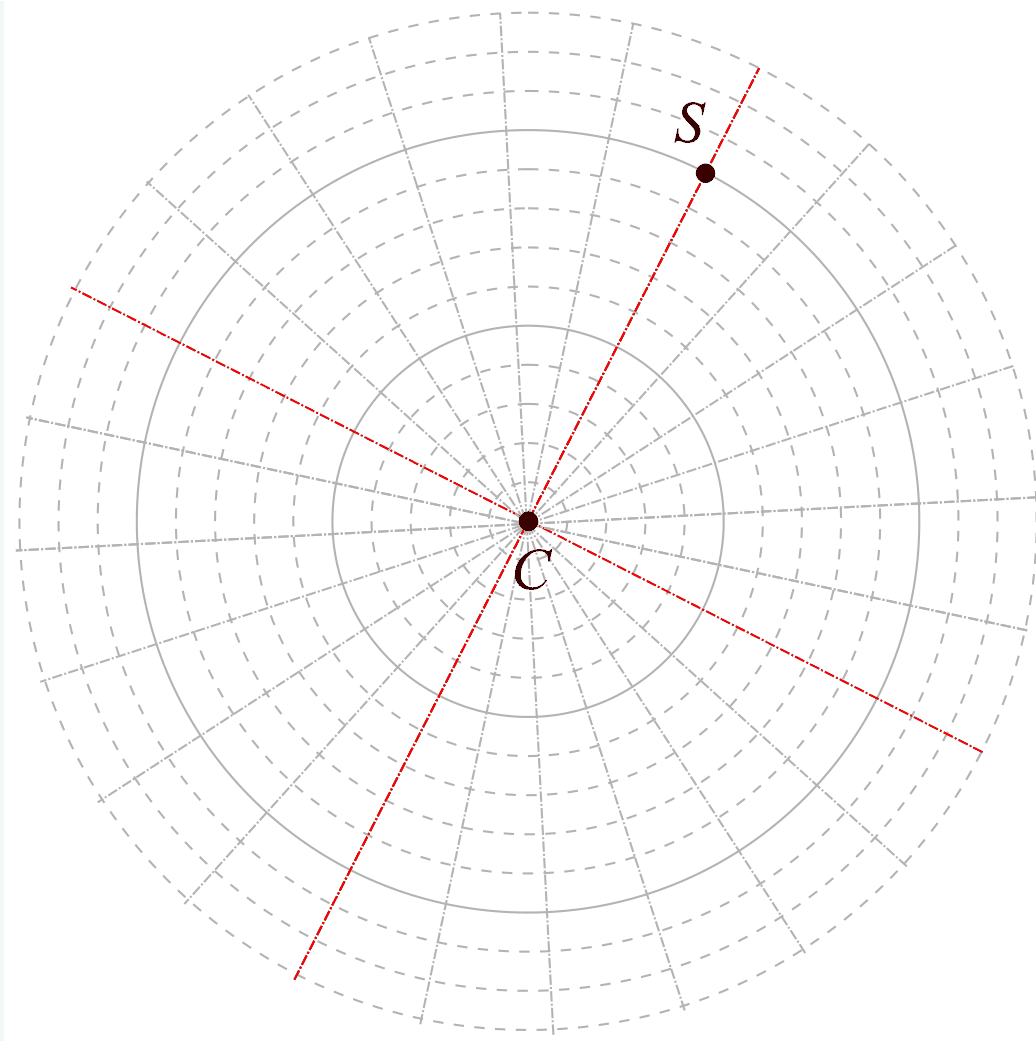
C.4 — 0.6pt

In our galaxy, $v_\odot = 220 \text{ km} \cdot \text{s}^{-1}$. Determine the values of the relative radial velocity (with 3 significant digits) and the distance from the galactic center (with 2 significant digits) of the 3 sources observed in Fig. 3. Distances should be expressed as multiples of $R_\odot$.

C.5 — 0.6pt

On the top view of our galaxy (in the answer box), indicate the positions of the sources observed in Fig. 3.

What could be deduced from repeated measurements changing ℓ ?



Part D - Tully-Fisher relation and MOND theory

The flat external velocity curve of NGC 6946 in Fig. 1 is a common property of spiral galaxies, as can be seen in Fig. 4 (left). Plotting the external constant velocity value $v_{c,\infty}$ as a function of the measured total mass M_{tot} of each galaxy gives an interesting correlation called the Tully-Fisher relation, see Fig. 4 (right).

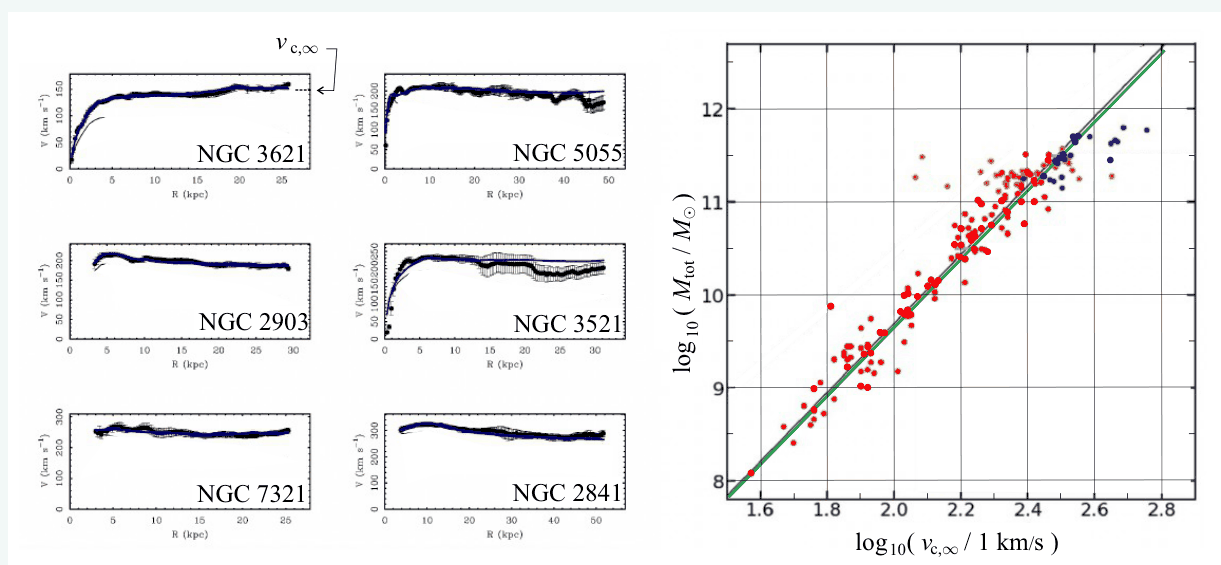


Fig. 4. Left: Rotation curves for typical spiral galaxies – Right: $\log_{10}(M_{\text{tot}})$ as a function of $\log_{10}(v_{c,\infty})$ on linear scales. Colored dots correspond to different galaxies and different surveys. The green line is the Tully-Fischer relation which is in very good agreement with the best fit line of the data (in black).

D.1 — 0.4pt

Assuming that the radius R of a galaxy doesn't depend on its mass, show that the model of Eq. 1 (part B) gives a relation of the form $M_{\text{tot}} = \eta v_{c,\infty}^\gamma$ where γ and η should be specified.

Compare this expression to the Tully-Fischer relation by computing γ_{TF} .

In the extremely low acceleration regime, of the order of $a_0 = 10^{-10} \text{ m} \cdot \text{s}^{-2}$, the MODified Newtonian Dynamics (MOND) theory suggests that one can modify Newton's second law using

$$\vec{F} = m\mu\left(\frac{a}{a_0}\right)\vec{a}$$

where $a = \|\vec{a}\|$ is the modulus of the acceleration and the μ function is defined by

$$\mu(x) = \frac{x}{1+x}.$$

D.2 — 0.2pt

Using data for NGC 6946 in Fig. 1, estimate, within Newton's theory, the modulus of the acceleration a_m of a mass in the outer regions of NGC 6946.

D.3 — 0.8pt

Let m be a mass on a circular orbit of radius r with velocity $v_{c,\infty}$ in the gravity field of a fixed mass M .

Within the MOND theory, with $a \ll a_0$, determine the Tully-Fischer exponent.

Using data for NGC 6946 and/or Tully-Fischer law, calculate a_0 to show that MOND operates in the correct regime.

D.4 — 0.9pt

Considering relevant cases, determine $v_c(r)$ for all values of r in the MOND theory in the case of a gravitational field due to a homogeneously distributed mass M with radius R_b .

General Data Sheet

Constant	Notation	Value	Unit
Speed of light in vacuum	c	$2.997\,924\,58 \times 10^8$	$\text{m} \cdot \text{s}^{-1}$
Planck constant	h	$6.626\,070\,15 \times 10^{-34}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Reduced Planck constant	$\hbar = \frac{h}{2\pi}$	$1.054\,571\,818 \times 10^{-34}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Boltzmann constant	k_B	$1.380\,649 \times 10^{-23}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$	mol^{-1}
Molar gas constant	R	$8.314\,462\,618\,153\,24$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	$\text{A} \cdot \text{s}$

Constant	Notation	Value	Unit
Universal constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	$\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Standard acceleration due to gravity	g	9.806 65	$\text{m} \cdot \text{s}^{-2}$
Stefan Boltzmann constant	σ	$5.670\,374\,419 \times 10^{-8}$	$\text{kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$
Vacuum permeability (magnetic constant)	μ_0	$1.256\,637\,061\,27(20) \times 10^{-6}$	$\text{kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity (electrical constant)	ϵ_0	$8.854\,187\,818\,8(14) \times 10^{-12}$	$\text{A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$
Rydberg constant	R_∞	$1.097\,373\,156\,815\,7(12) \times 10^7$	m^{-1}
Mass of the electron	m_e	$9.109\,383\,713\,9(28) \times 10^{-31}$	kg
Mass of the proton	m_p	$1.672\,621\,925\,95(52) \times 10^{-27}$	kg
Mass of the neutron	m_n	$1.674\,927\,500\,56(85) \times 10^{-27}$	kg
Atomic mass constant	m_u	$1.660\,539\,068\,92(52) \times 10^{-27}$	kg
Electronvolt	eV	$1.602\,176\,634 \times 10^{-19}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$

Official English text of the problem.

Answers

- A.1

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}}.$$

- A.2

$$r_n = n^2 r_1, \quad r_1 = \frac{\hbar}{m_e \alpha c} = 5.29 \times 10^{-11} \text{ m}, \quad v_1 = \alpha c.$$

- A.3

$$E_n = -\frac{e^2}{8\pi\epsilon_0 r_1 n^2}, \quad E_1 = -\frac{1}{2} m_e \alpha^2 c^2 = -13.6 \text{ eV}.$$

- A.4

$$B_1 = \frac{\mu_0 e \alpha c}{4\pi r_1^2}.$$

- A.5

$$\Delta E_F = -4\alpha^2 E_1, \quad \lambda_{\text{HF}} = \frac{\pi m_p \hbar}{3.72 m_e^2 \alpha^4 c} \simeq 0.21 \text{ m}.$$

- B.1

$$v_c = \sqrt{r \frac{d\varphi}{dr}}.$$

- B.2

$$M_b \sim 1 \times 10^9 M_\odot.$$

- B.3

$$k_1 = 4\pi G C_m, \quad k_2 = 4\pi G C_m r_m,$$

$$v_{c,m}(r \ll r_m) \simeq \sqrt{\frac{4\pi G C_m}{3} \frac{r}{r_m}}, \quad v_{c,m}(r \gg r_m) \simeq \sqrt{4\pi G C_m},$$

$$M_m(r \gg r_m) \simeq 4\pi C_m r, \quad M_{\text{in the figure}} \simeq 6.1 \times 10^{10} M_\odot.$$

- C.1

$$\ddot{z} = \frac{2\varphi_0}{z_0^2} \ln\left(\frac{r}{r_0}\right) z e^{-z^2/z_0^2}, \quad \omega_0 = \sqrt{\frac{2\varphi_0}{z_0^2} \ln\left(\frac{r_0}{r}\right)}.$$

- C.2

$$r \gg r_m, \quad \varphi_0 = 4\pi G C_m, \quad v_c(r) = \sqrt{\varphi_0}.$$

- C.3

$$v_{rE/S} = v_\odot \left(\frac{R_\odot}{R} - 1 \right) \sin \ell, \quad R = \frac{R_\odot v_\odot \sin \ell}{v_\odot \sin \ell + v_{rE/S}}.$$

- C.4 Numbering the peaks by increasing frequency shift:

$$(v_{r,1}, v_{r,2}, v_{r,3}) = (6.33, 31.7, 54.9) \text{ km s}^{-1},$$

$$(R_1, R_2, R_3) = (0.95, 0.78, 0.67) R_\odot.$$

- C.5 On the $\ell = 30^\circ$ sightline, mark both intersections with each of the three circles of radii $0.95R_\odot$, $0.78R_\odot$, and $0.67R_\odot$. The near and far distances from the Sun are

$$\frac{(d_{1,-}, d_{2,-}, d_{3,-})}{R_\odot} \simeq (0.0635, 0.272, 0.424),$$

$$\frac{(d_{1,+}, d_{2,+}, d_{3,+})}{R_\odot} \simeq (1.67, 1.46, 1.31).$$

Repeated longitude scans reveal the hydrogen distribution and spiral structure, subject to this near–far ambiguity.

- D.1

$$\eta = \frac{R}{G}, \quad \gamma = 2, \quad \gamma_{TF} \approx 3.7 \text{ (graphically close to 4)}.$$

- D.2

$$a_m \simeq 9.4 \times 10^{-11} \text{ m s}^{-2}.$$

- D.3

$$v_{c,\infty}^4 = GMa_0, \quad \gamma_{\text{MOND}} = 4, \quad a_0 \sim 2 \times 10^{-10} \text{ m s}^{-2}.$$

This is the relevant acceleration scale; the plotted NGC 6946 data alone do not establish the stronger inequality $a \ll a_0$.

- D.4

$$v_c(r) = \begin{cases} \left[\frac{GMr^2}{2R_b^3} \left(1 + \sqrt{1 + \frac{4a_0 R_b^3}{GMr}} \right) \right]^{1/2}, & 0 < r \leq R_b, \\ \left[\frac{GM}{2r} \left(1 + \sqrt{1 + \frac{4a_0 r^2}{GM}} \right) \right]^{1/2}, & r \geq R_b. \end{cases}$$

The continuous limit is $v_c(0) = 0$.

Solution

For the atomic numerical values, I evaluate α from its defining expression and the supplied constants:

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = 7.29735 \times 10^{-3}.$$

This differs slightly from the approximation printed in the statement.

