

IPhO 2023

Theory problems and solutions · Tokyo, Japan · 3 theory problems with complete solutions

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Problem 1: Characterization of Soil Colloids

Colloidal science is useful to characterize soil particles because many of them can be regarded as colloidal particles of micrometer size. For example, Brownian motion (random motion of colloidal particles) can be used to measure particle sizes.

Part A. Motions of colloidal particles (1.6 points)

We analyze the one-dimensional Brownian motion of a colloidal particle with mass M . The equation of motion for its velocity $v(t)$ reads:

$$M\dot{v} = -\gamma v(t) + F(t) + F_{\text{ext}}(t), \quad (1)$$

where γ is the friction coefficient, $F(t)$ is a force due to random collisions with water molecules, and $F_{\text{ext}}(t)$ is an external force. In Part A, we assume $F_{\text{ext}}(t) = 0$.

A.1 — 0.8pt

Consider that a water molecule collides with the particle at $t = t_0$, giving impulse I_0 , and $F(t) = 0$ afterward. If $v(t) = 0$ before the collision, $v(t) = v_0 e^{-(t-t_0)/\tau}$ for $t > t_0$. Determine v_0 and τ , using I_0 and necessary parameters in Eq.(1).

In the following, you may use τ in your answers.

A.2 — 0.8pt

Actually, water molecules collide with the particle one after another. Suppose the i th collision gives the impulse I_i at time t_i and determine $v(t)$ on condition that $t > 0$ and $v(0) = 0$. Also give the inequality specifying the range of t_i that needs to be considered for a given t . In the answer sheet, it is not necessary to specify this range in the expression for $v(t)$.

Part B. Effective equation of motion (1.8 points)

Results so far imply that particle velocities $v(t)$ and $v(t')$ may be regarded as uncorrelated random quantities if $|t - t'| \gg \tau$. On this basis, we introduce a theoretical model to approximately describe the one-dimensional Brownian motion, where the velocity changes randomly at each time interval δ ($\gg \tau$), i.e.,

$$v(t) = v_n \quad (t_{n-1} < t \leq t_n), \quad (2)$$

with $t_n = n\delta$ ($n = 0, 1, 2, \dots$) and a random quantity v_n . It satisfies

$$\langle v_n \rangle = 0, \quad \langle v_n v_m \rangle = \begin{cases} C & (n = m), \\ 0 & (n \neq m), \end{cases} \quad (3)$$

with a parameter C depending on δ . Here $\langle X \rangle$ indicates the expectation value of X . That is, if you draw random numbers X infinite times, the mean will be $\langle X \rangle$.

Now we consider the particle displacement $\Delta x(t) = x(t) - x(0)$ for $t = N\delta$ with an integer N .

B.1 — 1.0pt

Determine $\langle \Delta x(t) \rangle$ and $\langle \Delta x(t)^2 \rangle$ using C , δ , and t .

B.2 — 0.8pt

The quantity $\langle \Delta x(t)^2 \rangle$ is called the mean square displacement (MSD). It is a characteristic observable of the Brownian motion, which corresponds to the limiting case $\delta \rightarrow 0$. From this, we can show $C \propto \delta^\alpha$ and $\langle \Delta x(t)^2 \rangle \propto t^\beta$. Determine the values of α and β .

Part C. Electrophoresis (2.7 points)

Here we discuss electrophoresis, i.e., transport of charged particles by an electric field.

Suspension of colloidal particles with mass M and charge Q (> 0) is put in a narrow channel with a cross-section A (Fig.1(a)). We ignore the interaction between particles, effects of the wall, the fluid, the ions therein, and gravity.

