

IPhO 2024

Theory problems and solutions · Isfahan, Iran · 3 theory problems with complete solutions

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Problem 1: The Greenhouse Effect

In 2021, Syukuro Manabe and Klaus Hasselmann were awarded half of the Nobel Prize in Physics for their work in modeling Earth's climate and accurately predicting the global warming caused by human industrial activities. In this problem, we will examine a simple model of global warming due to the greenhouse effect. The greenhouse gases alter the optical properties of the Earth's atmosphere in transmitting or absorbing Earth's infrared radiation, resulting in a rise in the average temperature of the planet.

All objects, at different temperatures, emit thermal radiation. The quantity $u(\lambda, T)d\lambda$ indicates the thermal radiative power per unit area of an object at temperature T between the wavelengths λ and $\lambda + d\lambda$. According to Planck's theory of blackbody radiation, we have

$$u(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}, \quad (1)$$

in which $hc = 1.24 \times 10^3 \text{ eV} \cdot \text{nm}$ and $k_B = 8.62 \times 10^{-5} \text{ eV/K}$. The wavelength corresponding to the maximum of $u(\lambda, T)$ comes from the relation $\lambda_{\max} T = b$ (Wien's displacement law). Indeed, using equation (1), it can be shown that $b = \frac{hc}{x_m k_B}$, where the dimensionless quantity x_m is the non-trivial root of an equation of the form $f(x) = 0$; you are asked to find the function $f(x)$ in one of the following tasks. Total radiative power per unit area of a blackbody in all wavelengths is given by the Stephan-Boltzmann law as $U(T) = \sigma T^4$ where $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$. Moreover, according to Kirchhoff's law of radiation, at thermal equilibrium a body absorbing a certain fraction of the incident radiation at a specific wavelength, will radiate the same fraction of the blackbody radiation at that same wavelength.

Throughout this problem assume that the Sun is a blackbody at its average surface temperature of $T_S = 5.77 \times 10^3 \text{ K}$. The Sun's radius is $R_S = 6.96 \times 10^8 \text{ m}$ and the average distance between the Earth and the Sun is $d = 1.50 \times 10^{11} \text{ m}$. We denote by $\tilde{u}_S(\lambda)$, the spectral solar power radiated into a unit area of the Earth normal to the direction of radiation. The integral of this quantity over all wavelengths, i.e. $S_0 = \int \tilde{u}_S(\lambda) d\lambda$, is called the solar constant.

In this problem assume that the Earth is in thermal equilibrium and has the same temperature at all points on its surface. In all parts of the problem, express the desired quantity in parametric form in terms of the data given in the problem and then find its numerical value accurate to three significant figures. The required units are indicated on the answer sheet.

A. Earth as a Blackbody

In this part, consider the Earth's surface as a blackbody and neglect the Earth's atmosphere.

Part	Task	Points
A-1	Find the solar constant, S_0 .	0.6 pt
A-2	Find the Earth's temperature, T_E .	0.6 pt
A-3	Find the function $f(x)$.	0.4 pt
A-4	Calculate the numerical value of x_m , and from this value x_m , find the value of b .	0.4 pt
A-5	Find $\lambda_{\max}$ for the Sun and the Earth.	0.2 pt

In figure 1 the functions $\gamma\tilde{u}_S(\lambda)$ and $u(\lambda, T_E)$ are plotted versus λ , where γ is a dimensionless coefficient to rescale $\tilde{u}_S(\lambda)$ such that the values of the two peaks coincide.

Part	Task	Points
A-6	Determine γ .	0.8 pt

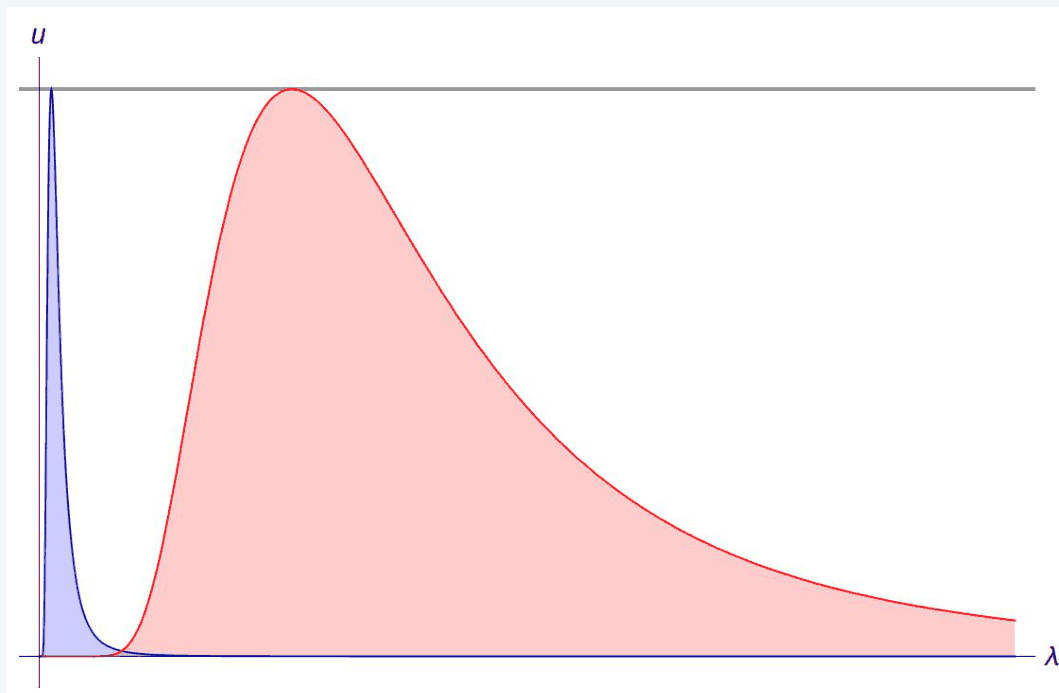


Figure 1 - The plot of $u(\lambda, T_E)$ (red) and $\gamma\tilde{u}_S(\lambda)$ (blue) versus λ

B. The Greenhouse Effect

In this part, we introduce a simple model in which the Earth's atmosphere is modeled as a thin layer at a small distance above the Earth's surface so that the difference between the area of the atmosphere's layer and the area of the Earth's surface can be neglected (see figure 2). In what follows assume that the major part of the thermal radiation from the Earth and the Sun are emitted at wavelengths near the $\lambda_{\max}$ for each one. Also assume that the "atmosphere layer" reflects a fraction $r_A = 0.255$ of the **visible-ultraviolet** radiation incident from above or below, and completely transmits the rest. Assume that the atmosphere does not reflect any part of the **infrared** radiation, however, it absorbs a fraction ε of the **infrared** radiation and transmits the rest. This behavior, known as the greenhouse effect, changes the average temperature of the Earth. The Earth's surface, on the other hand, reflects a fraction r_E of the **visible-ultraviolet** radiation and absorbs the rest of this radiation and all the **infrared** radiation.

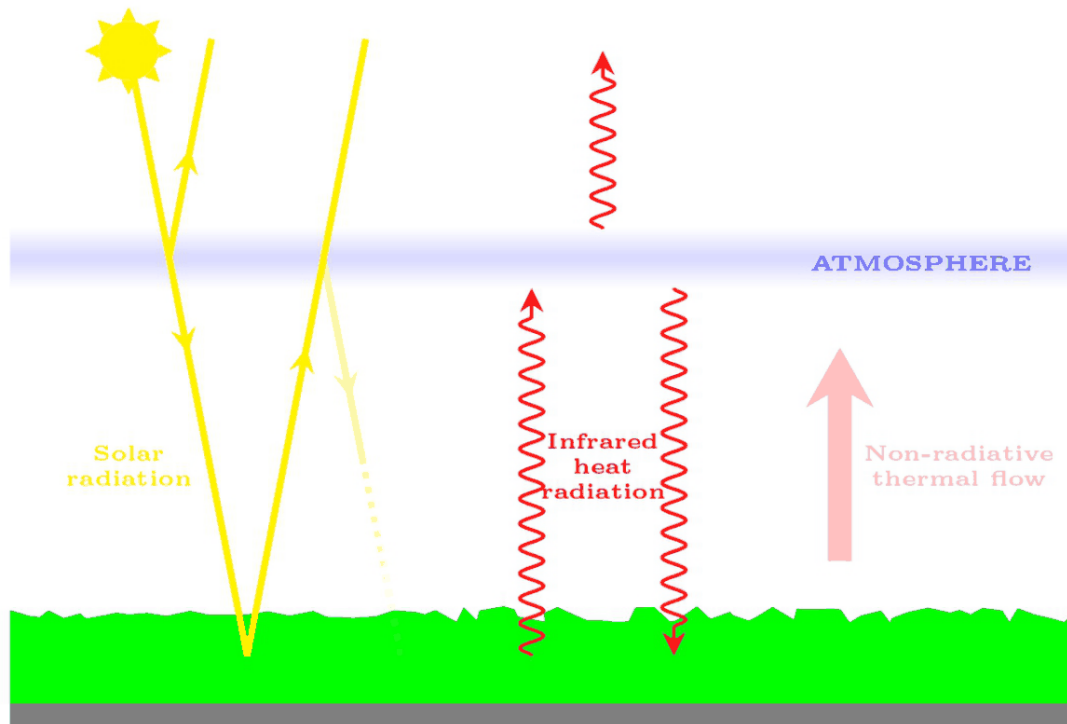


Figure 2 - Thermal flows between the Earth and the atmosphere

Part	Task	Points
B-1	Assume that $\varepsilon = 1$ and $r_E = 0$, and calculate the Earth's temperature T_E and the atmosphere's temperature T_A .	1.0 pt

Now assume that $r_E \neq 0$. In this case, the combined system of "Earth + atmosphere" reflects a different fraction of the solar radiation, called "albedo" and denoted by α .

Part	Task	Points
B-2	Determine the albedo, α , in terms of r_E and r_A . Then calculate its numerical value assuming $r_E = 0.102$ (and $r_A = 0.255$).	1.6 pt
B-3	a) Express the Earth's temperature in terms of σ , α , S_0 , and ε . b) Using the given data and the calculated albedo, find the numerical value of ε which leads to the current average temperature of $T_E = 288$ K for the Earth.	1.0 pt
B-4	Find $\frac{dT_E}{d\varepsilon}$ and determine by how much the Earth's temperature increases if ε increases by one percent.	0.8 pt

Assume $T_A = 245$ K and $T_E = 288$ K. These values come from real data and may differ from the results which you have obtained in the previous tasks. Now suppose that a non-radiative (e. g. convective) thermal flow $J_{NR} = k(T_E - T_A)$ is maintained from the Earth to the atmosphere, where k is a constant. The quantity, J_{NR} , is the transmitted power per unit area.

Part	Task	Points
B-5	Calculate ε and k in terms of T_E , T_A , σ , α , and S_0 .	1.6 pt

Part	Task	Points
B-6	a) Differentiating the equations obtained in part B-5 with respect to ε, find the two algebraic equations satisfied by $\frac{dT_A}{d\varepsilon}$ and $\frac{dT_E}{d\varepsilon}$. b) Use these equations to find the numerical value of change in the Earth's temperature as a result of a one percent increase in the value of ε.	1.0 pt

Official English text of the problem.

Answers

Define the globally averaged absorbed solar flux by

$$Q = \frac{(1 - \alpha)S_0}{4}.$$

A one-percent increase is taken to mean $\Delta\varepsilon = 0.01\varepsilon$. Numerical results use unrounded intermediate values.