A.1

The attractive Coulomb force supplies the centripetal force:

$$\frac{m_e v^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2}.$$

Therefore,

$$v = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e r}}.$$

A.2

Bohr quantization gives

$$m_e v_n r_n = n\hbar.$$

Squaring this and using the force balance from A.1,

$$n^2 \hbar^2 = m_e^2 v_n^2 r_n^2 = \frac{m_e e^2 r_n}{4\pi\epsilon_0}.$$

Consequently,

$$r_n = \frac{4\pi\epsilon_0 n^2 \hbar^2}{m_e e^2} = n^2 \frac{\hbar}{m_e \alpha c}.$$

Thus

$$r_1 = \frac{\hbar}{m_e \alpha c} = 5.29 \times 10^{-11} \text{ m}.$$

For the ground orbit,

$$v_1 = \frac{\hbar}{m_e r_1} = \boxed{\alpha c}.$$

More generally, $v_n = \alpha c/n$. Since $\alpha \ll 1$, the non-relativistic approximation is consistent.

A.3

Taking the electrostatic potential energy to vanish at infinity,

$$U(r) = -\frac{e^2}{4\pi\epsilon_0 r}.$$

The force balance gives

$$K = \frac{1}{2} m_e v^2 = \frac{e^2}{8\pi\epsilon_0 r}.$$

Hence the total mechanical energy is

$$E = K + U = -\frac{e^2}{8\pi\epsilon_0 r}.$$

At $r_n = n^2 r_1$,

$$E_n = -\frac{e^2}{8\pi\epsilon_0 r_1 n^2}.$$

In particular,

$$E_1 = -\frac{\alpha \hbar c}{2 r_1} = \boxed{-\frac{1}{2} m_e \alpha^2 c^2}.$$

Numerically,

$$E_1 = -2.17987 \times 10^{-18} \text{ J} = \boxed{-13.6 \text{ eV}}.$$

A.4

In the prescribed electron-frame picture, the proton circulates with speed $v_1 = \alpha c$. Its orbital period and equivalent current are

$$T = \frac{2\pi r_1}{v_1}, \quad I = \frac{e}{T} = \frac{e v_1}{2\pi r_1}.$$

The magnetic field at the centre of a circular current loop is

$$B_1 = \frac{\mu_0 I}{2 r_1}.$$

Therefore,

$$B_1 = \frac{\mu_0 e \alpha c}{4\pi r_1^2}.$$

A.5

A magnetic dipole in a field has energy

$$U = -\vec{\mathcal{M}}_s \cdot \vec{B}_1.$$

The parallel and anti-parallel states consequently have energies $-\mathcal{M}_s B_1$ and $+\mathcal{M}_s B_1$. Their positive energy separation is

$$\Delta E_F = 2\mathcal{M}_s B_1.$$

Using the magnetic moment specified in the problem,

$$\Delta E_F = 2 \frac{e\hbar}{m_e} \frac{\mu_0 e \alpha c}{4\pi r_1^2}.$$

With $\mu_0 \epsilon_0 c^2 = 1$ and $e^2/(4\pi\epsilon_0) = \alpha\hbar c$,

$$\Delta E_F = \frac{2\alpha^2 \hbar^2}{m_e r_1^2} = 2m_e \alpha^4 c^2.$$

Since $E_1 = -m_e \alpha^2 c^2/2$,

$$\Delta E_F = -4\alpha^2 E_1.$$

The hyperfine photon satisfies

$$\frac{2\pi\hbar c}{\lambda_{\text{HF}}} = \Delta E_{\text{HF}} \simeq 3.72 \frac{m_e}{m_p} \Delta E_F.$$

Thus

$$\lambda_{\text{HF}} = \frac{\pi m_p \hbar}{3.72 m_e^2 \alpha^4 c}.$$

The numerical intermediate values are

$$\Delta E_F \simeq 2.898 \times 10^{-3} \text{ eV}, \quad \lambda_{\text{HF}} \simeq 0.21116 \text{ m}.$$

To two significant digits,

$$\lambda_{\text{HF}} = 0.21 \text{ m} = 21 \text{ cm}.$$

B.1

The gravitational acceleration is

$$\vec{g} = -\frac{d\varphi}{dr} \hat{r}.$$

Equating this to the inward acceleration of a circular orbit,

$$-\frac{v_c^2}{r} \hat{r} = -\frac{d\varphi}{dr} \hat{r}.$$

Therefore,

$$v_c = \sqrt{r \frac{d\varphi}{dr}}.$$

B.2

Outside a spherical bulge, the shell theorem gives

$$\varphi(r) = -\frac{GM_b}{r} + \text{constant}.$$

It follows that

$$v_c^2 = \frac{GM_b}{r}, \quad M_b = \frac{rv_c^2}{G}.$$

A convenient approximate reading of the red curve is

$$r \simeq 10 \text{ kpc}, \quad v_c \simeq 20 \text{ km s}^{-1}.$$

Hence

$$M_b \simeq \frac{(10 \times 3.09 \times 10^{19})(2.0 \times 10^4)^2}{6.67430 \times 10^{-11}} = 1.85 \times 10^{39} \text{ kg}.$$

In solar units this is approximately $9.3 \times 10^8 M_\odot$. Given the graphical precision,

$$\boxed{M_b \sim 10^9 M_\odot}.$$

B.3

The mass enclosed within radius r is

$$M_m(r) = 4\pi \int_0^r \rho_m(x) x^2 dx = 4\pi C_m \int_0^r \frac{x^2}{r_m^2 + x^2} dx.$$

Applying the supplied integral,

$$\boxed{M_m(r) = 4\pi C_m \left[r - r_m \arctan\left(\frac{r}{r_m}\right) \right]}.$$

By spherical symmetry,

$$v_{c,m}^2(r) = \frac{GM_m(r)}{r} = 4\pi G C_m \left[1 - \frac{r_m}{r} \arctan\left(\frac{r}{r_m}\right) \right].$$

Comparison with the requested form gives

$$\boxed{k_1 = 4\pi G C_m, \quad k_2 = 4\pi G C_m r_m}.$$

For $r \ll r_m$, set $x = r/r_m$. Then

$$1 - \frac{\arctan x}{x} \simeq 1 - \frac{x - x^3/3}{x} = \frac{x^2}{3}.$$

Therefore,

$$\boxed{v_{c,m}(r) \simeq \sqrt{\frac{4\pi G C_m}{3}} \frac{r}{r_m}}.$$

The central rotation curve rises linearly, as expected for approximately uniform density.

For $r \gg r_m$, $\arctan(r/r_m) \rightarrow \pi/2$, while $r_m/r \rightarrow 0$. Thus

$$\boxed{v_{c,m}(r) \simeq \sqrt{4\pi G C_m} \equiv v_{c,\infty}},$$

and

$$M_m(r) \simeq 4\pi C_m r = \frac{v_{c,\infty}^2 r}{G}.$$

The pictured diameter is approximately 18 kpc, so take the enclosing radius as 9 kpc. The observed outer speed is approximately 170 km s^{-1} . Hence

$$M_{\text{in the figure}} \simeq \frac{(1.70 \times 10^5)^2 (9 \times 3.09 \times 10^{19})}{6.67430 \times 10^{-11}} \simeq 1.20 \times 10^{41} \text{ kg}.$$

Therefore,

$$M_{\text{in the figure}} \simeq 6.1 \times 10^{10} M_{\odot}.$$

C.1

For fixed r , the vertical equation of motion is

$$m\ddot{z} = -m \frac{\partial \varphi_G}{\partial z}.$$

Differentiating the prescribed potential gives

$$\ddot{z} = \frac{2\varphi_0}{z_0^2} \ln\left(\frac{r}{r_0}\right) z e^{-z^2/z_0^2}.$$

At $z = 0$, the force vanishes. For small oscillations, $|z| \ll z_0$, so

$$e^{-z^2/z_0^2} \simeq 1.$$

The linearized equation becomes

$$\ddot{z} + \frac{2\varphi_0}{z_0^2} \ln\left(\frac{r_0}{r}\right) z = 0.$$

For $r < r_0$, its coefficient is positive. Thus the plane is a stable equilibrium, with

$$\omega_0 = \sqrt{\frac{2\varphi_0}{z_0^2} \ln\left(\frac{r_0}{r}\right)}.$$

C.2

For $r \gg r_m$, B.3 gives

$$M_m(r) \simeq 4\pi C_m r.$$

The radial potential gradient is therefore

$$\frac{d\varphi}{dr} = \frac{GM_m(r)}{r^2} \simeq \frac{4\pi G C_m}{r}.$$

Integrating and choosing the additive constant appropriately,

$$\varphi(r) \simeq 4\pi G C_m \ln\left(\frac{r}{r_0}\right).$$

Thus the required regime and coefficient are

$$r \gg r_m, \quad \varphi_0 = 4\pi G C_m.$$

In the plane,

$$v_c^2 = r \frac{\partial \varphi_G(r, 0)}{\partial r} = r \frac{\varphi_0}{r} = \varphi_0,$$

so

$$\boxed{v_c(r) = \sqrt{\varphi_0}}.$$

C.3

Let T be the foot of the perpendicular from C to the line of sight. The geometry gives

$$CT = R_\odot \sin \ell.$$

The Sun's tangential velocity projects onto the sightline as

$$\vec{v}_\odot \cdot \hat{u}_v = v_\odot \sin \ell.$$

For the emitter, its tangential velocity is perpendicular to CE . Its projection onto the sightline is

$$\vec{v}_E \cdot \hat{u}_v = v_E \frac{CT}{R} = v_E \frac{R_\odot}{R} \sin \ell.$$

The flat rotation curve implies $v_E = v_\odot$. Consequently,

$$v_{rE/S} = (\vec{v}_E - \vec{v}_\odot) \cdot \hat{u}_v = \boxed{v_\odot \left(\frac{R_\odot}{R} - 1 \right) \sin \ell}.$$

Solving for R ,

$$\boxed{R = \frac{R_\odot v_\odot \sin \ell}{v_\odot \sin \ell + v_{rE/S}}}.$$

C.4

Following the intended convention, which is consistent with the answer sheet, use

$$v_r = +c \frac{f - f_0}{f_0},$$

with positive values denoting recession. The printed $f - f_0$ axis therefore has the opposite sign from the usual frequency shift for positive recession.

The three peaks occur approximately at

$$\Delta f_1 = 0.030 \text{ MHz}, \quad \Delta f_2 = 0.150 \text{ MHz}, \quad \Delta f_3 = 0.260 \text{ MHz}.$$

Since

$$v_\odot \sin \ell = 220 \sin 30^\circ = 110 \text{ km s}^{-1},$$

the radius formula becomes

$$\frac{R}{R_\odot} = \frac{110}{110 + v_r},$$

where v_r is expressed in km s^{-1} .

Source, in increasing peak frequency	Δf (MHz)	v_r (km s^{-1}), 3 significant digits	$R/R_\odot$, 2 significant digits
1	0.030	+6.33	0.95

Source, in increasing peak frequency	Δf (MHz)	v_r (km s ⁻¹), 3 significant digits	$R/R_\odot$, 2 significant digits
2	0.150	+31.7	0.78
3	0.260	+54.9	0.67

C.5

On the answer-sheet diagram, draw the observation ray from S , making 30° with SC on the side indicated by Fig. 2. Draw, or identify, the circles centred at C with radii

$$0.95R_\odot, \quad 0.78R_\odot, \quad 0.67R_\odot.$$

Each circle intersects the observation ray twice.

To specify the positions quantitatively, let $d = SE$. The cosine rule gives

$$R^2 = R_\odot^2 + d^2 - 2R_\odot d \cos \ell.$$

Thus

$$d_\pm = R_\odot \cos \ell \pm \sqrt{R^2 - R_\odot^2 \sin^2 \ell}.$$

At $\ell = 30^\circ$, the sightline's closest approach to the centre is

$$CT = 0.50R_\odot, \quad ST = R_\odot \cos 30^\circ = 0.866R_\odot.$$

All three inferred radii exceed CT and are smaller than $R_\odot$, so both intersections lie forward of the Sun.

Using the unrounded radii from C.4:

Source	Near position $d_-/R_\odot$	Far position $d_+/R_\odot$
1	0.0635	1.67
2	0.272	1.46
3	0.424	1.31

Moving away from S along the ray, the candidate points occur in the order

$$1_{\text{near}}, \quad 2_{\text{near}}, \quad 3_{\text{near}}, \quad T, \quad 3_{\text{far}}, \quad 2_{\text{far}}, \quad 1_{\text{far}}.$$

The spectrum determines the galactocentric radius of each source but cannot distinguish its near and far positions. Repeating the observations at different longitudes maps hydrogen-rich structures and reveals the galaxy's spiral arms; additional information is needed to resolve individual near–far ambiguities.

D.1

In the outer regime of B.3,

$$M_{\text{tot}} \simeq 4\pi C_m R, \quad v_{c,\infty}^2 = 4\pi G C_m.$$

Eliminating C_m ,

$$M_{\text{tot}} \simeq \frac{R}{G} v_{c,\infty}^2.$$

For a mass-independent radius,

$$\eta = \frac{R}{G}, \quad \gamma = 2.$$

For an empirical power law, taking logarithms gives

$$\log_{10} \left(\frac{M_{\text{tot}}}{M_{\odot}} \right) = \gamma_{TF} \log_{10} \left(\frac{v_{c,\infty}}{1 \text{ km s}^{-1}} \right) + \text{constant}.$$

Thus γ_{TF} is the slope in Fig. 4.

Approximate readings of the green line are

$$(x, y) \simeq (1.6, 8.2), \quad (2.8, 12.6).$$

They give

$$\gamma_{TF} \simeq \frac{12.6 - 8.2}{2.8 - 1.6} \simeq 3.7.$$

The observed scaling is close to a fourth-power law and substantially steeper than the second-power prediction of the fixed-radius Newtonian model.

D.2

The magnitude of the acceleration in a circular orbit is

$$a_m = \frac{v_c^2}{r}.$$

Using an outer observed point,

$$v_c \simeq 170 \text{ km s}^{-1}, \quad r \simeq 10 \text{ kpc},$$

gives

$$a_m \simeq \frac{(1.70 \times 10^5)^2}{10 \times 3.09 \times 10^{19}} = \boxed{9.4 \times 10^{-11} \text{ m s}^{-2}}.$$

D.3

In the deep-MOND limit $a \ll a_0$,

$$\mu \left(\frac{a}{a_0} \right) = \frac{a/a_0}{1 + a/a_0} \simeq \frac{a}{a_0}.$$

The modified equation of motion in the Newtonian gravitational field of M becomes

$$\frac{GMm}{r^2} \simeq m \frac{a^2}{a_0}.$$

For a circular orbit, $a = v_{c,\infty}^2/r$, so

$$\frac{GM}{r^2} = \frac{v_{c,\infty}^4}{a_0 r^2}.$$

The radius cancels:

$$\boxed{v_{c,\infty}^4 = GMa_0}.$$

Consequently,

$$M = \frac{1}{Ga_0} v_{c,\infty}^4, \quad \boxed{\gamma_{\text{MOND}} = 4}.$$

This also explains why the asymptotic circular speed is independent of radius.