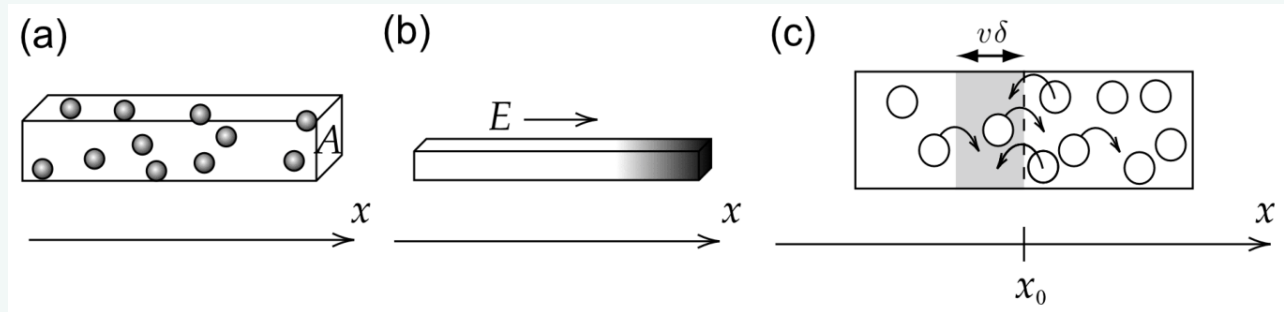


Fig.1: Setting for Part C.

By applying a uniform electric field E in the x -direction, particles are transported and their concentration $n(x)$ (particle number per unit volume) becomes non-uniform (Fig.1(b)). When E is removed, this non-uniformity gradually disappears. This is due to Brownian motion of particles. If $n(x)$ is not uniform, the numbers of right-going and left-going particles may differ (Fig.1(c)). This generates a particle flux $J_D(x)$, the mean number of particles flowing at x along the x -axis per unit cross-sectional area and unit time. This flux is known to satisfy

$$J_D(x) = -D \frac{dn}{dx}(x), \quad (4)$$

where D is called the diffusion coefficient.

Now let's assume, for simplicity, that half of the particles have velocity $+v$ and the other half have velocity $-v$. Let $N_+(x_0)$ be the number of particles with velocity $+v$ that cross x_0 from left to right per unit cross-sectional area and unit time. For particles with velocity $+v$ to cross x_0 in the time interval δ , they should be in the shaded region of Fig.1(c). Since δ is small, we have $n(x) \simeq n(x_0) + (x - x_0) \frac{dn}{dx}(x_0)$ in this region.

C.1 — 0.5pt

Express $N_+(x_0)$ using necessary quantities from v , δ , $n(x_0)$, and $\frac{dn}{dx}(x_0)$.

We define $N_-(x_0)$ as the counterpart of $N_+(x_0)$ for the velocity $-v$. With this, we have $J_D(x_0) = \langle N_+(x_0) - N_-(x_0) \rangle$. According to Eq.(3), we have $\langle v^2 \rangle = C$.

C.2 — 0.7pt

Determine $J_D(x_0)$ using necessary quantities from C , δ , $n(x_0)$, and $\frac{dn}{dx}(x_0)$. Using this and Eq. (4), express D in terms of C and δ , and $\langle \Delta x(t)^2 \rangle$ in terms of D and t .

Now we discuss the effect of osmotic pressure Π . It is given by $\Pi = \frac{n}{N_A}RT = nkT$ with the Avogadro constant N_A , the gas constant R , temperature T , and the Boltzmann constant $k = \frac{R}{N_A}$. Let us consider the non-uniform concentration formed under the electric field E (Fig.1(b)). Since $n(x)$ depends on x , so does $\Pi(x)$. Then the forces due to $\Pi(x)$ and $\Pi(x + \Delta x)$ must be balanced with the total force from the field E acting on the particles (Fig.2). Here we consider small Δx , so that $n(x)$ can be regarded as constant over this range, while $n(x + \Delta x) - n(x) \simeq \Delta x \frac{dn}{dx}(x)$.

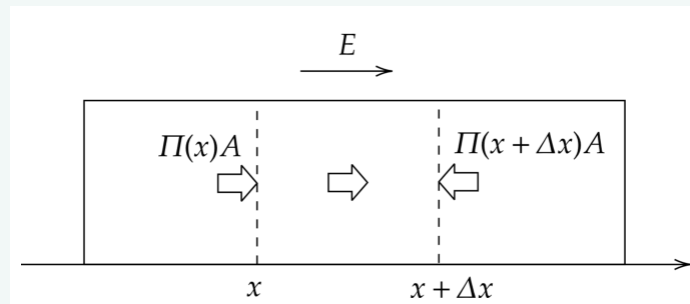


Fig.2: Force balance.