- A-1

$$S_0 = \sigma T_S^4 \left(\frac{R_S}{d} \right)^2 = 1.35 \times 10^3 \text{ W m}^{-2}.$$

- A-2

$$T_E = \left(\frac{S_0}{4\sigma} \right)^{1/4} = T_S \sqrt{\frac{R_S}{2d}} = 278 \text{ K}.$$

- A-3

$$f(x) = 5(1 - e^{-x}) - x.$$

- A-4

$$x_m = 4.97, \quad b = \frac{hc}{x_m k_B} = 2.90 \times 10^{-3} \text{ m K}.$$

- A-5

$$\lambda_{\max,S} = \frac{b}{T_S} = 502 \text{ nm}, \quad \lambda_{\max,E} = \frac{b}{T_E} = 10.4 \text{ }\mu\text{m}.$$

- A-6

$$\gamma = \left(\frac{d}{R_S} \right)^2 \left(\frac{T_E}{T_S} \right)^5 = \frac{1}{4} \sqrt{\frac{R_S}{2d}} = 1.20 \times 10^{-2}.$$

• B-1

$$T_E = \left[\frac{(1 - r_A)S_0}{2\sigma} \right]^{1/4} = 307 \text{ K},$$

$$T_A = \left[\frac{(1 - r_A)S_0}{4\sigma} \right]^{1/4} = 258 \text{ K}.$$

• B-2

$$\alpha = r_A + \frac{r_E(1 - r_A)^2}{1 - r_A r_E} = 0.313.$$

• B-3(a)

$$T_E = \left[\frac{(1 - \alpha)S_0}{4\sigma(1 - \varepsilon/2)} \right]^{1/4}.$$

• B-3(b)

$$\varepsilon = 2 - \frac{(1 - \alpha)S_0}{2\sigma T_E^4} = 0.809.$$

• B-4

$$\frac{dT_E}{d\varepsilon} = \frac{T_E}{4(2 - \varepsilon)} = 60.4 \text{ K},$$

$$\Delta T_E \simeq 0.489 \text{ K}.$$

The temperature change is the first-order result; the exact finite change is 0.491 K.

• B-5

$$\varepsilon = \frac{\sigma T_E^4 - Q}{\sigma(T_E^4 - T_A^4)} = 0.849,$$

$$k = \frac{(\sigma T_E^4 - Q)(2T_A^4 - T_E^4)}{(T_E^4 - T_A^4)(T_E - T_A)} = 0.365 \text{ W m}^{-2} \text{ K}^{-1}.$$

• B-6(a) With $X = dT_E/d\varepsilon$ and $Y = dT_A/d\varepsilon$,

$$(4\sigma T_E^3 + k)X - (4\varepsilon\sigma T_A^3 + k)Y = \sigma T_A^4,$$

$$-(4\varepsilon\sigma T_E^3 + k)X + (8\varepsilon\sigma T_A^3 + k)Y = \sigma(T_E^4 - 2T_A^4).$$

• B-6(b)

$$X = 61.7 \text{ K}, \quad Y = 47.8 \text{ K}, \quad \Delta T_E \simeq 0.524 \text{ K}.$$

The temperature change is the first-order result; the exact finite change is 0.527 K.

Solution

A-1

The Sun's total radiated power follows from the Stefan–Boltzmann law:

$$L_S = 4\pi R_S^2 \sigma T_S^4.$$

Conservation of radiative power distributes this luminosity over a sphere of radius d . Consequently,

$$S_0 = \frac{L_S}{4\pi d^2} = \sigma T_S^4 \left(\frac{R_S}{d} \right)^2.$$

Substitution gives

$$S_0 = 1353.076 \dots \text{ W m}^{-2} = \boxed{1.35 \times 10^3 \text{ W m}^{-2}}.$$

The same geometrical dilution applies at every wavelength:

$$\tilde{u}_S(\lambda) = \left(\frac{R_S}{d} \right)^2 u(\lambda, T_S).$$

A-2

Let R_E denote the Earth's radius. The Earth intercepts sunlight over its projected area πR_E^2 , while its entire surface $4\pi R_E^2$ emits thermal radiation.

For a blackbody Earth in equilibrium,

$$\pi R_E^2 S_0 = 4\pi R_E^2 \sigma T_E^4.$$

Thus

$$T_E = \left(\frac{S_0}{4\sigma} \right)^{1/4} = T_S \sqrt{\frac{R_S}{2d}} = 277.920 \dots \text{ K}.$$

Therefore,

$$\boxed{T_E = 278 \text{ K}}.$$

A-3

At fixed T , introduce

$$x = \frac{hc}{\lambda k_B T}.$$

Logarithmic differentiation of Planck's law gives

$$\frac{1}{u} \frac{\partial u}{\partial \lambda} = \frac{1}{\lambda} \left[-5 + \frac{x e^x}{e^x - 1} \right].$$

At a finite-wavelength maximum this derivative vanishes, so

$$5 = \frac{x}{1 - e^{-x}}.$$

Hence a suitable function is

$$\boxed{f(x) = 5(1 - e^{-x}) - x}.$$

The algebraic root $x = 0$ corresponds to $\lambda \rightarrow \infty$, rather than a finite-wavelength maximum. The required root is positive and nonzero.

A-4

The nonzero root can be obtained, for example, using Newton's method:

$$x_{n+1} = x_n - \frac{5(1 - e^{-x_n}) - x_n}{5e^{-x_n} - 1}.$$

Starting at $x_0 = 5$ gives

$$x_m = 4.965114 \dots$$

Since $x_m = hc/(\lambda_{\max} k_B T)$,

$$\lambda_{\max} T = \frac{hc}{x_m k_B} = b.$$

Using the supplied constants,

$$b = \frac{1.24 \times 10^3}{(4.965114 \dots)(8.62 \times 10^{-5})} \text{ nm K} = 2.897244 \dots \times 10^{-3} \text{ m K}.$$

Therefore,

$$x_m = 4.97, \quad b = 2.90 \times 10^{-3} \text{ m K}.$$

Planck's function tends to zero at both wavelength endpoints, and this is its unique finite positive stationary point, so it is the maximum.

A-5

Applying Wien's displacement law to each body,

$$\lambda_{\max, S} = \frac{b}{T_S} = 502.122 \dots \text{ nm},$$

$$\lambda_{\max, E} = \frac{b}{T_E} = 10.4247 \dots \mu\text{m}.$$

Thus

$$\lambda_{\max, S} = 502 \text{ nm}, \quad \lambda_{\max, E} = 10.4 \mu\text{m}.$$

The solar peak is in the visible range, while the Earth's peak is in the infrared.

A-6

At the maximum, $\lambda_{\max} = b/T$ and the exponential argument equals x_m . Therefore,

$$u_{\max}(T) = \frac{2\pi hc^2}{b^5(e^{x_m} - 1)} T^5.$$

Thus the peak spectral power per unit wavelength scales as T^5 .

Equality of the two plotted peak heights requires

$$\gamma \left(\frac{R_S}{d} \right)^2 u_{\max}(T_S) = u_{\max}(T_E),$$

which gives

$$\gamma = \left(\frac{d}{R_S} \right)^2 \left(\frac{T_E}{T_S} \right)^5.$$

From A-2,

$$\left(\frac{T_E}{T_S} \right)^4 = \frac{1}{4} \left(\frac{R_S}{d} \right)^2.$$

Consequently,

$$\gamma = \frac{T_E}{4T_S} = \frac{1}{4} \sqrt{\frac{R_S}{2d}} = \boxed{1.20 \times 10^{-2}}.$$

B-1

In part B, the stated spectral approximation treats solar radiation as visible-ultraviolet and thermal emission from the Earth and atmosphere as infrared. The atmosphere absorbs no solar radiation. By Kirchhoff's law, its infrared emissivity equals ε , so each face emits a flux $\varepsilon\sigma T_A^4$. The Earth's infrared emissivity is unity.

Here $\varepsilon = 1$ and $r_E = 0$. The globally averaged solar flux absorbed at the surface is

$$Q_1 = \frac{(1 - r_A)S_0}{4}.$$

Atmospheric equilibrium requires

$$\sigma T_E^4 = 2\sigma T_A^4,$$

because the atmosphere absorbs the Earth's upward emission and emits in both directions.

The surface balance is

$$Q_1 + \sigma T_A^4 = \sigma T_E^4.$$

Combining the equations,

$$\sigma T_A^4 = Q_1, \quad \sigma T_E^4 = 2Q_1.$$

Therefore,

$$T_A = \left[\frac{(1 - r_A)S_0}{4\sigma} \right]^{1/4} = \boxed{258 \text{ K}},$$

$$T_E = \left[\frac{(1 - r_A)S_0}{2\sigma} \right]^{1/4} = \boxed{307 \text{ K}}.$$

B-2

Normalize the incoming solar power to unity. The atmosphere immediately reflects a fraction r_A . The fraction escaping to space after one reflection from the surface is

$$(1 - r_A)r_E(1 - r_A).$$

Each additional round trip between the surface and atmosphere multiplies the remaining reflected power by $r_A r_E$.

Summing all contributions,

$$\begin{aligned}\alpha &= r_A + r_E(1 - r_A)^2 \sum_{n=0}^{\infty} (r_A r_E)^n \\ &= r_A + \frac{r_E(1 - r_A)^2}{1 - r_A r_E}.\end{aligned}$$

Equivalently,

$$1 - \alpha = \frac{(1 - r_A)(1 - r_E)}{1 - r_A r_E}.$$

With the given reflectivities,

$$\alpha = 0.313124\dots, \quad \boxed{\alpha = 0.313}.$$

All solar power that is not reflected is absorbed by the surface.

B-3

(a) Define

$$Q = \frac{(1 - \alpha)S_0}{4}.$$

Without non-radiative heat transport, the surface and atmospheric balances are

$$\begin{aligned}Q + \varepsilon \sigma T_A^4 &= \sigma T_E^4, \\ \varepsilon \sigma T_E^4 &= 2\varepsilon \sigma T_A^4.\end{aligned}$$

For $\varepsilon > 0$, the second equation gives

$$T_A^4 = \frac{T_E^4}{2}.$$

Substituting into the first equation,

$$Q = \sigma T_E^4 \left(1 - \frac{\varepsilon}{2}\right).$$

Hence

$$\boxed{T_E = \left[\frac{(1 - \alpha)S_0}{4\sigma(1 - \varepsilon/2)} \right]^{1/4}}.$$

This expression also extends continuously to $\varepsilon = 0$.