For a graphical estimate of a_0 , Fig. 4 gives approximately

$$v_{c,\infty} \simeq 100 \text{ km s}^{-1}, \quad M \simeq 4 \times 10^9 M_\odot.$$

Therefore,

$$a_0 \simeq \frac{(1.0 \times 10^5)^4}{(6.67430 \times 10^{-11})(4 \times 10^9)(1.99 \times 10^{30})} \simeq 1.9 \times 10^{-10} \text{ m s}^{-2}.$$

At the precision of the plot,

$$\boxed{a_0 \sim 2 \times 10^{-10} \text{ m s}^{-2}}.$$

This is of the stated order $10^{-10} \text{ m s}^{-2}$. Comparing with D.2 gives $a_m/a_0 \sim 0.5$: the displayed outer-galaxy data probe the MOND acceleration scale, but do not establish $a_m \ll a_0$.

For an asymptotically flat speed of 170 km s^{-1} , the strict deep-MOND condition is

$$r \gg \frac{v_{c,\infty}^2}{a_0} \sim 5 \text{ kpc}.$$

It becomes increasingly accurate farther out. Near the transition, the full interpolation function must be used, as in D.4.

D.4

The homogeneous sphere has density

$$\rho = \frac{3M}{4\pi R_b^3}.$$

Its enclosed mass is

$$M(< r) = \begin{cases} M(r/R_b)^3, & r \leq R_b, \\ M, & r \geq R_b. \end{cases}$$

The gravitational force remains Newtonian in the modified-second-law formulation specified in the problem. Define its magnitude per unit test mass as

$$g_N(r) = \frac{GM(< r)}{r^2} = \begin{cases} \frac{GMr}{R_b^3}, & r \leq R_b, \\ \frac{GM}{r^2}, & r \geq R_b. \end{cases}$$

Using the full interpolation function,

$$g_N = \mu\left(\frac{a}{a_0}\right)a = \frac{a^2}{a + a_0}.$$

Hence

$$a^2 - g_N a - g_N a_0 = 0.$$

The physical, non-negative root is

$$a = \frac{g_N + \sqrt{g_N^2 + 4a_0g_N}}{2}.$$

Since $v_c^2 = ra$,

$$v_c(r) = \left[\frac{r}{2} \left(g_N + \sqrt{g_N^2 + 4a_0g_N} \right) \right]^{1/2}.$$

Substitution gives the expression valid throughout both regions:

$$v_c(r) = \begin{cases} \left[\frac{GM r^2}{2R_b^3} \left(1 + \sqrt{1 + \frac{4a_0 R_b^3}{GM r}} \right) \right]^{1/2}, & 0 < r \leq R_b, \\ \left[\frac{GM}{2r} \left(1 + \sqrt{1 + \frac{4a_0 r^2}{GM}} \right) \right]^{1/2}, & r \geq R_b. \end{cases}$$

The two expressions agree at $r = R_b$, and the limit at the centre is $v_c(0) = 0$.

The limiting cases clarify the shape of the rotation curve:

Region	Acceleration regime	Circular speed
Inside, $r < R_b$	$g_N \gg a_0$	$v_c \simeq \sqrt{\frac{GM}{R_b^3}} r$
Inside, $r < R_b$	$g_N \ll a_0$	$v_c \simeq \left(\frac{GM a_0}{R_b^3} \right)^{1/4} r^{3/4}$
Outside, $r > R_b$	$g_N \gg a_0$	$v_c \simeq \sqrt{\frac{GM}{r}}$
Outside, $r > R_b$	$g_N \ll a_0$	$v_c \simeq (GM a_0)^{1/4}$

Thus the curve starts at zero, behaves as $r^{3/4}$ sufficiently close to the centre, and approaches a constant far outside. Newtonian intermediate regions occur only if the sphere's surface acceleration GM/R_b^2 is sufficiently larger than a_0 .

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Cox's Timepiece

In 1765, British clockmaker James Cox invented a clock whose only source of energy is the fluctuations in atmospheric pressure. Cox's clock used two vessels containing mercury. Changes in atmospheric pressure caused mercury to move between the vessels, and the two vessels to move relative to each other. This movement acted as an energy source for the actual clock.

We propose an analysis of this device. Throughout, we assume that

- the Earth's gravitational field $\vec{g} = -g\vec{u}_z$ is uniform with $g = 9.8 \text{ m} \cdot \text{s}^{-2}$ and $\vec{u}_z$ a unit vector;
- all liquids are incompressible and their density is denoted ρ ;
- no surface tension effects will be considered;
- the variations of atmospheric pressure with altitude are neglected;
- the surrounding temperature T_a is uniform and all transformations are isothermal.



Fig. 1. Artistic view of Cox's clock ¹

Part A - Pulling on a submerged tube

We first consider a bath of water that occupies the semi-infinite space $z \leq 0$. The air above it is at a pressure $P_a = P_0$. A cylindrical vertical tube of length $H = 1 \text{ m}$, cross-sectional area $S = 10 \text{ cm}^2$ and mass $m = 0.5 \text{ kg}$ is dipped into the bath. The bottom end of the tube is open, and the top end of the tube is closed. We denote h the altitude of the top of the tube and z_ℓ that of the water inside the tube. The thickness of the tube walls is neglected.

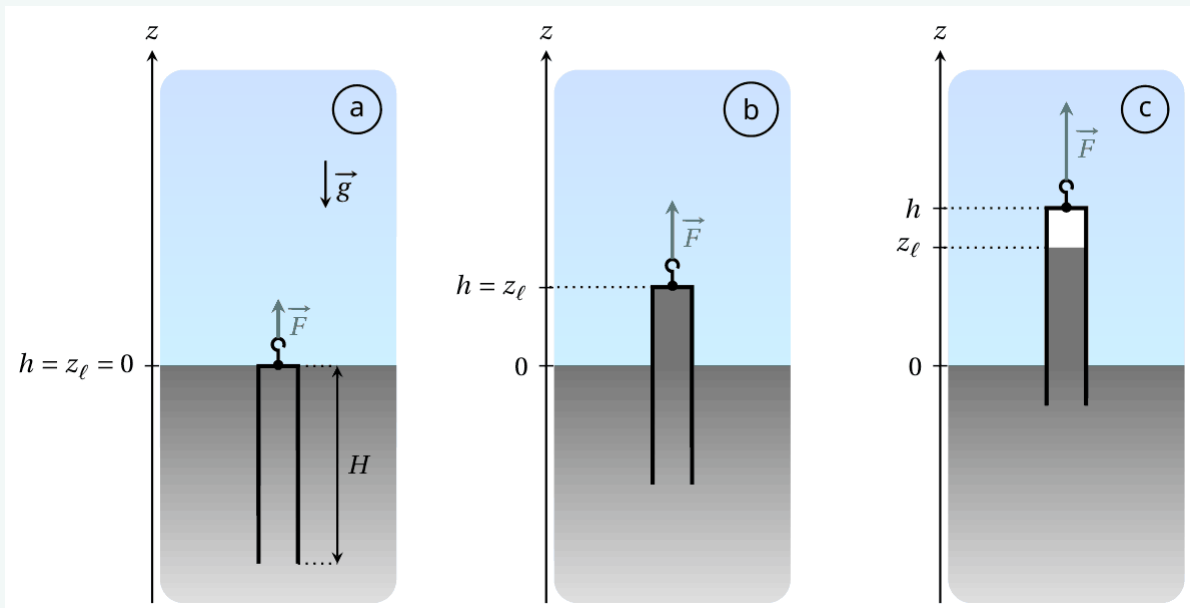


Fig. 2. Sketch of the tube in different configurations

We start from the situation where the tube in Fig. 2 contains no gas and its top is at the bath level: in other words, $h = 0$ and $z_\ell = 0$ (case a). The tube is then slowly lifted until its bottom end reaches the bath level. The pulling force exerted on the tube is denoted $\vec{F} = F\vec{u}_z$.

A.1 — 0.2pt

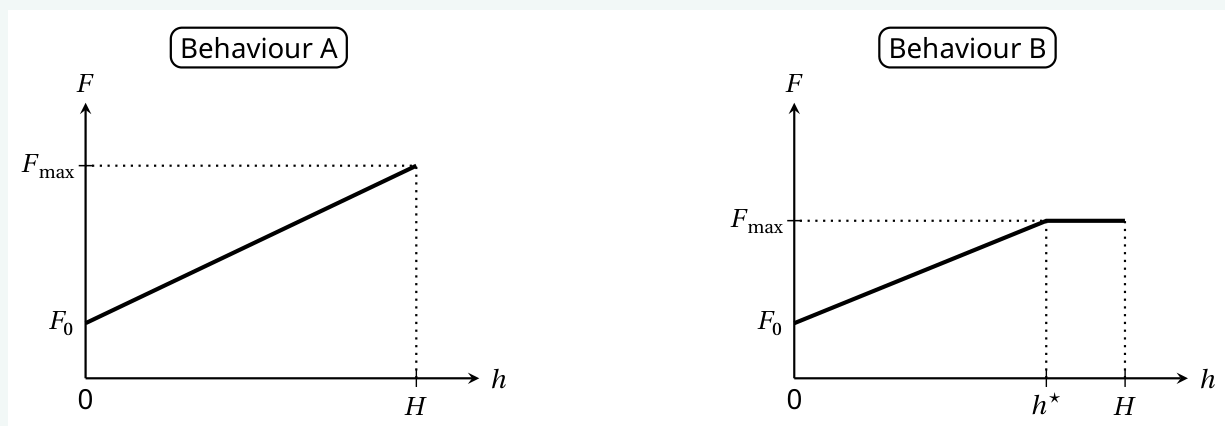
For the configuration shown in Fig. 2 (case b), express the pressure P_w in the water at the top of the tube. Also express the force $\vec{F}$ necessary to maintain the tube at this position. Expressions must be written in terms of P_0, ρ, m, S, h, g and $\vec{u}_z$.

Three experiments are performed. In each, the tube is lifted from the initial state shown in Fig. 2(a) under the conditions specified in Table 1.

Experiment	Liquid	T_a (°C)	ρ (kg · m ⁻³)	P_{sat} (Pa)
1	Water	20	1.00×10^3	2.34×10^3
2	Water	80	0.97×10^3	47.4×10^3
3	Water	99	0.96×10^3	99.8×10^3

Table 1. Experimental conditions and numerical values of physical quantities for each experiment (P_{sat} designates the saturated vapour pressure of the pure fluid)

In each case, we study the evolution of the force F that must be applied in order to maintain the tube in equilibrium at an altitude h , the external pressure being fixed at $P_a = P_0 = 1.000 \times 10^5$ Pa. Two different behaviours are possible



A.2 — 0.8pt

For each experiment, complete the table in the answer sheet to indicate the expected behaviour and the numerical values for F_{max} and for h^* (when pertinent), where F_{max} and h^* are defined in the figures illustrating the two behaviours.

Experiment	Behaviour (A or B ?)	h^* (cm)	F_{max} (N)
1			
2			
3			

When we replace the water with liquid mercury (whose properties are given below), behaviour B is observed.

Liquid	T_a (°C)	ρ (kg · m ⁻³)	P_{sat} (Pa)
Mercury	20	13.5×10^3	0.163

A.3 — 0.3pt

Express the relative error, denoted ε , committed when we evaluate the maximal force F_{max} neglecting P_{sat} compared to P_0 . Give the numerical value of ε .

Part B - Two-part barometric tube

From now on, we work with mercury (density $\rho = 13.5 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$) at the ambient temperature $T_a = 20^\circ\text{C}$ and we take $P_{\text{sat}} = 0$.

Let us consider a tube with a reservoir on top, modeled as two superposed cylinders of different dimensions, as shown in Fig. 3.

- the bottom part (still called the tube) has cross-sectional area S_t and height $H_t = 80 \text{ cm}$;
- the top part (called the bulb) has cross-sectional area $S_b > S_t$ and height $H_b = 20 \text{ cm}$.

This two-part tube is dipped into a semi-infinite liquid bath.

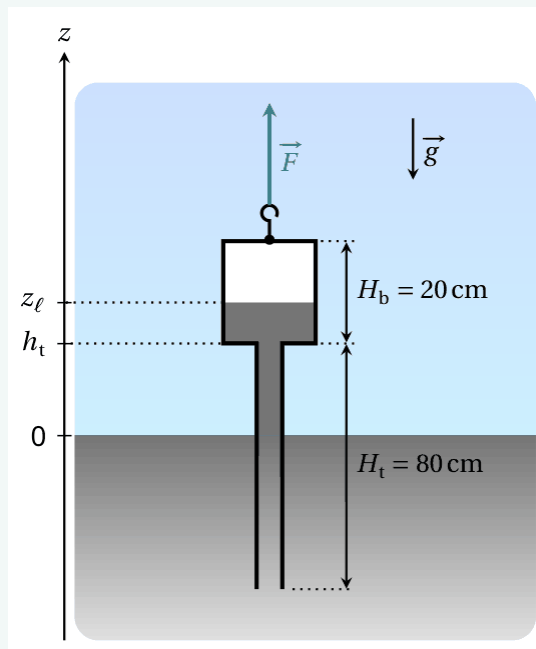


Fig. 3. Sketch of the two-part barometric tube

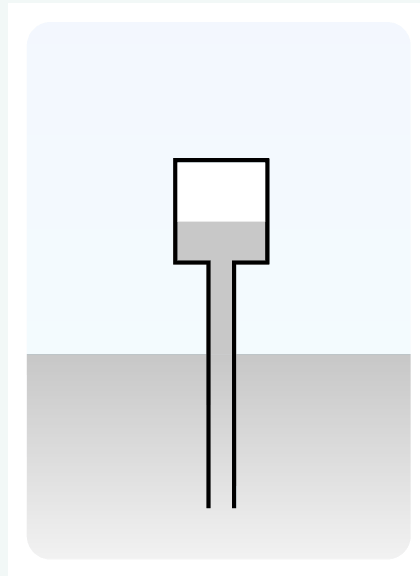
As in Part A, the system is prepared such that the tube contains no air. We identify the vertical position of the tube by the altitude h_t of the junction between the tube and the bulb. The height of the column of mercury is again denoted z_l . The force $\vec{F}$ that must be exerted to maintain the tube in equilibrium in the configuration shown in Fig. 3 can now be written as

$$\vec{F} = (m_{\text{tb}} + m_{\text{add}})g\vec{u}_z \quad (1)$$

where m_{tb} is the total mass of the two-part tube (when empty of mercury).