C.3 — 0.5pt

Express $\frac{dn}{dx}(x)$ using $n(x)$, T , Q , E , and k .

Let us discuss the balance of the flux now. Besides the flux $J_D(x)$ due to the Brownian motion, there is also a flux due to the electric field, $J_Q(x)$. It is given by

$$J_Q(x) = n(x)u, \quad (5)$$

where u is the terminal velocity of particles driven by the field.

C.4 — 0.5pt

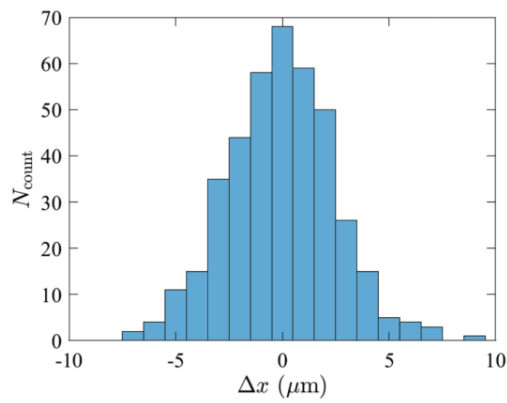
To determine u , we use Eq.(1) with $F_{\text{ext}}(t) = QE$. Since $v(t)$ is fluctuating, we consider $\langle v(t) \rangle$. Assuming $\langle v(0) \rangle = 0$ and using $\langle F(t) \rangle = 0$, evaluate $\langle v(t) \rangle$ and obtain $u = \lim_{t \rightarrow \infty} \langle v(t) \rangle$.

C.5 — 0.5pt

The flux balance reads $J_D(x) + J_Q(x) = 0$. Express the diffusion coefficient D in terms of k , γ , and T .

Part D. Mean square displacement (2.4 points)

Suppose we observed the Brownian motion of an isolated, spherical colloidal particle with radius $a = 5.0 \mu\text{m}$ in water. Figure 3 shows the histogram of displacements Δx measured in the x -direction at every interval $\Delta t = 60 \text{ s}$. The friction coefficient is given by $\gamma = 6\pi a\eta$ with water viscosity $\eta = 8.9 \times 10^{-4} \text{ Pa} \cdot \text{s}$ and the temperature was $T = 25^\circ \text{C}$.



Δx (μm)	-10	-9	-8	-7	-6	-5	-4
N_{count}	0	0	0	2	4	11	15

Δx (μm)	-3	-2	-1	0	1	2	3
N_{count}	35	44	58	68	59	50	26

Δx (μm)	4	5	6	7	8	9	10
N_{count}	15	5	4	3	0	1	0

Fig.3: Histogram of displacements.

Δx (μm)	-10	-9	-8	-7	-6	-5	-4
N_{count}	0	0	0	2	4	11	15

Δx (μm)	-3	-2	-1	0	1	2	3
N_{count}	35	44	58	68	59	50	26

Δx (μm)	4	5	6	7	8	9	10
N_{count}	15	5	4	3	0	1	0

D.1 – 1.0pt

Estimate the value of N_A without using the fact that it is the Avogadro constant, up to two significant digits from the data in Fig.3. The gas constant is $R = 8.31 \text{ J/K} \cdot \text{mol}$. Do not use the value of the Boltzmann constant k given in General Instructions. As for the Avogadro constant, you might obtain a value different from that in General Instructions.

Now we extend the model in Part B to describe the motion of a particle with charge Q under an electric field E . The particle velocity $v(t)$ considered in Eq.(2) should be replaced by $v(t) = u + v_n$ ($t_{n-1} < t \leq t_n$) with v_n satisfying Eq. (3) and u being the terminal velocity considered in Eq.(5).

D.2 – 0.8pt

Express the MSD $\langle \Delta x(t)^2 \rangle$ in terms of u , D , and t . Obtain approximate power laws for small t and large t , as well as the characteristic time t_* where this change occurs. Draw a rough graph of MSD in a log-log plot, indicating the approximate location of t_* .

Next, we consider swimming microbes (Fig.4(a)), in one dimension for simplicity (Fig.4(b)). These are spherical particles with radius a . They swim at velocity either $+u_0$ or $-u_0$, the sign chosen randomly at every time interval δ_0 without correlation. The observed motion is a combination of displacements due to swimming and those due to the Brownian motion of a spherical particle.