(b) Rearranging for the absorptivity,

$$\varepsilon = 2 \left(1 - \frac{Q}{\sigma T_E^4}\right) = 2 - \frac{(1 - \alpha)S_0}{2\sigma T_E^4}.$$

The unrounded fluxes are

$$Q = 232.348778\dots \text{ W m}^{-2}, \quad \sigma(288 \text{ K})^4 = 390.079395\dots \text{ W m}^{-2}.$$

Thus

$$\varepsilon = 0.808710\dots, \quad \boxed{\varepsilon = 0.809}.$$

B-4

Hold S_0 and α fixed. Logarithmic differentiation of the B-3 result gives

$$\frac{1}{T_E} \frac{dT_E}{d\varepsilon} = \frac{1}{8(1 - \varepsilon/2)} = \frac{1}{4(2 - \varepsilon)}.$$

Therefore,

$$\boxed{\frac{dT_E}{d\varepsilon} = \frac{T_E}{4(2 - \varepsilon)}}.$$

At the B-3 equilibrium,

$$\frac{dT_E}{d\varepsilon} = 60.4387 \dots \text{ K} = \boxed{60.4 \text{ K}}.$$

For a relative one-percent increase,

$$\Delta\varepsilon = 0.01\varepsilon.$$

To first order,

$$\Delta T_E \simeq \frac{dT_E}{d\varepsilon} \Delta\varepsilon = \frac{0.01\varepsilon T_E}{4(2 - \varepsilon)} = \boxed{0.489 \text{ K}}.$$

For comparison, evaluating the finite change directly gives

$$\Delta T_E^{\text{finite}} = T_E \left[\left(\frac{2 - \varepsilon}{2 - 1.01\varepsilon} \right)^{1/4} - 1 \right] = \boxed{0.491 \text{ K}}.$$

B-5

Include the upward non-radiative flux

$$J_{\text{NR}} = k(T_E - T_A).$$

The surface loses this energy, while the atmosphere gains it. Their equilibrium equations become

$$Q + \varepsilon\sigma T_A^4 = \sigma T_E^4 + k(T_E - T_A), \quad (2)$$

$$\varepsilon\sigma T_E^4 + k(T_E - T_A) = 2\varepsilon\sigma T_A^4. \quad (3)$$

For brevity, write

$$E = \sigma T_E^4, \quad A = \sigma T_A^4.$$

Adding the two energy balances eliminates the internal heat transfer:

$$Q = (1 - \varepsilon)E + \varepsilon A.$$

The right-hand side is the radiation escaping to space: transmitted surface radiation plus upward atmospheric emission. Solving,

$$\boxed{\varepsilon = \frac{E - Q}{E - A} = \frac{\sigma T_E^4 - Q}{\sigma(T_E^4 - T_A^4)}}.$$

Equation (3) gives

$$k(T_E - T_A) = \varepsilon(2A - E).$$

Therefore,

$$\boxed{k = \frac{\varepsilon\sigma(2T_A^4 - T_E^4)}{T_E - T_A} = \frac{(\sigma T_E^4 - Q)(2T_A^4 - T_E^4)}{(T_E^4 - T_A^4)(T_E - T_A)}}.$$

At $T_E = 288 \text{ K}$ and $T_A = 245 \text{ K}$,

$$E = 390.079395 \dots \text{ W m}^{-2}, \quad A = 204.290135 \dots \text{ W m}^{-2}.$$

Consequently,

$$\boxed{\varepsilon = 0.849, \quad k = 0.365 \text{ W m}^{-2} \text{ K}^{-1}}.$$

The resulting heat flux is

$$J_{\text{NR}} = 15.7 \text{ W m}^{-2},$$

which is positive in the stipulated upward direction.

B-6

(a) When ε varies, hold S_0 , α , and k fixed, and allow both temperatures to change. Set

$$X = \frac{dT_E}{d\varepsilon}, \quad Y = \frac{dT_A}{d\varepsilon}.$$

Differentiating equation (2) gives

$$\sigma T_A^4 + 4\varepsilon \sigma T_A^3 Y = 4\sigma T_E^3 X + k(X - Y).$$

Hence

$$\boxed{(4\sigma T_E^3 + k)X - (4\varepsilon \sigma T_A^3 + k)Y = \sigma T_A^4}.$$

Differentiating equation (3) gives

$$\sigma T_E^4 + 4\varepsilon \sigma T_E^3 X + k(X - Y) = 2\sigma T_A^4 + 8\varepsilon \sigma T_A^3 Y.$$

Thus

$$\boxed{-(4\varepsilon \sigma T_E^3 + k)X + (8\varepsilon \sigma T_A^3 + k)Y = \sigma(T_E^4 - 2T_A^4)}.$$

(b) To give an explicit parametric expression, define

$$p = 4\sigma T_E^3, \quad q = 4\sigma T_A^3,$$

and retain $E = \sigma T_E^4$, $A = \sigma T_A^4$. The determinant of the system is

$$D = (p + k)(2\varepsilon q + k) - (\varepsilon q + k)(\varepsilon p + k) = \varepsilon(2 - \varepsilon)pq + k[(1 - \varepsilon)p + \varepsilon q].$$

Eliminating Y gives

$$\boxed{X = \frac{\varepsilon q E + k(E - A)}{D}}.$$

Similarly,

$$Y = \frac{(p + k)(E - 2A) + (\varepsilon p + k)A}{D}.$$

Using the equilibrium values from B-5,

$$X = 61.7352 \dots \text{ K}, \quad Y = 47.7734 \dots \text{ K}.$$

For $\Delta\varepsilon = 0.01\varepsilon$,

$$\Delta T_E \simeq X \Delta\varepsilon = (61.7352 \dots)(0.01)(0.84897597 \dots) \text{ K}.$$

Therefore,

$$\boxed{\Delta T_E \simeq 0.524 \text{ K}}.$$

Solving the original nonlinear balances at $\varepsilon_{\text{new}} = 1.01\varepsilon$, with the same k , gives the finite change

$$\Delta T_{\text{E}}^{\text{finite}} = 0.527 \text{ K}.$$

The solution was checked against the official solution.

Problem 2: Trapping Ions and Cooling Atoms

In recent decades, trapping and cooling atoms and ions has been a fascinating topic for physicists, with several Nobel prizes awarded for work in this area. In the first part of this question, we will explore a technique for trapping ions, known as the "Paul trap". Wolfgang Paul and Hans Dehmelt received one half of the 1989 Nobel Prize in Physics for this work. Next, we investigate the Doppler cooling technique, one of the works cited in the press release for the 1997 Nobel Prize in Physics awarded to Steven Chu, Claude Cohen-Tannoudji, and William Daniel Phillips "for developments of methods to cool and trap atoms with laser light".

A. The Paul Trap

It is known that with electrostatic fields, it is not possible to create a stable equilibrium for a charged particle. Therefore, creating a stable equilibrium point for ions requires more sophisticated techniques. The Paul trap is one of these techniques.

Consider a ring of charge with a radius R and a uniform positive linear charge density λ . A positive point charge Q with mass m is placed at the center of the ring.

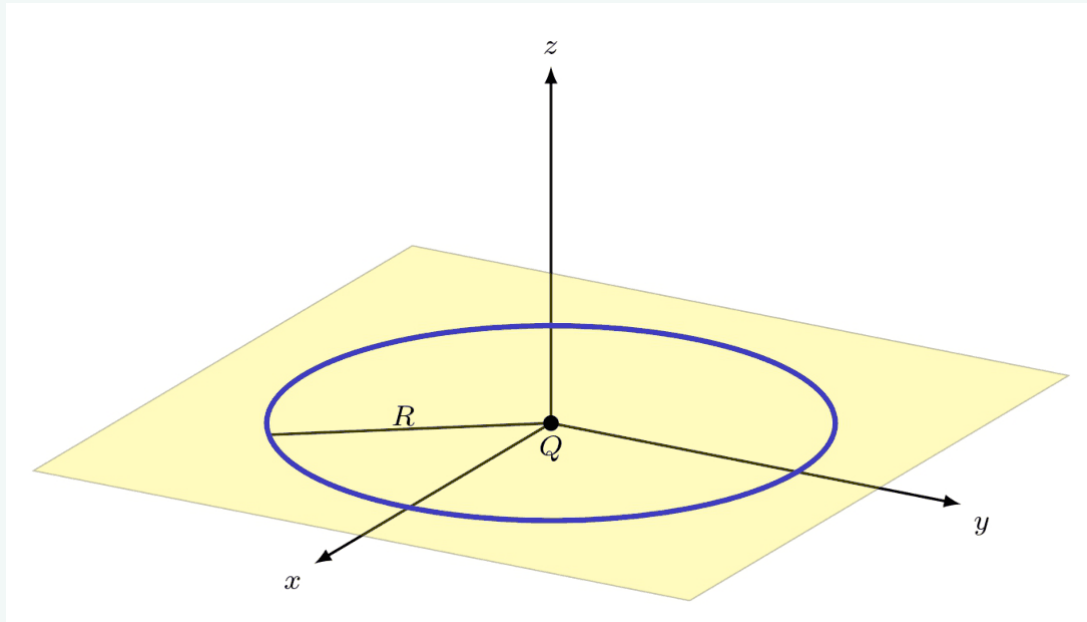


Figure 1 - A positively charged ring with a uniform linear charge density λ and radius R : the origin of the coordinate system is at the center of the ring.

A-
1

- In cartesian coordinates (x, y, z) , obtain the electric field due to the charged ring in the vicinity of the ring's center to the first order in x/R , y/R , and z/R .
- Find the angular frequency of small oscillations of the charged particle around the center of the ring in the directions for which a stable equilibrium exists.

1.5
pt

In order to trap the charge Q fully, we would like to apply alternating fields to produce a dynamic equilibrium. Assume that the charge density is $\lambda = \lambda_0 + u \cos \Omega t$ in which λ_0 , u , and Ω are adjustable. We shall ignore radiative effects. Then the equation of motion for small displacements from the center of the ring, along the direction perpendicular to the plane of the ring will turn out to be:

$$\ddot{z} = (+k^2 + a\Omega^2 \cos \Omega t)z \quad (1)$$

A-2	Write a and k in terms of the known parameters.	0.4 pt
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We would like to obtain an approximate solution to Equation (1) by making the following simplifying assumptions: $a \ll 1$, $\Omega \gg k$, and $a\Omega^2 \gg k^2$. With these assumptions, it can be shown that the solution of this equation can be split into two parts: $z(t) = p(t) + q(t)$, where $p(t)$ is a slowly varying component and $q(t)$ is a small-amplitude rapidly-varying component with a mean value of zero. In other words, $p(t)$ may be assumed constant over a few oscillations of $q(t)$ (see Figure 2).

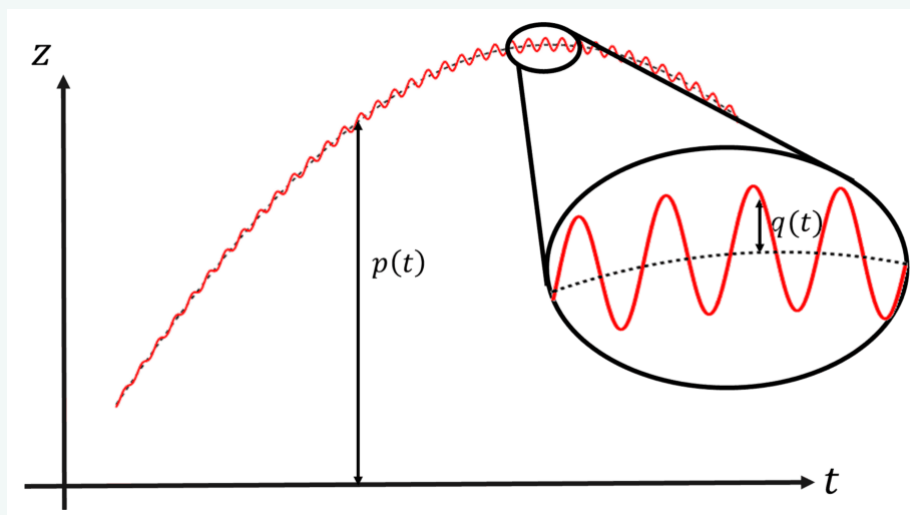


Figure 2 - A typical solution for the equation of motion of the charged particle: $p(t)$ gives the overall motion, and $q(t)$ represents small oscillations around this trajectory. The ellipse on the right is a magnification of a part of this trajectory.