B.1 — 0.3pt

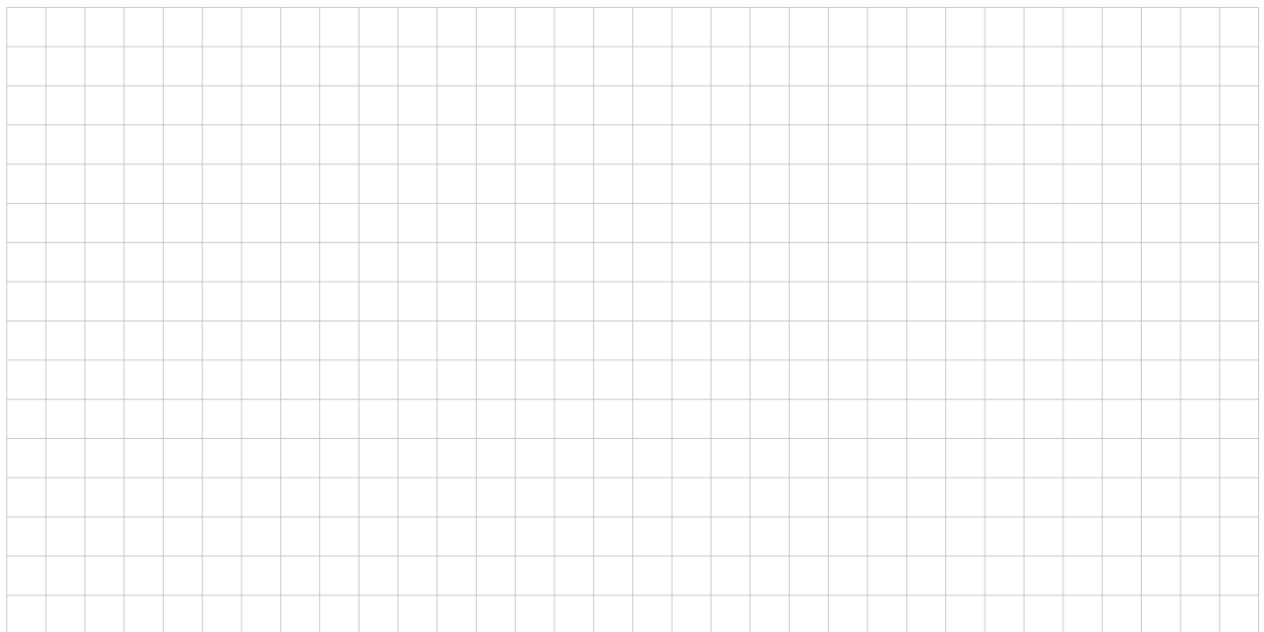
On the answer sheet, color the area corresponding to the volume of liquid mercury that is responsible for the term m_{add} appearing in equation (1).



The mass m_{add} depends both on the height h_t and the atmospheric pressure P_a . For the next question, assume that the atmospheric pressure is fixed at $P_a = P_0 = 1.000 \times 10^5 \text{ Pa}$. Starting from the situation where the system is completely submerged, the tube is slowly lifted until its base is flush with the liquid bath.

B.2 — 1.4pt

Sketch the evolution of the mass m_{add} as a function of h_t for $h_t \in [-H_b, H_t]$. On the graph, provide the expression for the slopes of the different segments, as well as the h_t analytical value of any angular points, in terms of $P_0, \rho, g, S_b, S_t, H_b$ and H_t .



As the system is lifted while $P_a = P_0 = 10^5 \text{ Pa}$, we stop when the free surface of the liquid is in the middle of the bulb. The value of h_t is fixed and then we observe variations in the mass m_{add} due to variations in the atmospheric pressure described by

$$P_a(t) = P_0 + P_1(t) \quad (2)$$

where P_0 designates the average value and P_1 is a perturbative term. We model P_1 by a periodic triangular function of amplitude $A = 5 \times 10^2 \text{ Pa}$ and period τ_1 of 1 week.

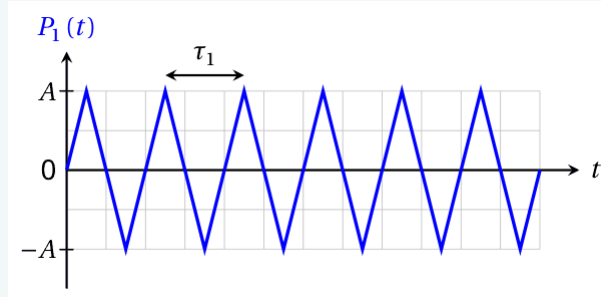


Fig. 4. Simplified model of the perturbative term $P_1(t)$

B.3 — 0.3pt

Given that $S_t = 5 \text{ cm}^2$ and $S_b = 200 \text{ cm}^2$, express the amplitude Δm_{add} of the variations of the mass m_{add} over time, then give its numerical value. Assume that the liquid surface always stays in the bulb.

Part C - Cox's timepiece

The real mechanism developed by Cox is complex (Fig. 5). We study a simplified version, depicted in Fig. 6, and described below

- a cylindrical bottom cistern containing a mercury bath;
- a two-part barometric tube identical to that studied in part B, which is still completely emptied of any air, is dipped into the bath;
- the cistern and the two-part tube are each suspended by a cable. Both cables (assumed to be inextensible and of negligible mass) pass through a system of ideal pulleys and finish attached to either side of the same mass M , which can slide on a horizontal surface;
- the total volume of liquid mercury contained in the system is $V_\ell = 5 \text{ L}$.

The height, cross-section and masses of each part are given in Table 2. The position of mass M is referenced by the coordinate x of its center of mass. We consider solid friction between the horizontal support and the mass M , without distinction between static and dynamic coefficients; the magnitude of this force when sliding occurs is denoted F_s .

Two stops limit the displacement of the mass M such that $-X \leq x \leq X$ (with $X > 0$). Assume that the value of X guarantees that

- the bottom of the two-part tube never touches the bottom of the cistern nor comes out of the liquid bath;
- the altitude z_ℓ of the mercury column is always in the upper bulb.



Fig. 5. Real Cox's timepiece ² (without mercury)

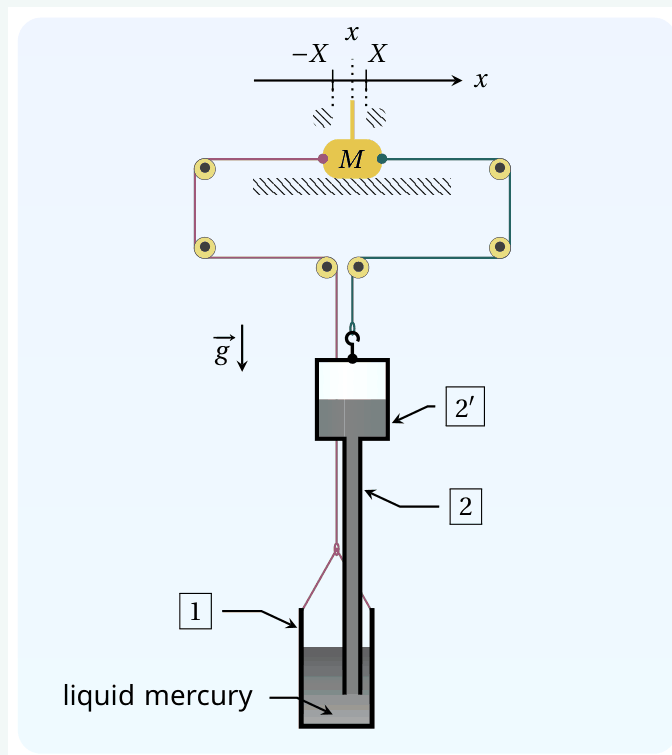


Fig. 6. Sketch of the system modeling the timepiece

Reference	Name	Height	Cross section area	Empty mass
[1]	cistern	$H_c = 30 \text{ cm}$	$S_c = 210 \text{ cm}^2$	m_c

Reference	Name	Height	Cross section area	Empty mass
2	tubular part of the barometric tube	$H_t = 80 \text{ cm}$	$S_t = 5 \text{ cm}^2$	total mass of the barometric tube: m_{tb}
2'	bulb of the barometric tube	$H_b = 20 \text{ cm}$	$S_b = 200 \text{ cm}^2$	

Table 2. Dimensions and notations for the model system

The system evolves in contact with the atmosphere, whose pressure fluctuates as in Fig. 4 (still with amplitude $A = 5 \times 10^2 \text{ Pa}$ and period $\tau_1 = 1 \text{ week}$). At the start $t = 0$, the mass M is at rest at $x = 0$ and the tensions exerted by the two cables on either side of the mass M are in balance while $P_1(0) = 0$. We define

$$\xi = \frac{S_b + S_c - S_t}{S_b S_c} \frac{F_s}{A} \simeq \frac{S_b + S_c}{S_b S_c} \frac{F_s}{A} \quad (3)$$

where the last expression uses that $S_t \ll S_b, S_c$ (which we will assume is valid until the end of the problem).

C.1 – 1pt

Determine the threshold ξ^* such that M remains indefinitely at rest when $\xi > \xi^*$.

For the next question only, suppose that the mass M is temporarily blocked at $x = X$.

C.2 – 1pt

Give an expression for the total tension force $\vec{T} = T\vec{u}_x$ acting on the mass M due to the tension in two cables at this position, when $P_1 = 0$, in terms of ρ, g, X and pertinent cross-sections.

When $\xi < \xi^*$, starting again from $x = 0$ and $P_1 = 0$, two different behaviours can be observed for $t \geq 0$. To distinguish them, we need to introduce another parameter

$$\lambda = \frac{2(S_b - S_t)}{S_b} \frac{\rho g X}{A} \simeq \frac{2\rho g X}{A} \quad (4)$$

C.3 – 2pt

Complete the table in the answer sheet to indicate the condition under which each regime is obtained. Conditions must be expressed as inequalities on ξ and/or λ . In addition, sketch the variations of $x(t)/X$ for $t \in [0, 3\tau_1]$ that are consistent with the variations of $P_1(t)/A$ already present. *Specification of remarkable points coordinates is not required.*

	Condition for observation	Graph of $x(t)/X$
Regime 1		
Regime 2		

In the real Cox's timepiece, energy provided by the mechanism is stored using a system of ratchets and used to raise a counterweight, like in a traditional clock. In the simplified model studied here, the energy recovered by the clock corresponds to the energy dissipated by the friction force exerted by the horizontal surface on the mass M . From now on, we assume that the system is dimensioned such that to work in the regime that allows the clock to recuperate energy. We also assume that the permanent regime is established. We denote W the energy dissipated by the solid friction force during a period τ_1 , which can be expressed only in terms of F_s and X .

All else equal, F_s and X can be adjusted to maximize the energy W ; we denote F_s^* and X^* their respective values in the optimal situation.

C.4 — 1pt

Considering $S_b \simeq S_c$ and $S_t \ll S_b$, determine the expressions for F_s^* and X^* as functions of ρ, g, S_c and A . Express the corresponding maximum energy W^* , then calculate its numerical value with $A = 5 \times 10^2 \text{ Pa}$.

We denote W_{pr}^* the work of atmospheric pressure forces received by the system in the optimal situation during a period τ_1 .

C.5 — 1.7pt

Express W_{pr}^* , then calculate the ratio W^*/W_{pr}^* . It could be useful to represent the evolution of the system in a (P, V) diagram, where V is the system's volume.

Credits:

[1]: Bruno Vacaro;

[2]: Victoria and Albert Museum, London.

Official English text of the problem.

Answers

Define

$$L = \frac{P_0}{\rho g}, \quad D = S_b + S_c - S_t, \quad q = \lambda + \frac{\xi}{2}.$$

• A.1

$$P_w = P_0 - \rho g h, \quad \vec{F} = (mg + \rho g S h) \vec{u}_z.$$

• A.2

Experiment	Behaviour	h^* (cm), when pertinent	$F_{\max}$ (N)
1	A	—	14.7
2	A	—	14.4
3	B	2.13	5.10

• A.3

$$\varepsilon = \frac{S P_{\text{sat}}}{mg + S(P_0 - P_{\text{sat}})} = 1.55 \times 10^{-6}.$$

Neglecting mercury vapour pressure overestimates the force.

• B.1 Shade all mercury **inside the tube and bulb above the external bath level**. In the depicted configuration,

$$m_{\text{add}} = \rho [S_t h_t + S_b (z_\ell - h_t)].$$

• B.2

$$m_{\text{add}}(h_t) = \begin{cases} \rho S_b (h_t + H_b), & -H_b \leq h_t \leq 0, \\ \rho (S_b H_b + S_t h_t), & 0 \leq h_t \leq L - H_b, \\ \rho [S_b L - (S_b - S_t) h_t], & L - H_b \leq h_t \leq L, \\ \rho S_t L, & L \leq h_t \leq H_t. \end{cases}$$

The successive slopes are

$$\rho S_b, \quad \rho S_t, \quad -\rho (S_b - S_t), \quad 0.$$

The angular points occur at

$$h_t = 0, \quad L - H_b \simeq 0.556 \text{ m}, \quad L \simeq 0.756 \text{ m}.$$

• B.3

$$\Delta m_{\text{add}} = \frac{S_b A}{g} = 1.02 \text{ kg}.$$

• C.1

$$\xi^* = 2.$$

• C.2

$$\vec{T} = \frac{4\rho g S_c (S_b - S_t)}{S_b + S_c - S_t} X \vec{u}_x \simeq \frac{4\rho g S_b S_c}{S_b + S_c} X \vec{u}_x.$$

• C.3, for $\xi < 2$:

Regime	Condition	Description of $x(t)/X$
1: remains at one stop	$q > 1$, equivalently $\xi + 2\lambda > 2$	Initially 0, then rises rapidly to +1 and remains there.
2: repeated strokes	$q < 1$, equivalently $\xi + 2\lambda < 2$	Initially 0, then switches between plateaus at +1 and -1.

The first positive stroke begins when $P_1/A = \xi/2$. Subsequent negative strokes begin at $P_1/A = -q$ on falling pressure, and positive strokes at $P_1/A = q$ on rising pressure. The boundary $q = 1$ is marginal; ideal static friction permits the mass to remain at the stop.