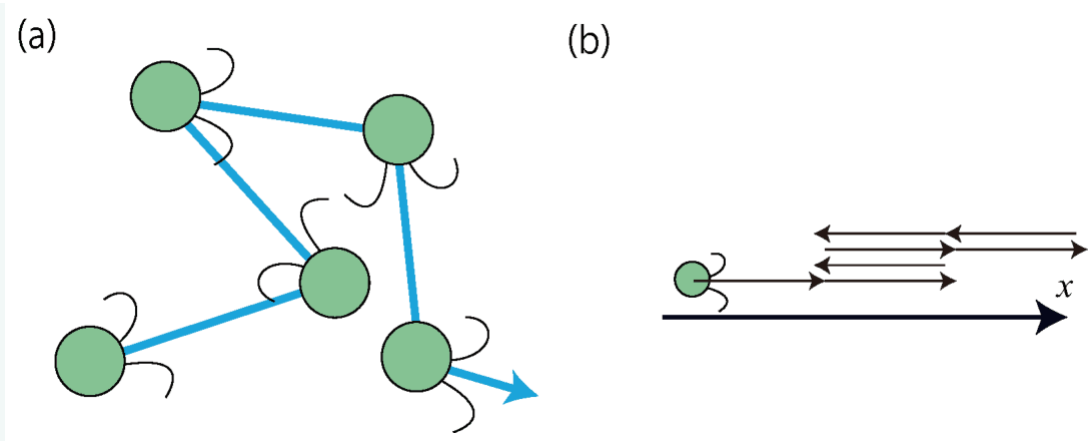


Fig.4: (a) Motion of microbes. (b) Its one-dimensional version.

D.3 — 0.6pt

Figure 5 displays the MSD $\langle \Delta x(t)^2 \rangle$ of those microbes, showing different power laws for small, large, and intermediate t , as indicated by dashed lines. Obtain the power law for each time range and express it using necessary quantities from D , u_0 , δ_0 , and t .

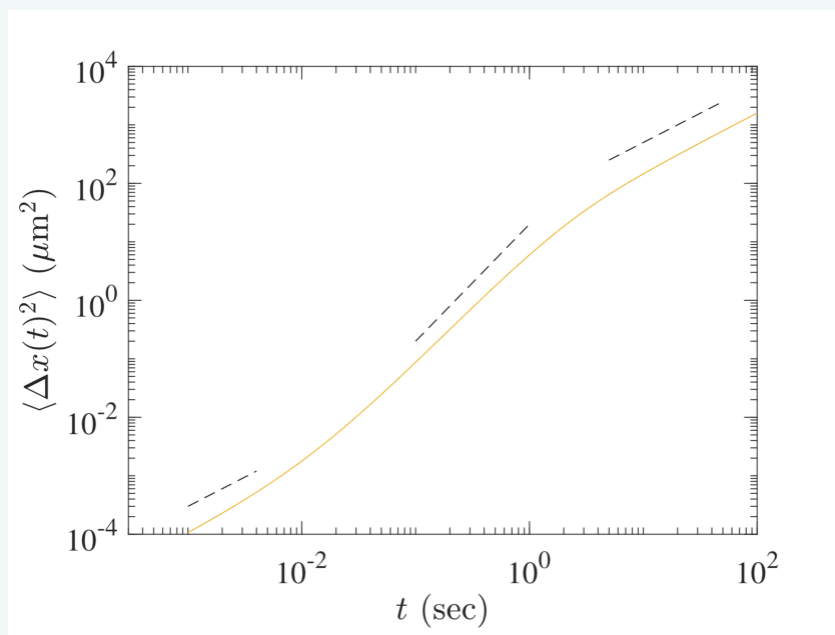


Fig.5: Mean square displacement of the microbes.