A-3	a) Using the approximations stated above, find the equation of motion for $q(t)$ in terms of a , Ω , and p .	1.8 pt
	b) Find the solution of this equation by considering appropriate initial conditions corresponding to the required properties of this function.	
A-4	a) Using the mean effect of the rapidly varying component and obtain an effective equation of motion for $p(t)$.	1.5 pt
	b) Investigate the stability of the equilibrium point and find the condition for a stable equilibrium.	

Assume that $\lambda_0 = 8 \times 10^{-9} \text{ C/m}$ and $R = 10 \text{ cm}$. We would like to use this device to trap a singly ionized atom 100 times heavier than a hydrogen atom.

A-5	Calculate k . Assume $a = 0.04$ and estimate the smallest frequency required to stabilize the motion of this ion. Use the data given at the end of the question.	0.4 pt
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B. Doppler Cooling

It may be necessary to cool a trapped atom or ion. Assume that a trapped atom of mass m , has two energy levels with an energy difference of $E_0 = \hbar\omega_A$. Electrons in the lower level may absorb a photon and jump to the higher level, but after a period τ they will return to the lower level and emit a photon with a frequency predominantly within $[\omega_A - \Gamma, \omega_A + \Gamma]$.

B-1 Use the Heisenberg's uncertainty principle to find Γ . 0.5 pt

With a similar reasoning, when we shine a laser light on the trapped atom, if the angular frequency of the laser, ω_L , falls in the interval $[\omega_A - \Gamma, \omega_A + \Gamma]$, the atom may absorb the photon. Assume that the frequency ω_L of the laser light is slightly lower than ω_A . For a particular device, the rate of photon absorption by an atom in the reference frame of the atom is given in Figure 3. The absorbed photon is then re-emitted in a random direction. To make things simple, we consider the problem in one dimension, i.e. we assume that the atoms can only move in the x -direction and the laser light shines on them both from the left and from the right. In the atom's reference frame, the light has a higher or lower frequency due to the motion of the atoms. Since the velocity v of the atoms is very small, we only include terms of order v/c and ignore all the higher-order terms. Moreover, we have $m \gg \hbar\omega_A/c^2$ so that the velocity of the atom nearly does not change after absorbing the photon. Also, the change in frequency due to the Doppler effect is so small compared to $\omega_A - \omega_L$, that the function for s in the diagram of Figure 3 may be approximated by the following linear function:

$$s(\omega) = s_L + \alpha(\omega - \omega_L)$$

where s is the number of absorbed photons per unit of time, s_L is the value of s for $\omega = \omega_L$, and α is the slope of the line tangent to the curve at ω_L . The frequency of the re-emitted photon is almost equal to the frequency of the incident photon, but it is emitted with equal probability in the positive or negative x -direction. In fact, up to the order considered here the two frequencies are identical. Note that we are considering the whole process in the atom's reference frame.

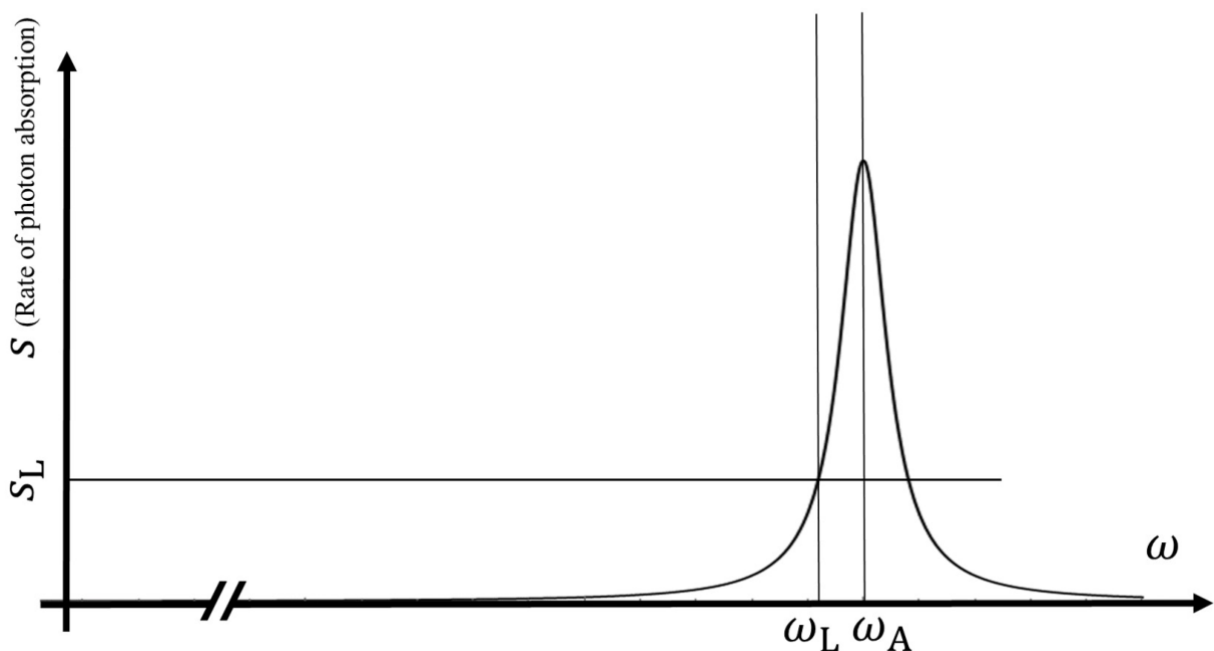


Figure 3 - The rate of photon absorption as a function of the frequency for a particular trap: the frequency corresponding to the energy difference between the two atomic levels is indicated by ω_A and the slightly smaller frequency of the laser is indicated by ω_L .

B-2	a) Assume that the trapped atom is moving with a velocity, $v = v_x$ in the lab frame. In the frame of reference of the atom, calculate the collision rate of the photons, incident from each of the two directions, with the atoms (denoted by s_+ and s_-) and the rate of absorption of momentum in each direction (denoted by π_+ and π_-). b) Determine the effective force on the atom as a function of v, $k_L = \omega_L/c$, $\hbar$, and α, in the reference frame of the laboratory. Assume $s_L \ll \alpha\omega_L$,	1.7 pt
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We would like to find the lowest temperature that can be achieved using this technique. Assume that the velocity of a particular atom has been reduced to zero exactly, and at this very moment it absorbs a photon (incident from any of the two directions), and re-emits the photon randomly in any of the two directions, with almost the same frequency. Assume that this process happens once every τ units of time.

B-3	Considering the momentum of the atom after such a process for the two possible outcomes, calculate the average power absorbed by the atom.	1.0 pt
B-4	Consider the force calculated in Task B-2 and calculate the output power. Then, calculate the average value of v^2 at equilibrium. Using your knowledge of the kinetic theory of gases estimate the temperature of the atoms.	0.8 pt
B-5	Estimate this temperature, for an atom 100 times heavier than a hydrogen atom. Assume that $\omega_L = 2 \times 10^{16}$ rad/s, $\tau = 5 \times 10^{-9}$ s, and $\alpha = 4$.	0.4 pt

mass of hydrogen atom:	$m_H = 1.674 \times 10^{-27}$ kg
charge of an electron:	$e = 1.602 \times 10^{-19}$ C
permittivity of free space:	$\epsilon_0 = 8.854 \times 10^{-12}$ F/m
Boltzmann constant:	$k_B = 1.381 \times 10^{-23}$ J/K
Planck constant:	$\hbar = 1.055 \times 10^{-34}$ J · s

Official English text of the problem.

Answers

• A-1(a):

$$\mathbf{E} \simeq \frac{\lambda}{4\epsilon_0 R^2} (-x \hat{\mathbf{x}} - y \hat{\mathbf{y}} + 2z \hat{\mathbf{z}}).$$

• A-1(b): All directions in the ring's plane are stable, with

$$\omega_x = \omega_y = \sqrt{\frac{Q\lambda}{4\epsilon_0 m R^2}}.$$

The z -direction is unstable.

- **A-2:**

$$a = \frac{Qu}{2\epsilon_0 m R^2 \Omega^2}, \quad k = \sqrt{\frac{Q\lambda_0}{2\epsilon_0 m R^2}}.$$

- **A-3(a):** $\ddot{q} \simeq a\Omega^2 p \cos \Omega t$.

- **A-3(b):** $q(t) \simeq -ap(t) \cos \Omega t$. For p treated as constant during a fast cycle, the appropriate initial conditions are $q(0) = -ap$, $\dot{q}(0) = 0$.

- **A-4(a):**

$$\ddot{p} + \left(\frac{a^2 \Omega^2}{2} - k^2 \right) p = 0.$$

- **A-4(b):** Stable trapping requires, within the stated approximation,

$$a^2 \Omega^2 > 2k^2.$$

The slow axial angular frequency is $\sqrt{a^2 \Omega^2 / 2 - k^2}$.

- **A-5:**

$$k \simeq 2.1 \times 10^5 \text{ s}^{-1}.$$

The estimated threshold is

$$\Omega_{\text{th}} \simeq 7.4 \times 10^6 \text{ rad/s}, \quad f_{\text{th}} \simeq 1.2 \text{ MHz}.$$

Choose a frequency above this threshold.

- **B-1:** The linewidth scale is $\Gamma \sim 1/\tau$; using the conventional lifetime half-width,

$$\Gamma \simeq \frac{1}{2\tau}.$$

- **B-2(a):** Let $+$ denote photons propagating along $+x$, and let $\pi_{\pm}$ be signed x -momentum transfer rates. Then

$$s_{\pm} = s_L \mp \alpha \omega_L \frac{v}{c},$$

$$\pi_{\pm} = \pm \hbar k_L \left[s_L \mp (s_L + \alpha \omega_L) \frac{v}{c} \right].$$

- **B-2(b):**

$$F_x = -2\alpha \hbar k_L^2 v.$$

- **B-3:**

$$P_{\text{in}} = \frac{\hbar^2 k_L^2}{m\tau}.$$

- **B-4:** The positive rate of energy removal is

$$P_{\text{out}} = 2\alpha\hbar k_L^2 v^2.$$

At equilibrium,

$$\langle v^2 \rangle = \frac{\hbar}{2\alpha m \tau}, \quad T = \frac{\hbar}{2\alpha k_B \tau}.$$

- **B-5:**

$$T \simeq 1.9 \times 10^{-4} \text{ K} = 0.19 \text{ mK}.$$

Solution

A-1(a)

The ring lies in the xy -plane and carries total charge $2\pi R\lambda$. By Coulomb's law, its electric field on the symmetry axis is

$$E_z(0, 0, z) = \frac{1}{4\pi\epsilon_0} \frac{2\pi R\lambda z}{(R^2 + z^2)^{3/2}}.$$

For $|z| \ll R$,

$$E_z(0, 0, z) \simeq \frac{\lambda}{2\epsilon_0 R^2} z.$$

Reflection and rotational symmetry require the general linear field near the center to have the form

$$E_x = Cx, \quad E_y = Cy, \quad E_z = Dz.$$

The axial result gives $D = \lambda/(2\epsilon_0 R^2)$. There is no charge in the neighborhood of the center, so Gauss's law gives

$$\nabla \cdot \mathbf{E} = 2C + D = 0.$$

Consequently,

$$\mathbf{E} \simeq \frac{\lambda}{4\epsilon_0 R^2} (-x \hat{\mathbf{x}} - y \hat{\mathbf{y}} + 2z \hat{\mathbf{z}}).$$

A-1(b)

Newton's second law in the plane gives

$$m\ddot{x} = -\frac{Q\lambda}{4\epsilon_0 R^2} x, \quad m\ddot{y} = -\frac{Q\lambda}{4\epsilon_0 R^2} y.$$

These are harmonic-oscillator equations. Thus every direction in the plane has the same angular frequency,

$$\omega_x = \omega_y = \sqrt{\frac{Q\lambda}{4\epsilon_0 m R^2}}.$$

Along the axis,

$$m\ddot{z} = \frac{Q\lambda}{2\epsilon_0 R^2} z.$$

For $Q, \lambda > 0$, this force points away from the center, so the axial equilibrium is unstable.