• C.4 The ideal limiting optimum is

$$F_s^* = \frac{S_c A}{2} \simeq 5.25 \text{ N}, \quad X^* = \frac{A}{4\rho g} \simeq 0.945 \text{ mm},$$

$$W^* = \frac{S_c A^2}{2\rho g} \simeq 1.98 \times 10^{-2} \text{ J}.$$

• C.5

$$W_{\text{pr}}^* = \frac{S_c A^2}{\rho g} \simeq 3.97 \times 10^{-2} \text{ J}, \quad \frac{W^*}{W_{\text{pr}}^*} = \frac{1}{2}.$$

Solution

A.1

Hydrostatic equilibrium gives

$$\frac{dP}{dz} = -\rho g.$$

The water is connected to the bath, where $P = P_0$ at $z = 0$. Consequently, while the water remains in contact with the closed top,

$$P_w = P_0 - \rho g h.$$

The pressure on the outside of the top produces a downward force $P_0 S$, while the water underneath produces an upward force $P_w S$. The vertical walls contribute no vertical pressure force, and their thickness is neglected. Equilibrium therefore requires

$$F + P_w S - P_0 S - mg = 0.$$

Thus

$$\vec{F} = (mg + \rho g S h) \vec{u}_z.$$

A.2

A vapour cavity appears when the pressure at the top reaches the saturated vapour pressure:

$$P_0 - \rho gh^* = P_{\text{sat}}.$$

Hence

$$h^* = \frac{P_0 - P_{\text{sat}}}{\rho g}.$$

Before this happens, $F = mg + \rho gSh$. After a vapour cavity has formed, its pressure remains P_{sat} at the prescribed temperature, so

$$F = mg + S(P_0 - P_{\text{sat}}).$$

These expressions combine into

$$F(h) = mg + S \min\{\rho gh, P_0 - P_{\text{sat}}\}, \quad 0 \leq h \leq H.$$

Therefore:

- if $h^* \geq H$, behaviour A occurs and $F_{\text{max}} = mg + \rho gSH$;
- if $h^* < H$, behaviour B occurs and $F_{\text{max}} = mg + S(P_0 - P_{\text{sat}})$.

Using $S = 10^{-3} \text{ m}^2$, $H = 1 \text{ m}$, and $mg = 4.90 \text{ N}$:

Experiment	Calculated saturation height	Comparison with H	Result
1	$\frac{100000 - 2340}{1000(9.8)} = 9.97 \text{ m}$	Above H	A; $F_{\text{max}} = 14.7 \text{ N}$
2	$\frac{100000 - 47400}{970(9.8)} = 5.53 \text{ m}$	Above H	A; $F_{\text{max}} = 14.406 \text{ N} \simeq 14.4 \text{ N}$
3	$\frac{100000 - 99800}{960(9.8)} = 0.02126 \text{ m}$	Below H	B; $h^* = 2.13 \text{ cm}$, $F_{\text{max}} = 5.10 \text{ N}$

Both graphs begin at $F_0 = mg = 4.90 \text{ N}$.

A.3

For mercury, the actual maximum force and its approximation are

$$F_{\text{max}} = mg + S(P_0 - P_{\text{sat}}), \quad F_{\text{max}}^{(0)} = mg + SP_0.$$

The approximation overestimates the force by SP_{sat} . Relative to the actual force,

$$\varepsilon = \frac{F_{\text{max}}^{(0)} - F_{\text{max}}}{F_{\text{max}}} = \frac{SP_{\text{sat}}}{mg + S(P_0 - P_{\text{sat}})}.$$

Numerically,

$$\varepsilon = \frac{10^{-3}(0.163)}{0.5(9.8) + 10^{-3}(100000 - 0.163)} = 1.55 \times 10^{-6}.$$

This is approximately 0.000155%.

B.1

The additional load is the weight of the mercury raised above the external bath surface. Mercury below that level is supported hydrostatically by the bath.

Thus, on the answer-sheet diagram, shade:

- the mercury in the narrow tube from the external bath level up to the tube–bulb junction;
- all the mercury in the bulb, from its bottom up to its internal free surface.

Do not shade the mercury below the external bath level. For the configuration shown,

$$m_{\text{add}} = \rho [S_t h_t + S_b (z_\ell - h_t)].$$

Since the pressure above the internal surface is zero,

$$z_\ell = \frac{P_a}{\rho g}.$$

An equivalent useful expression is therefore

$$m_{\text{add}} = \frac{S_b P_a}{g} - \rho (S_b - S_t) h_t$$

when the free surface lies in the bulb and the junction is above the external bath.

B.2

Introduce the barometric height

$$L = \frac{P_0}{\rho g} = \frac{10^5}{(13.5 \times 10^3)(9.8)} = 0.75586 \text{ m}.$$

For the given dimensions,

$$0 < L - H_b < L < H_t.$$

Until the top reaches L , the tube remains completely filled. Subsequently the internal free surface stays at $z_\ell = L$, while the tube continues to rise.

There are four geometrically distinct intervals.

$$1. -H_b \leq h_t \leq 0.$$

Only a height $h_t + H_b$ of the bulb is above the bath:

$$m_{\text{add}} = \rho S_b (h_t + H_b).$$

The graph starts at $(-H_b, 0)$, with slope ρS_b .

$$2. 0 \leq h_t \leq L - H_b.$$

The entire bulb and a length h_t of narrow tube are above the bath, all filled with mercury:

$$m_{\text{add}} = \rho (S_b H_b + S_t h_t).$$

The graph continues upward with the smaller slope ρS_t .

$$3. L - H_b \leq h_t \leq L.$$

A vacuum forms above the mercury, whose surface remains at L inside the bulb:

$$m_{\text{add}} = \rho [S_t h_t + S_b (L - h_t)].$$

The slope is $-\rho (S_b - S_t)$, so the graph decreases.

$$4. L \leq h_t \leq H_t.$$

The bulb is empty and the internal free surface lies in the narrow tube:

$$m_{\text{add}} = \rho S_t L = \frac{S_t P_0}{g}.$$

The graph is horizontal.

The graph is continuous, with the following vertices and endpoint:

h_t	m_{add}
$-H_b$	0
0	$\rho S_b H_b$
$L - H_b$	$\rho[S_b H_b + S_t(L - H_b)]$, the maximum
L	$\rho S_t L$
H_t	$\rho S_t L$

Its three angular points have horizontal coordinates

$$0, \quad \frac{P_0}{\rho g} - H_b, \quad \frac{P_0}{\rho g},$$

numerically 0, 55.6 cm, and 75.6 cm.

B.3

At the prescribed fixed position, the free surface remains inside the bulb. Thus

$$m_{\text{add}}(t) = \frac{S_b[P_0 + P_1(t)]}{g} - \rho(S_b - S_t)h_t.$$

Only the first term varies, so the amplitude is

$$\Delta m_{\text{add}} = \frac{S_b A}{g}.$$

With $S_b = 200 \text{ cm}^2 = 0.0200 \text{ m}^2$,

$$\Delta m_{\text{add}} = \frac{0.0200(500)}{9.8} = 1.02 \text{ kg}.$$

This is the amplitude about the mean; the peak-to-peak variation is 2.04 kg. The corresponding surface-height amplitude is

$$\frac{A}{\rho g} = 3.78 \text{ mm},$$

well below the 10 cm distance from the middle of the bulb to either end.

C.1

We first obtain the cable-force difference for arbitrary x and P_1 .

Let z_s denote the external mercury surface and z_c the cistern bottom. From the cable arrangement, displacement of the mass to the right by x :

- raises the cistern by x ;
- lowers the barometric tube by x .

Therefore

$$z_c = z_{c0} + x, \quad h_t = h_{t0} - x.$$

Hydrostatic equilibrium gives

$$z_\ell - z_s = \frac{P_a}{\rho g}.$$

The mercury above the external surface has mass

$$m_{\text{add}} = \rho [S_t(h_t - z_s) + S_b(z_\ell - h_t)] = \frac{S_b P_a}{g} - \rho(S_b - S_t)(h_t - z_s).$$

Because the tube walls have negligible volume, mercury conservation can be written as

$$V_\ell = S_c(z_s - z_c) + \frac{m_{\text{add}}}{\rho}.$$

Writing $\delta m = m_{\text{add}} - m_{\text{add},0}$, this implies

$$\delta z_s = x - \frac{\delta m}{\rho S_c}.$$

Subtracting the initial expression for m_{add} from its current value gives

$$\delta m = \frac{S_b P_1}{g} + \rho(S_b - S_t)(x + \delta z_s).$$

Consequently,

$$\boxed{\delta m = \frac{S_b S_c}{(S_b + S_c - S_t)g} P_1 + \frac{2\rho S_c(S_b - S_t)}{S_b + S_c - S_t} x}.$$

In a stationary configuration, the tube-side cable tension is

$$T_{\text{tb}} = (m_{\text{tb}} + m_{\text{add}})g,$$

whereas the cistern-side tension is

$$T_c = (m_c + \rho V_\ell - m_{\text{add}})g.$$

Their initial difference is zero. Hence the signed force on M , positive to the right, is

$$T = T_{\text{tb}} - T_c = 2g \delta m.$$

Define

$$K = \frac{S_b S_c}{S_b + S_c - S_t}, \quad \alpha = \frac{2\rho g(S_b - S_t)}{S_b}.$$

The resulting force law is

$$\boxed{T = 2K(P_1 + \alpha x)}.$$

The given parameters satisfy

$$F_s = KA\xi, \quad \alpha X = A\lambda.$$

At $x = 0$, the maximum force magnitude over the atmospheric cycle is $2KA$. Static friction can balance it throughout the cycle if

$$F_s \geq 2KA \quad \Longleftrightarrow \quad \xi \geq 2.$$

Thus the requested threshold is

$$\xi^* = 2.$$

C.2

Setting $P_1 = 0$ and $x = X$ in the force law,

$$\vec{T} = \frac{4\rho g S_c (S_b - S_t)}{S_b + S_c - S_t} X \vec{u}_x.$$

Using $S_t \ll S_b, S_c$,

$$\vec{T} \simeq \frac{4\rho g S_b S_c}{S_b + S_c} X \vec{u}_x.$$

The positive sign is significant: a displacement to the right produces a further force to the right. The equilibrium is mechanically unstable once friction can no longer hold the mass.

C.3

For the sketches, use the slow atmospheric driving limit: transfer between stops is rapid compared with the one-week pressure period. The stops arrest the moving system.

Write

$$p = \frac{P_1}{A}, \quad y = \frac{x}{X}.$$

Then

$$\frac{T}{KA} = 2(p + \lambda y), \quad \frac{F_s}{KA} = \xi.$$

Initially $y = 0$ and p increases from zero. Motion to the right starts as p exceeds

$$p = \frac{\xi}{2}.$$

Since $\partial T / \partial x = 2K\alpha > 0$, movement to the right increases the driving force. The mass rapidly reaches $y = +1$.

At the right stop, release to the left requires

$$T < -F_s.$$

Thus

$$2(p + \lambda) < -\xi \quad \Longleftrightarrow \quad p < -\left(\lambda + \frac{\xi}{2}\right).$$

Define

$$q = \lambda + \frac{\xi}{2}.$$

If $q > 1$, the pressure never becomes sufficiently low to release the mass. If $q < 1$, release occurs during every decreasing-pressure half-cycle. The analogous release condition at the left stop is $p > q$.

Therefore the two regimes are:

Regime	Condition	Precise sketch description
1: permanent residence at the right stop	$\xi < 2,$ $\xi + 2\lambda > 2$	Draw $x/X = 0$ initially, followed by a rapid rise to $+1$ on the first rising pressure branch at $p = \xi/2$. Continue horizontally at $+1$ through $3\tau_1$.
2: repeated motion between stops	$\xi < 2,$ $\xi + 2\lambda < 2$	Start with the same initial rise. Then draw horizontal plateaus at $+1$ and -1 , joined by rapid downward transitions whenever falling pressure reaches $p = -q$, and upward transitions whenever rising pressure reaches $p = q$.

For an explicit placement of the transitions, let $s = t/\tau_1$. The first positive transition is at

$$s = \frac{\xi}{8}.$$

In the cycling regime, the negative transitions within the requested interval are at

$$s = \frac{1}{2} + \frac{q}{4}, \quad \frac{3}{2} + \frac{q}{4}, \quad \frac{5}{2} + \frac{q}{4},$$

and subsequent positive transitions are at

$$s = 1 + \frac{q}{4}, \quad 2 + \frac{q}{4}.$$

These specify the requested three-period sketch completely.

At $q = 1$, the release threshold is only touched at a pressure extremum. Exact ideal static friction can keep the mass at the stop; sustained cycling is guaranteed by the strict inequality $q < 1$.

C.4

In the permanent cycling regime, the mass travels from one stop to the other and back during each period. Its total sliding distance is $4X$, so

$$\boxed{W = 4F_s X}.$$

With $S_b \simeq S_c$ and $S_t \ll S_b$,

$$\xi \simeq \frac{2F_s}{S_c A}, \quad \lambda \simeq \frac{2\rho g X}{A}.$$

The cycling condition becomes

$$\xi + 2\lambda < 2 \quad \Longleftrightarrow \quad F_s < S_c(A - 2\rho g X).$$

For a fixed X , the greatest attainable energy is approached as F_s approaches this upper bound. Its limiting envelope is

$$W(X) = 4S_c X(A - 2\rho g X).$$

Differentiating,

$$\frac{dW}{dX} = 4S_c(A - 4\rho g X).$$

The optimum is therefore

$$\boxed{X^* = \frac{A}{4\rho g}}, \quad \boxed{F_s^* = \frac{S_c A}{2}}.$$

Equivalently,

$$\xi_{\text{opt}} = 1, \quad \lambda_{\text{opt}} = \frac{1}{2}.$$

Thus

$$W^* = 4F_s^* X^* = \frac{S_c A^2}{2\rho g}.$$

With $S_c = 0.0210 \text{ m}^2$,

$$F_s^* = \frac{0.0210(500)}{2} = 5.25 \text{ N},$$

$$X^* = \frac{500}{4(13.5 \times 10^3)(9.8)} = 9.45 \times 10^{-4} \text{ m},$$

and

$$W^* = \frac{0.0210(500)^2}{2(13.5 \times 10^3)(9.8)} = 1.98 \times 10^{-2} \text{ J}.$$

Strictly, these values lie on the marginal switching boundary. They give the ideal limiting maximum, approached from the cycling side; reliable operation requires a small margin inside that boundary.

C.5

The volume relevant to atmospheric work includes the vacuum enclosed above the mercury. The liquid volume and the material volume of the vessels are constant, so

$$V = V_{\text{constant}} + V_{\text{vac}}, \quad V_{\text{vac}} = S_b(h_t + H_b - z_\ell).$$

Use the same approximation as in C.4:

$$S_b = S_c = S, \quad S_t = 0.$$

The conservation and hydrostatic relations derived in C.1 then give

$$\delta z_s = -\frac{P_1}{2\rho g}, \quad \delta z_\ell = \frac{P_1}{2\rho g}.$$

Since $\delta h_t = -x$,

$$V = V_{\text{ref}} - Sx - \frac{S}{2\rho g} P_1.$$

The atmospheric work received by the system is

$$W_{\text{pr}} = - \oint P_a dV.$$

Substitution yields

$$W_{\text{pr}} = S \oint P_a dx + \frac{S}{2\rho g} \oint P_a dP_1.$$

The second integral vanishes over a closed cycle. Also $P_0 \oint dx = 0$. Therefore

$$W_{\text{pr}} = S \oint P_1 dx.$$

At the limiting optimum, $q = 1$. The positive stroke occurs at $P_1 = +A$, with $\Delta x = 2X^*$, while the negative stroke occurs at $P_1 = -A$, with $\Delta x = -2X^*$. Consequently,

$$W_{\text{pr}}^* = S[A(2X^*) + (-A)(-2X^*)] = 4SAX^*.$$

Thus

$$W_{\text{pr}}^* = \frac{S_c A^2}{\rho g} = 3.97 \times 10^{-2} \text{ J}.$$

In a diagram with V horizontal and P_a vertical, the cycle is a parallelogram:

- at each fixed stop, $dV/dP_a = -S_c/(2\rho g)$, giving two parallel sloping branches;
- their horizontal separation is $2S_c X^*$;
- constant-pressure strokes connect them at $P_0 + A$ and $P_0 - A$;
- the cycle is traversed counterclockwise, giving positive work received equal to its area,

$$(2A)(2S_c X^*) = 4S_c A X^*.$$

Finally,

$$\frac{W^*}{W_{\text{pr}}^*} = \frac{S_c A^2 / (2\rho g)}{S_c A^2 / (\rho g)} = \frac{1}{2}.$$

The other half becomes kinetic energy during the rapid strokes and is dissipated when the moving system is arrested at the stops.

The solution was checked against the official solution.

Problem 3: Champagne!

Warning: Excessive alcohol consumption is harmful to health and drinking alcohol below legal age is prohibited.

Champagne is a French sparkling wine. Fermentation of sugars produces carbon dioxide (CO_2) in the bottle. The molar concentration of CO_2 in the liquid phase c_ℓ and the partial pressure P_{CO_2} in the gas phase are related by $c_\ell = k_H P_{\text{CO}_2}$, known as Henry's law and where k_H is called Henry's constant.

Data

- Surface tension of champagne $\sigma = 47 \times 10^{-3} \text{ J} \cdot \text{m}^{-2}$
- Density of the liquid $\rho_\ell = 1.0 \times 10^3 \text{ kg} \cdot \text{m}^{-3}$
- Henry's constant at $T_0 = 20^\circ\text{C}$, $k_H(20^\circ\text{C}) = 3.3 \times 10^{-4} \text{ mol} \cdot \text{m}^{-3} \cdot \text{Pa}^{-1}$
- Henry's constant at $T_0 = 6^\circ\text{C}$, $k_H(6^\circ\text{C}) = 5.4 \times 10^{-4} \text{ mol} \cdot \text{m}^{-3} \cdot \text{Pa}^{-1}$
- Atmospheric pressure $P_0 = 1 \text{ bar} = 1.0 \times 10^5 \text{ Pa}$
- Gases are ideal with an adiabatic coefficient $\gamma = 1.3$



Fig. 1. A glass filled with champagne.

Part A. Nucleation, growth and rise of bubbles

Immediately after opening a bottle of champagne at temperature $T_0 = 20^\circ\text{C}$, we fill a glass. The pressure in the liquid is P_0 and its temperature stays constant at T_0 . The concentration c_ℓ of dissolved CO_2 exceeds the equilibrium concentration and we study the nucleation of a CO_2 bubble. We note a its radius and P_b its inner pressure.

A.1 — 0.2pt

Express the pressure P_b in terms of P_0 , a and σ .

In the liquid, the concentration of dissolved CO_2 depends on the distance to the bubble. At long distance we recover the value c_ℓ and we note c_b the concentration close to the bubble surface. According to Henry's law, $c_b = k_H P_b$. We furthermore assume in all the problem that bubbles contain only CO_2 .

Since $c_\ell \neq c_b$, CO_2 molecules diffuse from areas of high to low concentration. We assume also that any molecule from the liquid phase reaching the bubble surface is transferred to the vapour.

A.2 — 0.5pt

Express the critical radius a_c above which a bubble is expected to grow in terms of P_0 , σ , c_ℓ and c_0 where $c_0 = k_H P_0$. Calculate numerically a_c for $c_\ell = 4c_0$.

In practice, bubbles mainly grow from pre-existing gas cavities. Consider then a bubble with initial radius $a_0 \approx 40 \mu\text{m}$. The number of moles of CO_2 transferred at the bubble's surface per unit area and time is noted j . Two models are possible for j .

- model (1) $j = \frac{D}{a}(c_\ell - c_b)$ where D is the diffusion coefficient of CO_2 in the liquid.
- model (2) $j = K(c_\ell - c_b)$ where K is a constant here.

Experimentally, the bubble radius $a(t)$ is found to depend on time as shown in **Fig. 2**. Here $c_\ell \approx 4c_0$, and since bubbles are large enough to be visible, the excess pressure due to surface tension can be neglected and $P_b \approx P_0$.

A.3 — 1.2pt

Express the number of CO_2 moles in the bubble n_c in terms of a , P_0 , T_0 and ideal gas constant R . Find $a(t)$ for both models. Indicate which model explains the experimental results in **Fig. 2**. Depending on your answer, calculate numerically K or D .

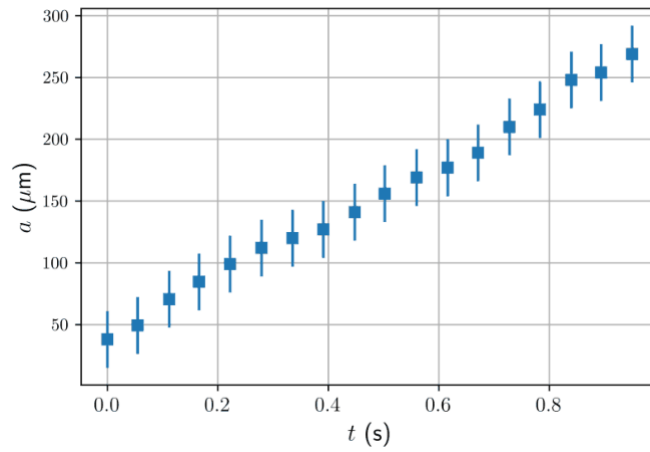


Fig. 2. Time evolution of CO_2 bubble radius in a glass of champagne (adapted from [1]).

Eventually bubbles detach from the bottom of the glass and continue to grow while rising. **Fig. 3.** shows a train of bubbles. The bubbles of the train have the same initial radius and are emitted at a constant frequency $f_b = 20 \text{ Hz}$.

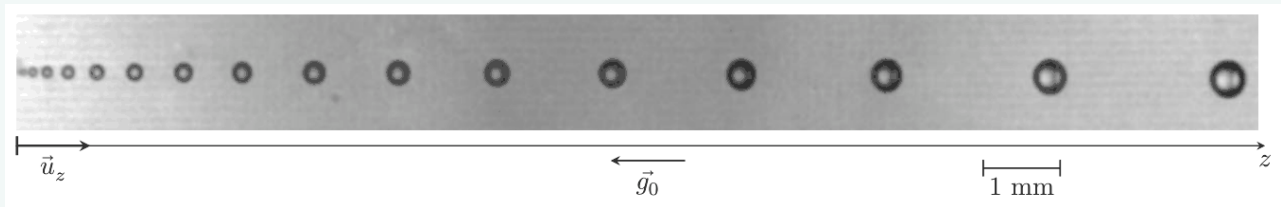


Fig. 3. A train of bubbles. The photo is rotated horizontally for the page layout (adapted from [1]).

For the range of velocities studied here, the drag force F on a bubble of radius a moving at velocity v in a liquid of dynamic viscosity η is given by Stokes' law $F = 6\pi\eta av$. Measurements show that at any moment in time, the bubble can be assumed to be travelling at its terminal velocity.

A.4 — 0.8pt

Give the expression of the main forces exerted on a vertically rising bubble. Obtain the expression of $v(a)$. Give a numerical estimate of η using ρ_ℓ , g_0 and quantities measured on **Fig. 3**.

The quasi-stationary growth of bubbles with rate $q_a = \frac{da}{dt}$ still applies during bubble rise.

A.5 — 0.5pt

Express the radius a_{H_ℓ} of a bubble reaching the free surface in terms of height travelled H_ℓ , growth rate $q_a = \frac{da}{dt}$, and any constants you may need. Assume $a_{H_\ell} \gg a_0$ and q_a constant, and give the numerical value of a_{H_ℓ} with $H_\ell = 10 \text{ cm}$ and q_a corresponding to **Fig. 2**.

There are N_b nucleation sites of bubbles. Assume that the bubbles are nucleated at a constant frequency f_b at the bottom of a glass of champagne (height H_ℓ for a volume V_ℓ), with a_0 still negligible. Neglect diffusion of CO_2 at the free surface.

A.6 — 1.1pt

Write the differential equation for $c_\ell(t)$. Obtain from this equation the characteristic time τ for the decay of the concentration of dissolved CO_2 in the liquid.

Part B. Acoustic emission of a bursting bubble

Small bubbles are nearly spherical as they reach the free surface. Once the liquid film separating the bubble from the air thins out sufficiently, a circular hole of radius r forms in the film and, driven by surface tension, opens very quickly (**Fig. 4. left**). The hole opens at constant speed v_f (**Fig. 4. right**). The film outside the rim remains still, with constant thickness h .

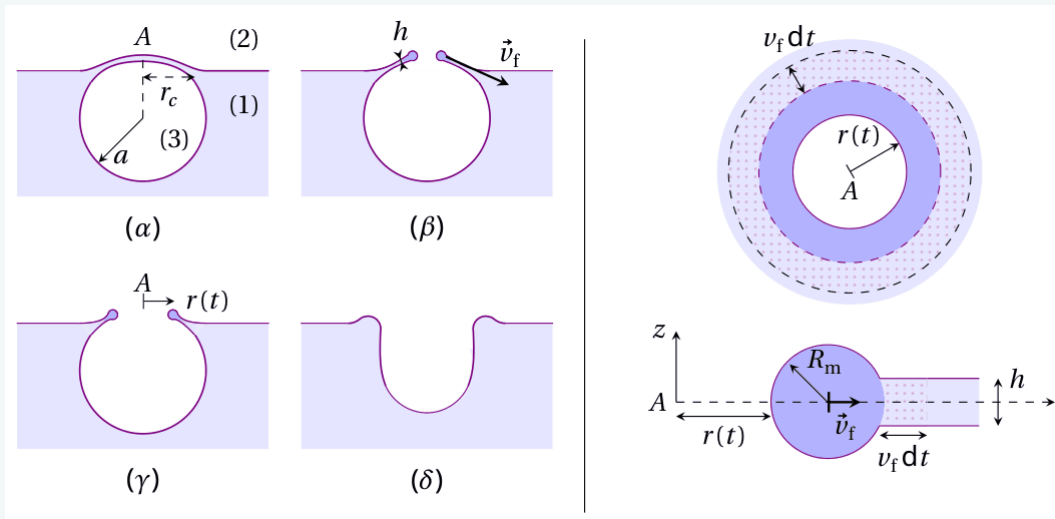


Fig. 4. (Left) (α) Bubble at the surface: (1) liquid, (2) air at pressure P_0 and (3), CO_2 at pressure P_b , (β) and (γ) retraction of the liquid film, where the rim is in dark blue, (δ) bubble collapse. (Right) Retraction of the liquid film at time t . Top: sketch of the pierced film seen from above. Bottom: cross-section of the rim and the retracting film. During dt the rim accumulates nearby liquid (dotted).

Due to dissipative processes, only half of the difference of the surface energy between t and $t + dt$ of the rim and the accumulated liquid is transformed into kinetic energy. We further assume that the variation of the surface of the rim is negligible compared to that of the film.

B.1 — 1.1pt

Express v_f in terms of ρ_ℓ , σ and h .

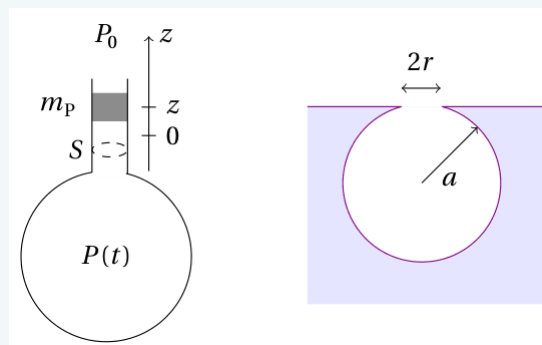


Fig. 5. (Left) a Helmholtz resonator. (Right) a bubble as an oscillator.

When the film bursts, it releases internal pressure and emits a sound. We model this acoustic emission by a Helmholtz resonator: a cavity open to the atmosphere at P_0 through a bottleneck aperture of area S (**Fig. 5. left**). In the neck, a mass m_p makes small amplitude position oscillations due to the pressure forces it experiences as the gas in the cavity expands or compresses adiabatically. The gravity force on m_p is negligible compared to pressure forces. Let V_0 be the volume of gas under the mass m_p for $P = P_0$ as $z = 0$.

B.2 — 1.1pt

Express the frequency of oscillation f_0 of m_p . Hint: for $\varepsilon \ll 1$, $(1 + \varepsilon)^\alpha \approx 1 + \alpha\varepsilon$.

The Helmholtz model may be used for a bubble of radius a . V_0 is the volume of the closed bubble. From literature, the mass of the equivalent of the piston is $m_p = 8\rho_g r^3/3$ where r is the radius of the circular aperture and $\rho_g = 1.8 \text{ kg} \cdot \text{m}^{-3}$ is the density of the gas (**Fig. 5. right**). During the bursting process, r goes from 0 to r_c , given by

$$r_c = \frac{2}{\sqrt{3}} a^2 \sqrt{\frac{\rho_l g_0}{\sigma}}.$$

At the same time, the frequency of emitted sound increases until a maximum value of 40 kHz and the bursting time is $t_b = 3 \times 10^{-2} \text{ ms}$.