A-2

Substituting the time-dependent charge density into the axial equation gives

$$\ddot{z} = \frac{Q}{2\varepsilon_0 m R^2} (\lambda_0 + u \cos \Omega t) z.$$

Comparison with Equation (1) yields

$$k^2 = \frac{Q\lambda_0}{2\varepsilon_0 m R^2}, \quad a = \frac{Qu}{2\varepsilon_0 m R^2 \Omega^2}.$$

Here a is dimensionless and k has units of inverse time.

A-3(a)

Insert $z = p + q$ into the equation of motion:

$$\ddot{p} + \ddot{q} = k^2 p + k^2 q + a\Omega^2 p \cos \Omega t + a\Omega^2 q \cos \Omega t.$$

Over a fast oscillation, p is approximately constant. Since q has small amplitude, $a \ll 1$, and $a\Omega^2 \gg k^2$, the leading rapidly varying equation is

$$\ddot{q} \simeq a\Omega^2 p \cos \Omega t.$$

A-3(b)

Holding p constant during the integration gives

$$q(t) = -ap \cos \Omega t + C_1 t + C_2.$$

A rapidly oscillating component cannot contain a secular drift, so $C_1 = 0$. Its mean must vanish, which requires $C_2 = 0$. Hence

$$q(t) \simeq -ap(t) \cos \Omega t.$$

For frozen p , the corresponding initial conditions are

$$q(0) = -ap, \quad \dot{q}(0) = 0.$$

The amplitude of q is a times that of the local slow displacement, consistent with $a \ll 1$. Corrections involving the slow variation of p are neglected at this order.

A-4(a)

Average the full equation over a drive period, treating p as constant during that period. Since

$$\langle q \rangle = 0, \quad \langle \cos \Omega t \rangle = 0, \quad \langle \cos^2 \Omega t \rangle = \frac{1}{2},$$

we obtain

$$\ddot{p} = k^2 p + a\Omega^2 \langle q \cos \Omega t \rangle.$$

Using $q = -ap \cos \Omega t$,

$$\langle q \cos \Omega t \rangle = -\frac{ap}{2}.$$

Therefore,

$$\ddot{p} = \left(k^2 - \frac{a^2 \Omega^2}{2} \right) p.$$

Although k^2 is small compared with $a\Omega^2$, it must be retained here: the alternating field contributes to the mean motion only at order $a^2\Omega^2$.

A-4(b)

Define

$$\omega_s^2 = \frac{a^2 \Omega^2}{2} - k^2.$$

If $\omega_s^2 > 0$, the slow displacement oscillates:

$$p(t) = A \cos \omega_s t + B \sin \omega_s t,$$

and the complete approximate motion,

$$z(t) \simeq p(t) (1 - a \cos \Omega t),$$

remains bounded. Thus the axial stability condition is

$$a^2 \Omega^2 > 2k^2.$$

For $a > 0$, this is

$$\Omega > \frac{\sqrt{2} k}{a}.$$

Below this threshold, the slow equation has an exponentially growing solution. At equality, $\ddot{p} = 0$, so a nonzero slow velocity produces unbounded drift; equality does not provide confinement.

For completeness, the transverse equation is

$$\ddot{x} = - \left(\frac{k^2}{2} + \frac{a\Omega^2}{2} \cos \Omega t \right) x.$$

The same averaging procedure gives, for its slow component X ,

$$\ddot{X} + \left(\frac{k^2}{2} + \frac{a^2 \Omega^2}{8} \right) X = 0,$$

and identically for y . These directions are stable within the approximation, so the axial condition supplies the requirement for trapping in all three directions.

A-5

The ion has

$$Q = e, \quad m = 100m_H = 1.674 \times 10^{-25} \text{ kg}, \quad R = 0.10 \text{ m}.$$

Thus

$$k = \sqrt{\frac{(1.602 \times 10^{-19})(8 \times 10^{-9})}{2(8.854 \times 10^{-12})(1.674 \times 10^{-25})(0.10)^2}} = 2.079 \times 10^5 \text{ s}^{-1}.$$

Therefore,

$$k \simeq 2.1 \times 10^5 \text{ s}^{-1}.$$

For the specified $a = 0.04$, the estimated threshold angular frequency is

$$\Omega_{\text{th}} = \frac{\sqrt{2} k}{a} = 7.351 \times 10^6 \text{ rad/s.}$$

The corresponding ordinary frequency is

$$f_{\text{th}} = \frac{\Omega_{\text{th}}}{2\pi} = 1.170 \times 10^6 \text{ Hz.}$$

Hence,

$$\boxed{\Omega_{\text{th}} \simeq 7.4 \times 10^6 \text{ rad/s,} \quad f_{\text{th}} \simeq 1.2 \text{ MHz.}}$$

A frequency above this estimated threshold is required. At the threshold,

$$\frac{\Omega_{\text{th}}}{k} \simeq 35, \quad \frac{a\Omega_{\text{th}}^2}{k^2} = 50,$$

consistent with the assumed separation of scales.

B-1

The excited state has a characteristic lifetime τ . The energy–time uncertainty estimate is

$$\Delta E \tau \gtrsim \frac{\hbar}{2}.$$

Identifying the characteristic energy half-width as $\Delta E \simeq \hbar\Gamma$ gives

$$\Gamma \tau \gtrsim \frac{1}{2}.$$

For a width set by the finite lifetime, the usual estimate is therefore

$$\boxed{\Gamma \simeq \frac{1}{2\tau}.$$

The robust conclusion from the uncertainty principle is the scale $\Gamma \sim 1/\tau$; the numerical factor depends on how the linewidth is defined.

B-2(a)

Let $+$ label photons propagating along $+x$, and $-$ label photons propagating along $-x$. For an atom moving at velocity v , the first-order Doppler shifts give

$$\omega'_+ = \omega_L \left(1 - \frac{v}{c}\right), \quad \omega'_- = \omega_L \left(1 + \frac{v}{c}\right).$$

A photon traveling in the same direction as the atom is redshifted; one traveling against it is blueshifted.

The supplied function $s(\omega)$ already specifies the absorption rate in the atom's frame. Evaluating it at these frequencies gives

$$s_+ = s_L + \alpha(\omega'_+ - \omega_L) = s_L - \alpha\omega_L \frac{v}{c},$$

$$s_- = s_L + \alpha(\omega'_- - \omega_L) = s_L + \alpha\omega_L \frac{v}{c}.$$

Thus,

$$s_{\pm} = s_L \mp \alpha \omega_L \frac{v}{c}.$$

Each absorbed photon has signed momentum $\pm \hbar \omega'_{\pm}/c$. Defining $\pi_{\pm}$ as signed x -momentum transfer rates,

$$\pi_+ = \frac{\hbar \omega'_+}{c} s_+, \quad \pi_- = -\frac{\hbar \omega'_-}{c} s_-.$$

Expanding and keeping only first-order terms in v/c ,

$$\pi_+ = \hbar k_L \left[s_L - (s_L + \alpha \omega_L) \frac{v}{c} \right],$$

$$\pi_- = -\hbar k_L \left[s_L + (s_L + \alpha \omega_L) \frac{v}{c} \right].$$

B-2(b)

In the atom's frame, re-emission has zero mean momentum because the two emission directions are equally probable. Therefore the mean force is

$$F'_x = \pi_+ + \pi_- = -\frac{2\hbar \omega_L}{c^2} (s_L + \alpha \omega_L) v.$$

Within the approximation used here, the absorbed and emitted energies are equal in the atom's frame. The completed cycle therefore has zero net energy transfer in that frame when recoil is neglected in calculating the mean force. Transforming to the laboratory leaves the longitudinal force unchanged to the retained order.

Using $s_L \ll \alpha \omega_L$, we obtain

$$F_x \simeq -\frac{2\alpha \hbar \omega_L^2}{c^2} v = -2\alpha \hbar k_L^2 v.$$

The laser frequency lies on the rising side of the absorption curve, so $\alpha > 0$. The force therefore opposes the velocity and removes kinetic energy.

B-3

Let the incident photon propagate along $+x$ and have momentum magnitude

$$p_{\gamma} = \hbar k_L.$$

The atom initially has zero momentum. Momentum conservation gives the two equally probable outcomes:

Emission direction	Final atomic momentum	Final kinetic energy	Probability
Along $+x$, the incident direction	0	0	1/2
Along $-x$, opposite the incident direction	$2p_{\gamma}$	$2p_{\gamma}^2/m$	1/2

Thus the mean kinetic-energy gain per absorption–emission cycle is

$$\langle \Delta K \rangle = \frac{1}{2}(0) + \frac{1}{2} \frac{(2p_{\gamma})^2}{2m} = \frac{p_{\gamma}^2}{m}.$$

An incident photon traveling along $-x$ gives the same mean energy gain. Since there is one complete cycle every τ ,

$$P_{\text{in}} = \frac{\langle \Delta K \rangle}{\tau} = \frac{\hbar^2 k_L^2}{m\tau}.$$

B-4

The rate at which the damping force changes the atom's mechanical energy is

$$F_x v = -2\alpha \hbar k_L^2 v^2.$$

Hence the positive output power, meaning the rate of energy removal, is

$$P_{\text{out}} = 2\alpha \hbar k_L^2 v^2.$$

At thermal equilibrium, the mean cooling power balances recoil heating:

$$2\alpha \hbar k_L^2 \langle v^2 \rangle = \frac{\hbar^2 k_L^2}{m\tau}.$$

Cancelling common factors gives

$$\langle v^2 \rangle = \frac{\hbar}{2\alpha m\tau}.$$

The problem restricts motion to one dimension, so equipartition gives

$$\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k_B T.$$

Consequently,

$$T = \frac{\hbar}{2\alpha k_B \tau}.$$

The atomic mass and laser wave number cancel from the equilibrium temperature in this model.

B-5

Using the given values,

$$T = \frac{1.055 \times 10^{-34}}{2(4)(1.381 \times 10^{-23})(5 \times 10^{-9})} = 1.9098 \times 10^{-4} \text{ K}.$$

Therefore,

$$T \simeq 1.9 \times 10^{-4} \text{ K} = 0.19 \text{ mK} \simeq 190 \mu\text{K}.$$

For the specified mass $m = 1.674 \times 10^{-25} \text{ kg}$, the corresponding velocity variance is

$$\langle v^2 \rangle = 1.58 \times 10^{-2} \text{ m}^2/\text{s}^2, \quad v_{\text{rms}} \simeq 0.126 \text{ m/s}.$$

The solution was checked against the official solution.