B.3 — 1.1pt

Find the radius a and the thickness h of the champagne film separating the bubble from the atmosphere.

Part C. Popping champagne

In a bottle, the total quantity of CO_2 is $n_T = 0.2 \text{ mol}$, either dissolved in the volume $V_L = 750 \text{ mL}$ of liquid champagne, or as a gas in the volume $V_G = 25 \text{ mL}$ under the cork (**Fig. 6. left**). V_G contains only CO_2 . The equilibrium between both CO_2 phases follows Henry's Law. We suppose that the fast gaseous CO_2 expansion when the bottle is opened, is adiabatic and reversible. Ambient temperature T_0 and pressure $P_0 = 1 \text{ bar}$ are constant.

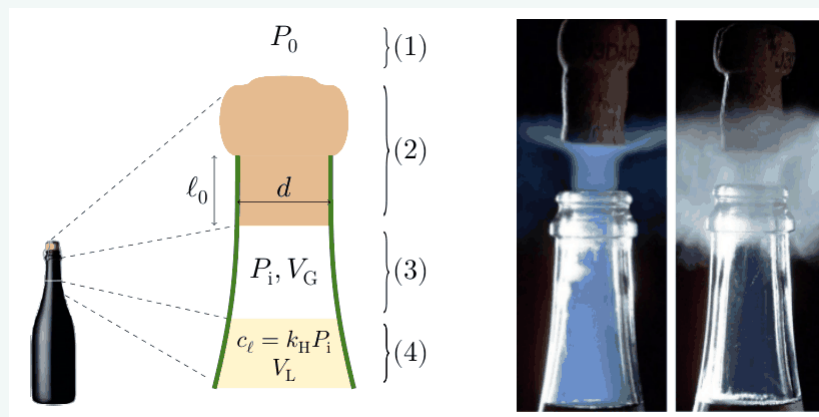


Fig. 6. Left: traditional bottleneck: (1) surrounding air, (2) cork stopper, (3) headspace, (4) liquid champagne. Right: Two phenomena observed while opening the bottle at two different temperatures (adapted from [2]).

C.1 — 0.4pt

Give the numerical value of the pressure P_1 of gaseous CO_2 in the bottle for $T_0 = 6^\circ\text{C}$ and $T_0 = 20^\circ\text{C}$.

Another step of champagne production (not described here) leads to the following values of P_1 that we will use for the next questions: $P_1 = 4.69$ bar at $T_0 = 6^\circ\text{C}$ and $P_1 = 7.45$ bar at $T_0 = 20^\circ\text{C}$.

During bottle opening, two different phenomena can be observed, depending on T_0 (**Fig. 6.** right).

- either a blue fog appears, due to the formation of solid CO_2 crystals (but water condensation is inhibited);
- or a grey-white fog appears, due to water vapor condensation in the air surrounding the bottleneck. In this latter case, there is no formation of CO_2 solid crystals.

The saturated vapor pressure $P_{\text{sat}}^{\text{CO}_2}$ for the CO_2 solid/gas transition follows:

$$\log_{10} \left(\frac{P_{\text{sat}}^{\text{CO}_2}}{P_0} \right) = A - \frac{B}{T + C}$$

with T in K, $A = 6.81$, $B = 1.30 \times 10^3$ K and $C = -3.49$ K.

C.2 — 0.7pt

Give the numerical value T_f of the CO_2 gas at the end of the expansion, after opening a bottle, if $T_0 = 6^\circ\text{C}$ and if $T_0 = 20^\circ\text{C}$, if no phase transition occurred.

Choose which statements are true (several statements possible):

1. At $T_0 = 6^\circ\text{C}$ a grey-white fog appears while opening the bottle.
2. At $T_0 = 6^\circ\text{C}$ a blue fog appears while opening the bottle.
3. At $T_0 = 20^\circ\text{C}$ a grey-white fog appears while opening the bottle.
4. At $T_0 = 20^\circ\text{C}$ a blue fog appears while opening the bottle.

During bottle opening, the cork stopper pops out. We now determine the maximum height H_c it reaches. Assume that the friction force F due to the bottleneck on the cork stopper is $F = \alpha A$ where A is the area of contact and α is a constant to determine. Initially, the pressure force slightly overcomes the friction force. The cork's mass is $m = 10$ g, its diameter $d = 1.8$ cm and the length of the cylindrical part initially stuck in the bottleneck is $\ell_0 = 2.5$ cm. Once the cork has left the bottleneck, you can neglect the net pressure force.

C.3 — 1.3pt

Give the numerical value of H_c if the external temperature is $T_0 = 6^\circ\text{C}$.

[1] Liger-Belair et al, Am. J. Enol. Vitic., Vol. 50, No. 3 (1999).

[2] Liger-Belair et al., Sc. Reports **7**, 10938 (2017).

General Data Sheet

Constant	Notation	Value	Unit
Speed of light in vacuum	c	$2.997\,924\,58 \times 10^8$	$\text{m} \cdot \text{s}^{-1}$

Constant	Notation	Value	Unit
Planck constant	h	$6.626\,070\,15 \times 10^{-34}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Reduced Planck constant	$\hbar = \frac{h}{2\pi}$	$1.054\,571\,818 \times 10^{-34}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Boltzmann constant	k_B	$1.380\,649 \times 10^{-23}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$
Avogadro constant	N_A	$6.022\,140\,76 \times 10^{23}$	mol^{-1}
Molar gas constant	R	$8.314\,462\,618\,153\,24$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	$\text{A} \cdot \text{s}$
Universal constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	$\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Standard acceleration due to gravity	g	$9.806\,65$	$\text{m} \cdot \text{s}^{-2}$
Stefan Boltzmann constant	σ	$5.670\,374\,419 \times 10^{-8}$	$\text{kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$
Vacuum permeability (magnetic constant)	μ_0	$1.256\,637\,061\,27(20) \times 10^{-6}$	$\text{kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity (electrical constant)	ϵ_0	$8.854\,187\,818\,8(14) \times 10^{-12}$	$\text{A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$
Rydberg constant	R_∞	$1.097\,373\,156\,815\,7(12) \times 10^7$	m^{-1}
Mass of the electron	m_e	$9.109\,383\,713\,9(28) \times 10^{-31}$	kg
Mass of the proton	m_p	$1.672\,621\,925\,95(52) \times 10^{-27}$	kg
Mass of the neutron	m_n	$1.674\,927\,500\,56(85) \times 10^{-27}$	kg
Atomic mass constant	m_u	$1.660\,539\,068\,92(52) \times 10^{-27}$	kg
Electronvolt	eV	$1.602\,176\,634 \times 10^{-19}$	$\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2}$

Official English text of the problem.

Answers

Here $\Delta c = c_\ell - c_0$, and g_0 is the gravitational acceleration given in the data sheet.

- A.1

$$P_b = P_0 + \frac{2\sigma}{a}.$$

- A.2

$$a_c = \frac{2\sigma c_0}{P_0(c_\ell - c_0)}.$$

For $c_\ell = 4c_0$,

$$a_c = 3.1 \times 10^{-7} \text{ m} = 0.31 \mu\text{m}.$$

- **A.3**

$$n_c = \frac{4\pi P_0 a^3}{3RT_0},$$

$$a_{(1)}(t) = \sqrt{a_0^2 + \frac{2DRT_0\Delta c}{P_0}t},$$

$$a_{(2)}(t) = a_0 + \frac{KRT_0\Delta c}{P_0}t.$$

Model (2) fits the approximately linear growth:

$$q_a \approx 2.4 \times 10^{-4} \text{ m s}^{-1}, \quad K \approx 1.0 \times 10^{-4} \text{ m s}^{-1}.$$

- **A.4** Buoyancy $4\pi\rho_\ell g_0 a^3/3$ acts upwards; Stokes drag $6\pi\eta a v$ and the negligible bubble weight act downwards.

$$v(a) \simeq \frac{2\rho_\ell g_0 a^2}{9\eta}, \quad \eta \approx 2.7 \times 10^{-3} \text{ Pa s}$$

from approximate measurements of Fig. 3.

- **A.5**

$$a_{H_\ell} \simeq \left(\frac{27\eta q_a H_\ell}{2\rho_\ell g_0} \right)^{1/3} \approx 4.5 \times 10^{-4} \text{ m} = 0.45 \text{ mm}.$$

- **A.6**

$$\frac{dc_\ell}{dt} = -\frac{18\pi\eta K H_\ell N_b f_b}{\rho_\ell g_0 V_\ell} (c_\ell - c_0), \quad \tau = \frac{\rho_\ell g_0 V_\ell}{18\pi\eta K H_\ell N_b f_b}.$$

- **B.1**

$$v_f = \sqrt{\frac{2\sigma}{\rho_\ell h}}.$$

- **B.2**

$$f_0 = \frac{1}{2\pi} \sqrt{\frac{\gamma P_0 S^2}{m_p V_0}}.$$

- **B.3**

$$a \approx 5.3 \times 10^{-4} \text{ m} = 0.53 \text{ mm}, \quad h \approx 3.8 \times 10^{-6} \text{ m} = 3.8 \mu\text{m}.$$

- **C.1**

$$P_i(6^\circ\text{C}) \approx 4.81 \text{ bar}, \quad P_i(20^\circ\text{C}) \approx 7.76 \text{ bar}.$$

- **C.2** Using the replacement pressures specified for this part:

$$T_f(6^\circ\text{C}) \approx 195.4 \text{ K } (-77.7^\circ\text{C}),$$

$$T_f(20^\circ\text{C}) \approx 184.4 \text{ K } (-88.7^\circ\text{C}).$$

The true statements are **1 and 4**: grey-white fog at 6°C , blue fog at 20°C .

$$H_c \approx 7.7 \text{ m}.$$

Solution

A.1

The Young–Laplace law gives the pressure jump across a curved liquid–gas interface:

$$P_b - P_0 = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a spherical bubble, both principal radii are a . Thus

$$P_b = P_0 + \frac{2\sigma}{a}.$$

There is one liquid–gas interface around the submerged bubble.

A.2

Henry's law at the bubble surface gives

$$c_b = k_H P_b = k_H \left(P_0 + \frac{2\sigma}{a} \right) = c_0 + \frac{2\sigma c_0}{P_0 a}.$$

Gas diffuses into the bubble, making it grow, when $c_\ell > c_b$. The threshold therefore satisfies

$$c_\ell - c_0 = \frac{2\sigma c_0}{P_0 a_c},$$

so

$$a_c = \frac{2\sigma c_0}{P_0 (c_\ell - c_0)}.$$

For a supersaturated liquid, bubbles with $a > a_c$ grow, whereas smaller bubbles tend to dissolve.

With $c_\ell = 4c_0$,

$$a_c = \frac{2\sigma}{3P_0} = \frac{2(47 \times 10^{-3})}{3(1.0 \times 10^5)} = 3.13 \times 10^{-7} \text{ m}.$$

Hence

$$a_c \approx 0.31 \text{ } \mu\text{m}.$$

A.3

Under the stated approximation $P_b \simeq P_0$, the ideal gas law gives

$$P_0 \frac{4\pi a^3}{3} = n_c R T_0,$$

and therefore

$$n_c = \frac{4\pi P_0 a^3}{3 R T_0}.$$

The molar influx through the bubble surface is

$$\frac{dn_c}{dt} = 4\pi a^2 j.$$

Differentiating the ideal-gas expression at constant P_0, T_0 ,

$$\frac{4\pi P_0 a^2}{RT_0} \frac{da}{dt} = 4\pi a^2 j,$$

which reduces to

$$\frac{da}{dt} = \frac{RT_0}{P_0} j.$$

For the short growth experiment, take the bulk concentration as constant and use $c_b \simeq c_0$.

For **model (1)**,

$$\frac{da}{dt} = \frac{DRT_0(c_\ell - c_0)}{P_0 a}.$$

Integrating with $a(0) = a_0$,

$$a(t) = \sqrt{a_0^2 + \frac{2DRT_0(c_\ell - c_0)}{P_0} t}.$$

This predicts a decreasing growth rate and an asymptotic $t^{1/2}$ dependence.

For **model (2)**,

$$\frac{da}{dt} = \frac{KRT_0(c_\ell - c_0)}{P_0} = q_a,$$

so

$$a(t) = a_0 + \frac{KRT_0(c_\ell - c_0)}{P_0} t.$$

Figure 2 shows an approximately constant slope, supporting **model (2)**. Reading roughly $a = 40 \mu\text{m}$ at $t = 0$ and $a = 270 \mu\text{m}$ at $t = 0.95 \text{ s}$,

$$q_a \approx \frac{(270 - 40) \times 10^{-6}}{0.95} \approx 2.4 \times 10^{-4} \text{ m s}^{-1}.$$

At 20°C ,

$$T_0 = 293.15 \text{ K}, \quad c_0 = k_H P_0 = 33 \text{ mol m}^{-3},$$

and $c_\ell - c_0 \simeq 3c_0 = 99 \text{ mol m}^{-3}$. Hence

$$K = \frac{P_0 q_a}{RT_0(c_\ell - c_0)} \approx \frac{10^5 (2.4 \times 10^{-4})}{8.31446 (293.15) (99)}.$$

Thus

$$K \approx 1.0 \times 10^{-4} \text{ m s}^{-1}.$$

A.4

Let $\mathbf{u}_z$ point vertically upwards. The forces are buoyancy,

$$\mathbf{F}_B = \rho_\ell \frac{4\pi a^3}{3} g_0 \mathbf{u}_z,$$

Stokes drag,

$$\mathbf{F}_D = -6\pi\eta a v \mathbf{u}_z,$$

and the bubble's weight,

$$\mathbf{W} = -\rho_g \frac{4\pi a^3}{3} g_0 \mathbf{u}_z.$$

At terminal velocity their sum vanishes:

$$6\pi\eta a v = (\rho_\ell - \rho_g) \frac{4\pi a^3}{3} g_0.$$

Since $\rho_g \ll \rho_\ell$,

$$v(a) = \frac{2(\rho_\ell - \rho_g)g_0 a^2}{9\eta} \simeq \frac{2\rho_\ell g_0 a^2}{9\eta}.$$

Adjacent bubbles follow the same trajectory with an age difference

$$\Delta t = \frac{1}{f_b}.$$

Their centre-to-centre separation Δz therefore gives

$$v \simeq \frac{\Delta z}{\Delta t} = f_b \Delta z,$$

evaluated approximately at the mean radius of the pair.