Problem 3: Black Widow Pulsar

A significant number of the observed stars are binaries. One or both of the stars may be neutron stars rotating with a high angular velocity and emitting electromagnetic waves; such stars are called pulsars. Sometimes a companion star is an expansive mass of gas that gradually falls down onto the neutron star and causes its mass to increase (Figure 1-a). In this way, a neutron star

gradually swallows up a portion of the mass of its companion star. For this reason, the neutron star has been compared to a black widow (or redback spider), a female spider which eats its mate after mating. The heating of the gas falling down onto the black widow generates radiation which can be observed. The heaviest neutron stars often are black widows and they serve as natural laboratories for testing fundamental physics. Figure 1-b shows the picture of the companion of the neutron star PSR J2215+5135, taken by the 3.4-meter optical telescope of the Iranian National Observatory. No neutron star can be seen in this image and the observed light is due to its companion.

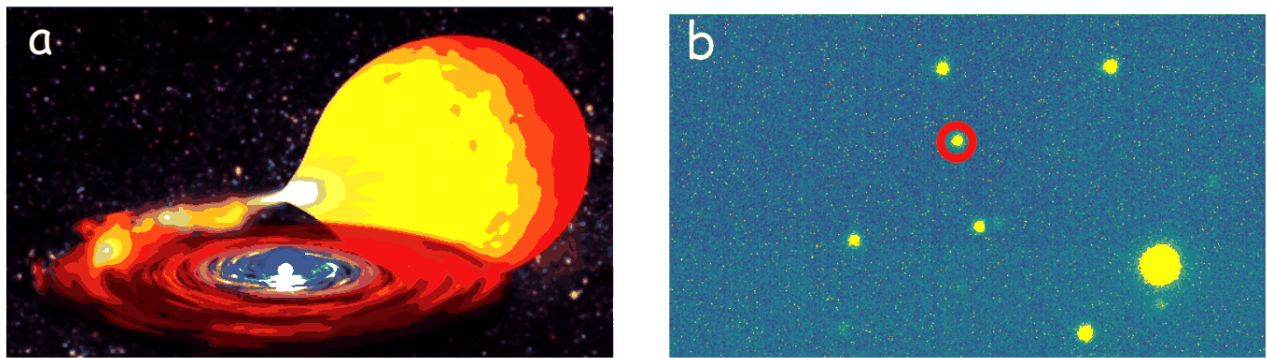


Figure 1 - (a) The falling gases of the companion star onto the neutron star - (b) The companion of the neutron star PSR J2215+5135

A. A Binary System

Consider a simple model in which the black widow and its companion star, are represented by two point masses M_1 and M_2 moving on a circular orbit around their center of mass. To investigate the dynamics of this system, consider a rotating coordinate system in which the two bodies are stationary. Take the center of mass to be the origin of the coordinate system. Assume that the two point masses lie on the x -axis on both sides of the origin at a distance a from each other, and that M_1 lies on the negative x -axis. At an arbitrary point (x, y) in the plane of motion, the effective potential $\varphi(x, y)$ for a unit test mass is the sum of the gravitational potentials of the two point masses plus the centrifugal potential.

A-1	Write $\varphi(x, y)$ in terms of M_1 , M_2 , G , and a .	1.0 pt
A-2	Assuming $M_1 > M_2$, plot the function $\varphi(x, 0)$ qualitatively.	0.7 pt

Suppose (just for task A-3) $M_2 = M_1/3$ and assume that M_2 is surrounded by a rarefied gas of very low density. The mass of this gas is insignificant and we ignore its gravitational effects. If the size of this gas envelope becomes greater than a specific limit, the gas will overflow onto M_1 . Suppose the overflow occurs through $x = x_0$ on the x -axis.

A-3	Find the numerical value of $\frac{x_0}{a}$, up to two significant figures. You may use the calculator.	0.5 pt
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Take the rotational period of the stars around their center of mass to be P . Assume that mass flows from M_2 to M_1 at a very small rate of $dM_1/dt = \beta$. This rate is so small that in each period of rotation, the distance between the two stars can be assumed to be constant. However, after a long period of time, the distance between the two stars changes, while the motion remains circular.

A-4	Calculate the rate of change of a and P in terms of β , M_1 , M_2 , G , and a .	0.6 pt
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The gas separated from M_2 forms a disk rotating around M_1 and heats up due to friction (Figure 1-a). As the gas loses energy, it spirals inward toward M_1 and finally falls onto it. In the steady state, the mass flows at the constant rate of β , from M_2 to the disc and from the disc onto M_1 . At the same time, the heated disk emits thermal radiation as a blackbody. This disk forms very close to the neutron star so the gravitational pull of the M_2 star can be ignored for the analysis of the disk's motion. Also, ignore the heat capacity of the gas.

A-5	Determine the temperature of the disc at distance r from the center of the star M_1 in terms of β , M_1 , G , and σ (Stefan-Boltzmann constant).	1.0 pt
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In the binary system PSR J2215+5135, the mass of the neutron star is $M_{\text{NS}} = 2.27 M_{\odot}$ and the mass of its companion star is $M_S = 0.33 M_{\odot}$, where $M_{\odot} = 1.98 \times 10^{30} \text{ kg}$ is the mass of the Sun. The rotational period is $P = 4.14 \text{ hr}$, and the Stefan-Boltzmann constant is $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$, and the gravitational constant is $G = 6.67 \times 10^{-11} \text{ m}^3/\text{kg s}^2$.

Assume that the mass flow rate to the neutron star is $\beta = \dot{M}_{\text{NS}} = 9 \times 10^{-10} M_{\odot} \text{ yr}^{-1}$.

A-6	Calculate the temperature of the disc at the radius $r = \frac{a}{10}$ in kelvins.	0.5 pt
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Assume that after a sudden explosion, the M_1 star ejects a part of its mass out of the binary system at a very high speed, and its mass becomes M'_1 . Take the magnitude of the velocity of M'_1 relative to M_2 to be v' after the explosion.

A-7	Determine the maximum value of v' , in terms of M'_1 , M_2 , G , and a , that allows the new binary system to stay bounded. Assuming that the explosion is isotropic, what is the minimum value of M'_1 for the binary system to remain bounded?	0.7 pt
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B. Analysis of the Stability of a Star

In this part we study the stability of a single star. Consider a star containing a specific kind of matter with the equation of state $p = K\rho^\gamma$ where K and γ are constants. Let $p(r)$ and $\rho(r)$ be the pressure and density at a distance r from the center of the star, respectively. The pressure and density at the center of the star are p_c and ρ_c , respectively. In all tasks of the part B, take all outward vectors to be positive.

B-1	Determine the gravitational acceleration $g(r)$ near the center of the star in terms of r and the constants G and ρ_c .	0.2 pt
B-2	Derive a (differential) equation for determining $\rho(r)$ at equilibrium, and write it in the following form: $\frac{d}{dr} \left[h_1(\rho, r) \frac{d\rho}{dr} \right] + h_2(r)\rho = 0$. Find the functions h_1 and h_2 .	0.6 pt
B-3	Construct a quantity r_0 of the form $r_0 = G^l p_c^m \rho_c^n$ with the dimension of length.	0.4 pt
B-4	Rewrite the (differential) equation of task B-2 in the following form: $\frac{d}{dx} \left[A_1(u, x) \frac{du}{dx} \right] + A_2(x)u(x) = 0$, where $x = \frac{r}{r_0}$ and $u = \frac{\rho}{\rho_c}$. Find the functions $A_1(u, x)$ and $A_2(x)$.	0.3 pt
B-5	For $\gamma = 2$ one finds $u(x) = \frac{f(x)}{x}$. Determine $f(x)$.	0.6 pt

Assume that for a particular star $\frac{du}{dx}$, as a function of x , is given by the curve given in Figure 2.

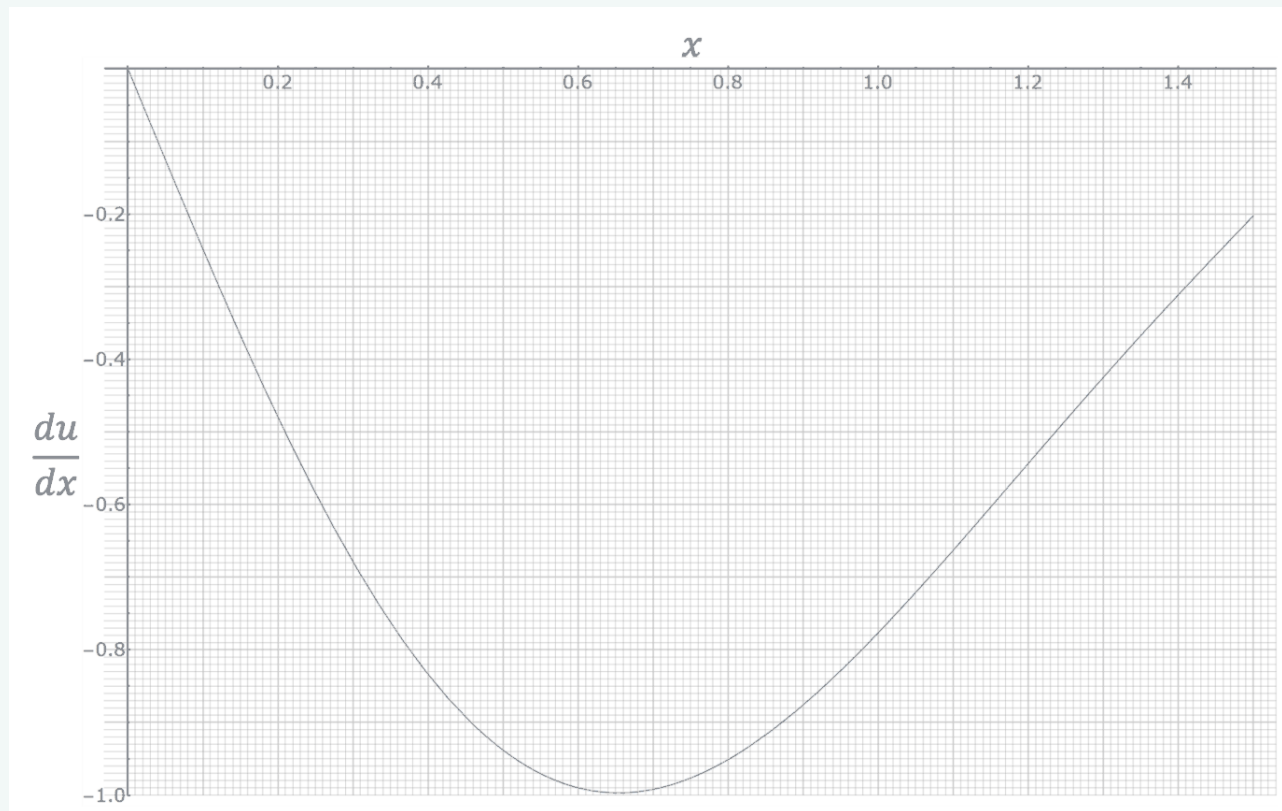


Figure 2 - The plot of $\frac{du}{dx}$

B-6	Use the behavior of the curve in Figure 2, in the vicinity of the point $x = 0$, to find γ up to 3 significant figures. Use the given ruler if necessary.	0.8 pt
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To analyze the stability of the system, we assume that the star deviates slightly from its equilibrium state: we assume that the spherical shell, which was in equilibrium at radius r , now has a radius $\tilde{r}$, similarly the parameters g , p , and ρ have changed to $\tilde{g}$, $\tilde{p}$, and $\tilde{\rho}$ respectively. For convenience, we shall only consider small r 's near the center of the star, for which we can assume that $\tilde{r} = r(1 + \varepsilon(t))$, where $\varepsilon(t) \ll 1$.

B-7	Find $\tilde{\rho}$ and $\tilde{g}$ in terms of ρ and g to the first order in ε .	0.9 pt
B-8	Using Newton's equation of motion for the spherical layer with the equilibrium radius of r find $\frac{d^2\tilde{r}}{dt^2}$ in terms of $\tilde{g}$, $\tilde{\rho}$, K , γ , and $\frac{\partial\tilde{\rho}}{\partial\tilde{r}}$ (By $\frac{\partial\tilde{\rho}}{\partial\tilde{r}}$ we mean derivative of $\tilde{\rho}$ with respect to $\tilde{r}$ at constant t .)	0.6 pt
B-9	Obtain $\frac{d^2\varepsilon}{dt^2}$ in terms of ε and the constants given in the problem. Find the minimum value of γ for a stable equilibrium, and find the oscillation's angular frequency of the star.	0.6 pt

Official English text of the problem.

Answers

Write $M = M_1 + M_2$, and let the stellar positions be $x_1 = -aM_2/M$ and $x_2 = aM_1/M$.

- **A-1**

$$\varphi(x, y) = -\frac{GM_1}{\sqrt{(x + aM_2/M)^2 + y^2}} - \frac{GM_2}{\sqrt{(x - aM_1/M)^2 + y^2}} - \frac{GM}{2a^3}(x^2 + y^2)$$

up to an additive constant.

- **A-2** There are three strictly concave branches, each with one maximum. The potential tends to $-\infty$ on both sides of x_1, x_2 , and as $x \rightarrow \pm\infty$. With the usual labels, the maxima occur in the order

$$x_{L_3} < x_1 < 0 < x_{L_1} < x_2 < x_{L_2},$$

and their heights satisfy

$$\varphi(L_1) < \varphi(L_2) < \varphi(L_3) < 0.$$

- **A-3**

$$\frac{x_0}{a} = 0.36.$$

- **A-4**

$$\dot{a} = 2\beta a \left(\frac{1}{M_2} - \frac{1}{M_1} \right), \quad \dot{P} = 6\pi\beta \sqrt{\frac{a^3}{GM}} \left(\frac{1}{M_2} - \frac{1}{M_1} \right).$$

- **A-5** Using local orbital-energy release and radiation from both faces,

$$T(r) = \left(\frac{GM_1\beta}{8\pi\sigma r^3} \right)^{1/4}.$$

- **A-6**

$$T(a/10) \simeq 8.9 \times 10^3 \text{ K}.$$

- **A-7** The escape-speed threshold is

$$v'_{\text{crit}} = \sqrt{\frac{2G(M'_1 + M_2)}{a}},$$

with a bound orbit requiring $v' < v'_{\text{crit}}$. For isotropic ejection,

$$M'_1 > \frac{M_1 - M_2}{2}.$$

Equality in either condition is the marginal, parabolic escape case.

- **B-1**

$$g(r) \simeq -\frac{4\pi}{3}G\rho_c r.$$

- **B-2** One convenient normalization is

$$h_1(\rho, r) = K\gamma r^2 \rho^{\gamma-2}, \quad h_2(r) = 4\pi G r^2.$$

- **B-3**

$$l = -\frac{1}{2}, \quad m = \frac{1}{2}, \quad n = -1, \quad r_0 = \sqrt{\frac{p_c}{G\rho_c^2}}.$$

- **B-4**

$$A_1(u, x) = \gamma x^2 u^{\gamma-2}, \quad A_2(x) = 4\pi x^2.$$

- **B-5**

$$f(x) = \frac{\sin(\sqrt{2\pi} x)}{\sqrt{2\pi}}.$$

- **B-6**

$$\gamma \simeq 1.67.$$

- **B-7**

$$\tilde{\rho} = \rho(1 - 3\varepsilon) + O(\varepsilon^2), \quad \tilde{g} = g(1 - 2\varepsilon) + O(\varepsilon^2).$$

- **B-8**

$$\frac{d^2 \tilde{r}}{dt^2} = \tilde{g} - K\gamma \tilde{\rho}^{\gamma-2} \frac{\partial \tilde{\rho}}{\partial \tilde{r}}.$$

- **B-9**

$$\ddot{\varepsilon} + \frac{4\pi G \rho_c}{3} (3\gamma - 4) \varepsilon = 0.$$

The critical value is $\gamma_{\text{crit}} = 4/3$, with restoring oscillations for $\gamma > 4/3$ and marginal behavior at equality. Their angular frequency is

$$\omega = \sqrt{\frac{4\pi G \rho_c}{3} (3\gamma - 4)}.$$

Solution

A-1

Let $M = M_1 + M_2$. The center-of-mass condition and the separation give

$$M_1 x_1 + M_2 x_2 = 0, \quad x_2 - x_1 = a,$$

so

$$x_1 = -\frac{M_2}{M} a, \quad x_2 = \frac{M_1}{M} a.$$

If the binary's angular velocity is Ω , Newton's equation for either star gives

$$\Omega^2 = \frac{GM}{a^3}.$$

The gravitational potential per unit test mass is the sum of the two point-mass potentials. In the uniformly rotating frame, the outward centrifugal acceleration is generated by the potential

$$\varphi_{\text{cf}} = -\frac{1}{2}\Omega^2(x^2 + y^2).$$

Consequently,

$$\varphi(x, y) = -\frac{GM_1}{\sqrt{(x + aM_2/M)^2 + y^2}} - \frac{GM_2}{\sqrt{(x - aM_1/M)^2 + y^2}} - \frac{GM}{2a^3}(x^2 + y^2).$$

A-2

On the x -axis,

$$\varphi(x, 0) = -\frac{GM_1}{|x - x_1|} - \frac{GM_2}{|x - x_2|} - \frac{1}{2}\Omega^2 x^2.$$

A precise description of the sketch is as follows:

- There are downward vertical singularities at $x = x_1$ and $x = x_2$: the potential approaches $-\infty$ from either side of each star.
- At large positive or negative x ,

$$\varphi(x, 0) \sim -\frac{1}{2}\Omega^2 x^2,$$

so both outer ends descend to $-\infty$.

- Each of the three intervals separated by the stars contains exactly one maximum.

To establish the last point, away from the masses,

$$\frac{d^2\varphi}{dx^2} = -\frac{2GM_1}{|x - x_1|^3} - \frac{2GM_2}{|x - x_2|^3} - \Omega^2 < 0.$$

Thus each branch is strictly concave. Its derivative changes from positive to negative between its two ends, giving one maximum.

Denote the left, inner, and right maxima by L_3, L_1, L_2 , respectively. Since $M_1 > M_2$, the heavier star is closer to the origin, and

$$\left. \frac{d\varphi}{dx} \right|_{x=0} = \frac{GM_1}{|x_1|^2} - \frac{GM_2}{x_2^2} > 0.$$

Therefore the inner maximum is to the right of the origin:

$$x_{L_3} < x_1 < 0 < x_{L_1} < x_2 < x_{L_2}.$$

The left outer maximum is highest, the right outer maximum is next, and the inner maximum is lowest:

$$\varphi(L_1) < \varphi(L_2) < \varphi(L_3).$$

With the additive constant chosen in A-1, the whole curve is negative and has no horizontal-axis intercept. Its vertical-axis intercept is

$$\varphi(0, 0) = -\frac{GM}{a} \left(\frac{M_1}{M_2} + \frac{M_2}{M_1} \right).$$

A-3

The gas first passes toward M_1 through the inner equilibrium point L_1 , where the effective force vanishes. This point is a maximum of $\varphi(x, 0)$ along the line joining the stars and a saddle of the two-dimensional effective potential.

Here

$$M_1 = \frac{3}{4}M, \quad M_2 = \frac{1}{4}M,$$

so

$$x_1 = -\frac{a}{4}, \quad x_2 = \frac{3a}{4}.$$

Writing $z = x_0/a$, the condition $\partial\varphi/\partial x = 0$ between the stars becomes

$$\frac{3/4}{(z + 1/4)^2} - \frac{1/4}{(3/4 - z)^2} - z = 0.$$

The root in $-1/4 < z < 3/4$ is

$$z = 0.360743 \dots$$

Thus, to two significant figures,

$$\boxed{\frac{x_0}{a} = 0.36}.$$

A-4

Treat the transfer as conservative: all mass lost by M_2 reaches M_1 , and there is no external torque. In the point-mass model, spin and angular momentum temporarily stored in the stream are neglected.

Hence

$$\dot{M}_1 = \beta, \quad \dot{M}_2 = -\beta, \quad \dot{M} = 0,$$

and the orbital angular momentum is conserved.

For a circular binary,

$$L = \frac{M_1 M_2}{M} a^2 \Omega = M_1 M_2 \sqrt{\frac{Ga}{M}}.$$

Taking its logarithmic derivative,

$$0 = \frac{\dot{L}}{L} = \frac{\beta}{M_1} - \frac{\beta}{M_2} + \frac{1}{2} \frac{\dot{a}}{a}.$$

Therefore

$$\boxed{\dot{a} = 2\beta a \left(\frac{1}{M_2} - \frac{1}{M_1} \right)}.$$

Kepler's relation gives

$$P = 2\pi \sqrt{\frac{a^3}{GM}}.$$

Since M is constant,

$$\frac{\dot{P}}{P} = \frac{3}{2} \frac{\dot{a}}{a} = 3\beta \left(\frac{1}{M_2} - \frac{1}{M_1} \right).$$

Thus

$$\dot{P} = 6\pi\beta \sqrt{\frac{a^3}{GM}} \left(\frac{1}{M_2} - \frac{1}{M_1} \right).$$

Both the separation and the period increase when mass flows from the lighter star to the heavier one.

A-5

Use a thin disk with nearly circular Keplerian orbits, negligible radial kinetic energy, and local radiation of the orbital mechanical energy lost while matter crosses an annulus.

The circular-orbit condition is

$$v^2 = \frac{GM_1}{r}.$$

The gas's mechanical energy per unit mass is consequently

$$e(r) = \frac{1}{2}v^2 - \frac{GM_1}{r} = -\frac{GM_1}{2r}.$$

Consider an annulus between r and $r + dr$, where $dr > 0$. Matter enters through its outer edge and leaves through its inner edge. At mass flow rate β , the released power is

$$dL = \beta[e(r + dr) - e(r)] = \frac{GM_1\beta}{2r^2} dr.$$

Both disk faces radiate. Their combined annular area is

$$dA = 2(2\pi r dr) = 4\pi r dr.$$

Neglecting heat storage and equating the local energy release to blackbody radiation,

$$4\pi r dr \sigma T^4 = \frac{GM_1\beta}{2r^2} dr.$$

Hence

$$T(r) = \left(\frac{GM_1\beta}{8\pi\sigma r^3} \right)^{1/4}.$$

In particular, $T \propto r^{-3/4}$.