Using the scale in Fig. 3, the two bubbles near the right-hand end give approximately

$$a \simeq 0.24 \text{ mm}, \quad \Delta z \simeq 2.3 \text{ mm}.$$

Consequently,

$$v \simeq 20(2.3 \times 10^{-3}) = 4.6 \times 10^{-2} \text{ m s}^{-1},$$

and

$$\eta \simeq \frac{2\rho_\ell g_0 a^2}{9f_b \Delta z} = \frac{2(10^3)(9.80665)(0.24 \times 10^{-3})^2}{9(20)(2.3 \times 10^{-3})}.$$

This gives the graphical estimate

$$\eta \approx 2.7 \times 10^{-3} \text{ Pa s}.$$

A.5

Combine the terminal rise speed with the constant radial growth rate:

$$\frac{dz}{dt} = \frac{2\rho_\ell g_0}{9\eta} a^2, \quad \frac{da}{dt} = q_a.$$

Eliminating time,

$$\frac{dz}{da} = \frac{2\rho_\ell g_0}{9\eta q_a} a^2.$$

Integrating from the bottom to the free surface,

$$H_\ell = \frac{2\rho_\ell g_0}{27\eta q_a} (a_{H_\ell}^3 - a_0^3).$$

Thus the expression retaining the initial radius is

$$a_{H_\ell} = \left(a_0^3 + \frac{27\eta q_a H_\ell}{2\rho_\ell g_0} \right)^{1/3}.$$

With $a_{H_\ell} \gg a_0$,

$$a_{H_\ell} \simeq \left(\frac{27\eta q_a H_\ell}{2\rho_\ell g_0} \right)^{1/3}.$$

Using $H_\ell = 0.10 \text{ m}$, $q_a \simeq 2.4 \times 10^{-4} \text{ m s}^{-1}$, and the viscosity estimated above,

$$a_{H_\ell} \approx 4.5 \times 10^{-4} \text{ m} = 0.45 \text{ mm}.$$

A.6

Use the quasi-stationary description of the bubble train on the concentration-depletion timescale. The liquid is represented by a uniform bulk concentration $c_\ell(t)$, and changes in the gas inventory of bubbles still rising are neglected in the loss balance.

Each bubble reaching the surface carries

$$n_{\text{out}} = \frac{4\pi P_0}{3RT_0} a_{H_\ell}^3.$$

The result of A.5, with negligible a_0 , gives

$$n_{\text{out}} = \frac{4\pi P_0}{3RT_0} \frac{27\eta q_a H_\ell}{2\rho_\ell g_0} = \frac{18\pi\eta H_\ell P_0}{\rho_\ell g_0 RT_0} q_a.$$

As the liquid loses dissolved gas, model (2) gives

$$q_a(t) = \frac{KRT_0}{P_0} (c_\ell(t) - c_0).$$

Therefore

$$n_{\text{out}} = \frac{18\pi\eta K H_\ell}{\rho_\ell g_0} (c_\ell - c_0).$$

There are $N_b f_b$ bubbles emitted per unit time. Conservation of CO_2 gives

$$V_\ell \frac{dc_\ell}{dt} = -N_b f_b n_{\text{out}},$$

hence

$$\frac{dc_\ell}{dt} = -\frac{18\pi\eta K H_\ell N_b f_b}{\rho_\ell g_0 V_\ell} (c_\ell - c_0).$$

The solution is

$$c_\ell(t) = c_0 + [c_\ell(0) - c_0] e^{-t/\tau},$$

where

$$\tau = \frac{\rho_\ell g_0 V_\ell}{18\pi\eta K H_\ell N_b f_b}.$$

It is the excess concentration above c_0 that decays exponentially.

B.1

During dt , the retracting rim sweeps out an annulus of area

$$dA = 2\pi r v_f dt.$$

The film has two liquid–gas surfaces, so the decrease in surface energy is

$$-dE_s = 2\sigma dA.$$

By the stated dissipation rule, half of this becomes kinetic energy:

$$dE_k = \sigma dA.$$

The newly accumulated mass is

$$dm = \rho_\ell h dA.$$

Since the rim speed remains constant, its kinetic-energy increase is

$$dE_k = \frac{1}{2} v_f^2 dm = \frac{1}{2} \rho_\ell h v_f^2 dA.$$

Equating the two expressions,

$$\frac{1}{2} \rho_\ell h v_f^2 = \sigma,$$

so

$$v_f = \sqrt{\frac{2\sigma}{\rho_\ell h}}.$$

B.2

An upward displacement z of the piston changes the enclosed gas volume to

$$V = V_0 + Sz.$$

For reversible adiabatic compression or expansion,

$$PV^\gamma = P_0 V_0^\gamma.$$

Thus

$$P = P_0 \left(1 + \frac{Sz}{V_0} \right)^{-\gamma}.$$

For $|Sz| \ll V_0$,

$$P \simeq P_0 \left(1 - \gamma \frac{Sz}{V_0} \right).$$

Neglecting the piston weight as specified, its equation of motion is

$$m_p \ddot{z} = (P - P_0)S = -\frac{\gamma P_0 S^2}{V_0} z.$$

This is simple harmonic motion with

$$\omega_0^2 = \frac{\gamma P_0 S^2}{m_p V_0}.$$

Therefore

$$f_0 = \frac{1}{2\pi} \sqrt{\frac{\gamma P_0 S^2}{m_p V_0}}.$$

B.3

For the bubble,

$$S = \pi r^2, \quad V_0 = \frac{4\pi a^3}{3}, \quad m_p = \frac{8\rho_g r^3}{3}.$$

Substitution into B.2 gives

$$f_0^2 = \frac{1}{4\pi^2} \frac{\gamma P_0 \pi^2 r^4}{(8\rho_g r^3/3)(4\pi a^3/3)} = \frac{9\gamma P_0 r}{128\pi \rho_g a^3}.$$

Thus $f_0 \propto \sqrt{r}$, and the maximum frequency occurs at $r = r_c$. Using the given expression for r_c ,

$$f_{\max}^2 = \frac{9\gamma P_0}{128\pi \rho_g a^3} \frac{2a^2}{\sqrt{3}} \sqrt{\frac{\rho_\ell g_0}{\sigma}} = \frac{3\sqrt{3}\gamma P_0}{64\pi \rho_g a} \sqrt{\frac{\rho_\ell g_0}{\sigma}}.$$

Therefore

$$a = \frac{3\sqrt{3}\gamma P_0}{64\pi \rho_g f_{\max}^2} \sqrt{\frac{\rho_\ell g_0}{\sigma}}.$$

With $f_{\max} = 4.0 \times 10^4$ Hz,

$$a = \frac{3\sqrt{3}(1.3)(10^5)}{64\pi(1.8)(4.0 \times 10^4)^2} \sqrt{\frac{10^3(9.80665)}{47 \times 10^{-3}}} = 5.33 \times 10^{-4} \text{ m}.$$

Hence

$$a \approx 0.53 \text{ mm}.$$

The final aperture radius is

$$r_c = \frac{2}{\sqrt{3}} a^2 \sqrt{\frac{\rho_\ell g_0}{\sigma}} \approx 1.50 \times 10^{-4} \text{ m}.$$

Because the aperture opens at constant speed,

$$v_f = \frac{r_c}{t_b}, \quad t_b = 3 \times 10^{-2} \text{ ms} = 3 \times 10^{-5} \text{ s},$$

so

$$v_f \approx 5.0 \text{ m s}^{-1}.$$

Finally, B.1 gives

$$h = \frac{2\sigma}{\rho_\ell v_f^2} = \frac{2\sigma t_b^2}{\rho_\ell r_c^2} \approx 3.77 \times 10^{-6} \text{ m}.$$

Thus

$$h \approx 3.8 \text{ } \mu\text{m}.$$

C.1

The amounts of dissolved and gaseous CO_2 are, respectively,

$$n_L = k_H P_i V_L, \quad n_G = \frac{P_i V_G}{RT_0}.$$

Conservation of the total amount gives

$$n_T = P_i \left(k_H V_L + \frac{V_G}{RT_0} \right).$$

Consequently,

$$\boxed{P_i = \frac{n_T}{k_H V_L + V_G/(RT_0)}}.$$

Use

$$V_L = 7.50 \times 10^{-4} \text{ m}^3, \quad V_G = 2.50 \times 10^{-5} \text{ m}^3.$$

At 6°C, $T_0 = 279.15 \text{ K}$:

$$P_i = \frac{0.2}{(5.4 \times 10^{-4})(7.50 \times 10^{-4}) + (2.50 \times 10^{-5})/(8.31446 \times 279.15)} \approx 4.81 \times 10^5 \text{ Pa}.$$

At 20°C, $T_0 = 293.15 \text{ K}$:

$$P_i = \frac{0.2}{(3.3 \times 10^{-4})(7.50 \times 10^{-4}) + (2.50 \times 10^{-5})/(8.31446 \times 293.15)} \approx 7.76 \times 10^5 \text{ Pa}.$$

These are absolute pressures:

$$\boxed{P_i(6^\circ\text{C}) \approx 4.81 \text{ bar}}, \quad \boxed{P_i(20^\circ\text{C}) \approx 7.76 \text{ bar}}.$$

C.2

For this part, the problem specifies the replacement pressures

$$P_i(6^\circ\text{C}) = 4.69 \text{ bar}, \quad P_i(20^\circ\text{C}) = 7.45 \text{ bar}.$$

For reversible adiabatic expansion of an ideal gas,

$$PV^\gamma = \text{constant}, \quad TV^{\gamma-1} = \text{constant}.$$

Eliminating V , and taking the final pressure to be P_0 ,

$$\boxed{T_f = T_0 \left(\frac{P_0}{P_i} \right)^{(\gamma-1)/\gamma}}.$$

Thus

$$T_f(6^\circ\text{C}) = 279.15 \left(\frac{1}{4.69} \right)^{0.3/1.3} \approx 195.4 \text{ K},$$

and

$$T_f(20^\circ\text{C}) = 293.15 \left(\frac{1}{7.45} \right)^{0.3/1.3} \approx 184.4 \text{ K}.$$

To determine whether solid CO_2 forms, find the sublimation-equilibrium temperature at P_0 . Setting $P_{\text{sat}}^{\text{CO}_2} = P_0$,

$$0 = A - \frac{B}{T_s + C},$$

so

$$T_s = \frac{B}{A} - C = \frac{1300}{6.81} + 3.49 \approx 194.4 \text{ K}.$$

For the initially cold bottle,

$$195.4 \text{ K} > T_s.$$

Indeed, the given vapor-pressure expression yields

$$P_{\text{sat}}^{\text{CO}_2}(195.4 \text{ K}) \approx 1.09 \text{ bar} > P_0.$$

The gas does not reach solid–gas saturation during this expansion. According to the stated alternatives, **grey-white fog** forms.

For the initially warmer bottle,

$$184.4 \text{ K} < T_s,$$

and

$$P_{\text{sat}}^{\text{CO}_2}(184.4 \text{ K}) \approx 0.422 \text{ bar} < P_0.$$

The phase-change-free expansion would produce supersaturated CO_2 , so solid crystals form, giving **blue fog**.

Therefore

Statements 1 and 4 are true.

The calculated T_f values are the requested hypothetical temperatures without phase change; solid formation modifies the actual thermal evolution in the warm-bottle case.

C.3

Use the specified pressure at 6°C :

$$P_i = 4.69 \times 10^5 \text{ Pa}.$$

Let x be the upward displacement of the cork from its initial position, and let

$$S = \frac{\pi d^2}{4}$$

be its cross-sectional area.

While $0 \leq x \leq \ell_0$, the contact area is

$$A(x) = \pi d(\ell_0 - x),$$

so the friction force is

$$F_{\text{fr}}(x) = \alpha \pi d(\ell_0 - x).$$

Initially, the net pressure force nearly balances friction:

$$(P_i - P_0)S \simeq \alpha \pi d \ell_0.$$

The cork's weight is only 0.098 N , compared with an initial net pressure force of about 94 N , so it can be neglected when estimating α . Hence

$$\alpha \simeq \frac{(P_i - P_0)d}{4\ell_0} \approx 6.64 \times 10^4 \text{ Pa}.$$

The rapid expansion involves the gas initially in the headspace. Its volume and pressure during the cork's motion are

$$V(x) = V_G + Sx, \quad P(x) = P_i \left(\frac{V_G}{V_G + Sx} \right)^\gamma.$$

When the cork leaves the neck,

$$V_e = V_G + S\ell_0.$$

The work done by the internal gas is

$$W_{\text{gas}} = \int_0^{\ell_0} P(x)S \, dx = \int_{V_G}^{V_e} P_i \left(\frac{V_G}{V} \right)^\gamma dV,$$

giving

$$W_{\text{gas}} = \frac{P_i V_G}{\gamma - 1} \left[1 - \left(\frac{V_G}{V_e} \right)^{\gamma-1} \right].$$

The work against atmospheric pressure is

$$W_{\text{atm}} = P_0 S \ell_0,$$

and the energy dissipated by friction is

$$W_{\text{fr}} = \int_0^{\ell_0} \alpha \pi d (\ell_0 - x) \, dx = \frac{1}{2} \alpha \pi d \ell_0^2 \simeq \frac{1}{2} (P_i - P_0) S \ell_0.$$

After the cork exits, the net pressure force is neglected as instructed. Neglecting air resistance, all the remaining mechanical energy becomes gravitational potential energy at the highest point. Taking H_c from the initial cork position,

$$mg_0 H_c = W_{\text{gas}} - W_{\text{atm}} - W_{\text{fr}}.$$

Therefore

$$H_c = \frac{1}{mg_0} \left\{ \frac{P_i V_G}{\gamma - 1} \left[1 - \left(\frac{V_G}{V_G + S\ell_0} \right)^{\gamma-1} \right] - P_0 S \ell_0 - \frac{1}{2} (P_i - P_0) S \ell_0 \right\}.$$

Numerically,

$$S = 2.5447 \times 10^{-4} \, \text{m}^2, \quad V_e = 3.1362 \times 10^{-5} \, \text{m}^3,$$

and

$$W_{\text{gas}} \approx 2.570 \, \text{J}, \quad W_{\text{atm}} \approx 0.636 \, \text{J}, \quad W_{\text{fr}} \approx 1.174 \, \text{J}.$$

Thus

$$H_c \approx \frac{2.570 - 0.636 - 1.174}{0.010(9.80665)} \approx 7.75 \, \text{m}.$$

To sensible precision,

$$\boxed{H_c \approx 7.7 \, \text{m}}.$$

If the height is measured from the cork's position when it clears the neck, subtract $\ell_0 = 0.025 \, \text{m}$; the rounded answer remains **7.7 m**.

The solution was checked against the official solution.