A-6

The total mass and period are

$$M = (2.27 + 0.33)(1.98 \times 10^{30}) = 5.148 \times 10^{30} \text{ kg},$$

$$P = 4.14(3600) = 14904 \text{ s}.$$

Kepler's relation gives

$$a = \left(\frac{GMP^2}{4\pi^2} \right)^{1/3} = 1.24548 \times 10^9 \text{ m}.$$

Therefore

$$r = \frac{a}{10} = 1.24548 \times 10^8 \text{ m.}$$

Using $1 \text{ yr} \simeq 3.156 \times 10^7 \text{ s}$,

$$\beta = \frac{9 \times 10^{-10} (1.98 \times 10^{30})}{3.156 \times 10^7} \simeq 5.65 \times 10^{13} \text{ kg s}^{-1}.$$

With $M_1 = 2.27(1.98 \times 10^{30}) \text{ kg}$,

$$T = \left[\frac{(6.67 \times 10^{-11})(2.27 \times 1.98 \times 10^{30})(5.65 \times 10^{13})}{8\pi(5.67 \times 10^{-8})(1.24548 \times 10^8)^3} \right]^{1/4}.$$

This yields

$$T \simeq 8.86 \times 10^3 \text{ K},$$

or

$$\boxed{T \simeq 8.9 \times 10^3 \text{ K}}.$$

A-7

The explosion is sudden, so the separation immediately afterward remains a . Once the rapidly ejected material has left, the relative orbital energy of the remaining stars is

$$E' = \frac{1}{2}\mu'v'^2 - \frac{GM'_1M_2}{a}, \quad \mu' = \frac{M'_1M_2}{M'_1 + M_2}.$$

A bound two-body orbit requires $E' < 0$. Thus

$$v'^2 < \frac{2G(M'_1 + M_2)}{a},$$

and the limiting speed is

$$\boxed{v'_{\text{crit}} = \sqrt{\frac{2G(M'_1 + M_2)}{a}}}.$$

At equality the orbit is parabolic and is not bound.

If the ejection is isotropic in the exploding star's rest frame, there is no recoil. The instantaneous velocities of both remaining stars are unchanged. Their relative speed therefore remains the original circular-orbit speed:

$$v'^2 = \Omega^2 a^2 = \frac{G(M_1 + M_2)}{a}.$$

Substitution into the binding condition gives

$$M_1 + M_2 < 2(M'_1 + M_2),$$

or

$$\boxed{M'_1 > \frac{M_1 - M_2}{2}}.$$

Equivalently, less than half of the binary's original total mass may be ejected:

$$M_1 - M'_1 < \frac{M_1 + M_2}{2}.$$

The stated minimum is a critical threshold; equality gives marginal escape.

B-1

By spherical symmetry, only the mass enclosed inside radius r contributes to the gravitational field there. Near the center, $\rho \simeq \rho_c$, so

$$m(r) \simeq \frac{4\pi}{3} \rho_c r^3.$$

Taking outward acceleration as positive,

$$g(r) = -\frac{Gm(r)}{r^2} \simeq -\frac{4\pi}{3} G \rho_c r.$$

Therefore

$$\boxed{g(r) \simeq -\frac{4\pi}{3} G \rho_c r.}$$

B-2

Hydrostatic equilibrium requires

$$\frac{dp}{dr} = \rho g.$$

The enclosed mass satisfies

$$g = -\frac{Gm(r)}{r^2}, \quad \frac{dm}{dr} = 4\pi r^2 \rho.$$

Consequently,

$$\frac{d}{dr}(r^2 g) = -4\pi G r^2 \rho.$$

Using $g = (1/\rho) dp/dr$,

$$\frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dp}{dr} \right) + 4\pi G r^2 \rho = 0.$$

From $p = K\rho^\gamma$,

$$\frac{dp}{dr} = K\gamma\rho^{\gamma-1} \frac{d\rho}{dr}.$$

The required density equation is therefore

$$\boxed{\frac{d}{dr} \left(K\gamma r^2 \rho^{\gamma-2} \frac{d\rho}{dr} \right) + 4\pi G r^2 \rho = 0.}$$

Thus one valid choice is

$$\boxed{h_1 = K\gamma r^2 \rho^{\gamma-2}}, \quad \boxed{h_2 = 4\pi G r^2}.$$

Multiplying both functions by the same nonzero constant gives an equivalent equation.

B-3

The relevant dimensions are

$$[G] = L^3 M^{-1} T^{-2}, \quad [p_c] = M L^{-1} T^{-2}, \quad [\rho_c] = M L^{-3}.$$

Requiring $G^l p_c^m \rho_c^n$ to have dimension L gives

$$-l + m + n = 0, \quad 3l - m - 3n = 1, \quad -2l - 2m = 0.$$

Solving,

$$l = -\frac{1}{2}, \quad m = \frac{1}{2}, \quad n = -1.$$

Hence

$$r_0 = \frac{\sqrt{p_c}}{\sqrt{G} \rho_c} = \sqrt{\frac{p_c}{G \rho_c^2}}.$$

Since $p_c = K \rho_c^\gamma$, it is also useful to write

$$r_0^2 = \frac{K \rho_c^{\gamma-2}}{G}.$$

B-4

Substitute

$$r = r_0 x, \quad \rho = \rho_c u, \quad \frac{d\rho}{dr} = \frac{\rho_c}{r_0} \frac{du}{dx}$$

into the equation from B-2:

$$K \gamma \rho_c^{\gamma-1} \frac{d}{dx} \left(x^2 u^{\gamma-2} \frac{du}{dx} \right) + 4\pi G r_0^2 \rho_c x^2 u = 0.$$

Because $G r_0^2 \rho_c = K \rho_c^{\gamma-1}$, division by $K \rho_c^{\gamma-1}$ gives

$$\frac{d}{dx} \left(\gamma x^2 u^{\gamma-2} \frac{du}{dx} \right) + 4\pi x^2 u = 0.$$

Thus

$$A_1(u, x) = \gamma x^2 u^{\gamma-2}, \quad A_2(x) = 4\pi x^2.$$

The central conditions are

$$u(0) = 1, \quad u'(0) = 0,$$

the latter following from regularity and spherical symmetry.

B-5

For $\gamma = 2$, the equation reduces to

$$\frac{d}{dx} (x^2 u') + 2\pi x^2 u = 0.$$

Let $u = f/x$. Then

$$x^2 u' = x f' - f, \quad \frac{d}{dx} (x^2 u') = x f''.$$

For $x > 0$,

$$f'' + 2\pi f = 0.$$

Writing $k = \sqrt{2\pi}$, the general solution is

$$f = A \sin(kx) + B \cos(kx).$$

Finiteness of $u = f/x$ at the center requires $B = 0$. The normalization $u(0) = 1$ then gives $Ak = 1$. Therefore

$$f(x) = \frac{\sin(\sqrt{2\pi} x)}{\sqrt{2\pi}},$$

and

$$u(x) = \frac{\sin(\sqrt{2\pi} x)}{\sqrt{2\pi} x}.$$

The physical density profile ends at its first zero, $x = \sqrt{\pi/2}$.

B-6

For this task, return to general γ . Near the center, regularity permits the expansion

$$u(x) = 1 + cx^2 + O(x^4), \quad u'(x) = 2cx + O(x^3).$$

Substitution into the dimensionless equilibrium equation gives, at order x^2 ,

$$6\gamma c + 4\pi = 0.$$

Hence

$$u'(x) = -\frac{4\pi}{3\gamma}x + O(x^3).$$

If s is the slope of the plotted u' -versus- x curve at the origin, then

$$s = u''(0) = -\frac{4\pi}{3\gamma}, \quad \gamma = -\frac{4\pi}{3s}.$$

The tangent at the origin in Figure 2 has slope approximately

$$s \simeq -2.51.$$

Thus

$$\gamma \simeq \frac{4\pi}{3(2.51)} \simeq 1.67.$$

This is consistent with $\gamma = 5/3$.

B-7

Follow a particular material shell. Its mass is conserved as it moves from r to

$$\tilde{r} = r(1 + \varepsilon).$$

Since the same scale factor applies to neighboring central shells,

$$d\tilde{r} = (1 + \varepsilon) dr.$$

Conservation of the mass in a thin shell gives

$$4\pi r^2 \rho dr = 4\pi \tilde{r}^2 \tilde{\rho} d\tilde{r},$$

so

$$\tilde{\rho} = \frac{\rho}{(1 + \varepsilon)^3} = \rho(1 - 3\varepsilon) + O(\varepsilon^2).$$

Therefore

$$\boxed{\tilde{\rho} \simeq \rho(1 - 3\varepsilon)}.$$

The mass enclosed by the moving shell is also conserved. Consequently,

$$\tilde{g} = -\frac{Gm(r)}{\tilde{r}^2} = \frac{g}{(1 + \varepsilon)^2},$$

giving

$$\boxed{\tilde{g} \simeq g(1 - 2\varepsilon)}.$$

Here g is negative, consistently with the prescribed outward-positive convention.

B-8

The outward pressure acceleration is

$$-\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{r}}.$$

Adding gravity, Newton's equation is

$$\frac{d^2 \tilde{r}}{dt^2} = \tilde{g} - \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{r}}.$$

Using the same equation of state for the perturbed matter,

$$\tilde{p} = K \tilde{\rho}^\gamma,$$

we obtain

$$\frac{\partial \tilde{p}}{\partial \tilde{r}} = K \gamma \tilde{\rho}^{\gamma-1} \frac{\partial \tilde{\rho}}{\partial \tilde{r}}.$$

Hence

$$\boxed{\frac{d^2 \tilde{r}}{dt^2} = \tilde{g} - K \gamma \tilde{\rho}^{\gamma-2} \frac{\partial \tilde{\rho}}{\partial \tilde{r}}}.$$

B-9

At fixed time, ε is independent of the central shell label r . Therefore

$$\tilde{\rho}(\tilde{r}, t) = (1 + \varepsilon)^{-3} \rho\left(\frac{\tilde{r}}{1 + \varepsilon}\right),$$

and

$$\frac{\partial \tilde{\rho}}{\partial \tilde{r}} = (1 + \varepsilon)^{-4} \frac{d\rho}{dr}.$$

The pressure term in B-8 consequently scales as

$$K \gamma \tilde{\rho}^{\gamma-2} \frac{\partial \tilde{\rho}}{\partial \tilde{r}} = K \gamma \rho^{\gamma-2} \frac{d\rho}{dr} (1 + \varepsilon)^{2-3\gamma}.$$

In the unperturbed equilibrium,

$$K \gamma \rho^{\gamma-2} \frac{d\rho}{dr} = g.$$

Together with $\tilde{g} = g(1 + \varepsilon)^{-2}$, this gives

$$\frac{d^2\tilde{r}}{dt^2} = g \left[(1 + \varepsilon)^{-2} - (1 + \varepsilon)^{2-3\gamma} \right].$$

To first order,

$$\begin{aligned} (1 + \varepsilon)^{-2} &= 1 - 2\varepsilon, \\ (1 + \varepsilon)^{2-3\gamma} &= 1 + (2 - 3\gamma)\varepsilon. \end{aligned}$$

Thus

$$\frac{d^2\tilde{r}}{dt^2} = g(3\gamma - 4)\varepsilon.$$

Since r labels the original shell and is constant,

$$\frac{d^2\tilde{r}}{dt^2} = r\ddot{\varepsilon}.$$

Using the near-center result $g/r = -4\pi G\rho_c/3$,

$$\ddot{\varepsilon} = -\frac{4\pi G\rho_c}{3}(3\gamma - 4)\varepsilon.$$

For a restoring acceleration, the coefficient multiplying $-\varepsilon$ must be positive. Hence

$$\gamma > \frac{4}{3}.$$

The critical value is therefore

$$\gamma_{\text{crit}} = \frac{4}{3}.$$

At equality there is no restoring force to this order; for $\gamma < 4/3$, a growing perturbation exists. For $\gamma > 4/3$, the motion is harmonic with

$$\omega = \sqrt{\frac{4\pi G\rho_c}{3}(3\gamma - 4)} = \sqrt{4\pi G\rho_c \left(\gamma - \frac{4}{3} \right)}.$$

The graph-derived value $\gamma \simeq 1.67$ lies in the stable range.

The solution was checked against the official solution.