Part E. Water purification (1.5 points)

Here we discuss the purification of water including colloid-like soil particles, by adding electrolytes to coagulate them. Particles interact through van der Waals force and electrostatic force, the latter including effects of both surface charges and the surrounding layer of oppositely charged ions (such ions and their layer are called counter-ions and the electric double layer, respectively; see Fig.6(a)). As a result, the interaction potential for particle distance d (Fig.6(b)) is given by

$$U(d) = -\frac{A}{d} + \frac{B\epsilon(kT)^2}{q^2} e^{-d/\lambda}, \quad (6)$$

where A and B are positive constants, ϵ is the dielectric constant of water, and λ is the thickness of the electric double layer. Assuming that charges of ions are $\pm q$, we have

$$\lambda = \sqrt{\frac{\epsilon kT}{2N_A q^2 c}}, \quad (7)$$

where c is the molar concentration of ion.

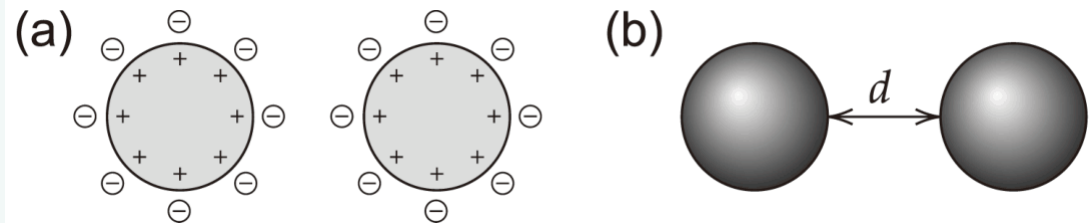


Fig.6: (a) Surface charges of colloidal particles and counter-ions. (b) Definition of the distance d .

E.1 – 1.5pt

Addition of sodium chloride (NaCl) to the suspension causes colloidal particles to coagulate. Determine the lowest concentration c of NaCl necessary for coagulation. It is sufficient to consider two particles without thermal fluctuations, i.e., $F(t) = 0$ in Eq.(1), and assume that the terminal velocity for the given potential force is reached instantaneously.

Official English text of the problem.

Answers

- A.1

$$v_0 = \frac{I_0}{M}, \quad \tau = \frac{M}{\gamma}.$$

- A.2

$$v(t) = \frac{1}{M} \sum_{0 < t_i < t} I_i e^{-(t-t_i)/\tau}, \quad 0 < t_i < t.$$

Immediately after a collision at the observation time, include $t_i = t$.

- B.1

$$\langle \Delta x(t) \rangle = 0, \quad \langle \Delta x(t)^2 \rangle = C \delta t.$$

- B.2

$$\alpha = -1, \quad \beta = 1.$$

- C.1

$$N_+(x_0) \simeq \frac{v}{2} n(x_0) - \frac{v^2 \delta}{4} \frac{dn}{dx}(x_0).$$

- C.2

$$J_D(x_0) = -\frac{C\delta}{2} \frac{dn}{dx}(x_0), \quad D = \frac{C\delta}{2}, \quad \langle \Delta x(t)^2 \rangle = 2Dt.$$

- C.3

$$\frac{dn}{dx}(x) = \frac{QE}{kT} n(x).$$

- C.4

$$\langle v(t) \rangle = \frac{QE}{\gamma} (1 - e^{-t/\tau}), \quad u = \frac{QE}{\gamma}.$$

- C.5

$$D = \frac{kT}{\gamma}.$$

- D.1

$$N_A \simeq 5.6 \times 10^{23} \text{ mol}^{-1}.$$

- D.2

$$\langle \Delta x(t)^2 \rangle = 2Dt + u^2 t^2, \quad t_* = \frac{2D}{u^2}.$$

For $t \ll t_*$, the MSD is approximately $2Dt$, with log-log slope 1. For $t \gg t_*$, it is approximately $u^2 t^2$, with slope 2.

- D.3 Define $t_s = 2D/u_0^2$. For the three distinct regimes shown, $t_s \ll \delta_0$, and

$$\langle \Delta x(t)^2 \rangle \simeq \begin{cases} 2Dt, & t \ll t_s, \\ u_0^2 t^2, & t_s \ll t \ll \delta_0, \\ (2D + u_0^2 \delta_0)t \simeq u_0^2 \delta_0 t, & t \gg \delta_0. \end{cases}$$

The respective log-log slopes are 1, 2, 1.

- E.1 The critical concentration is

$$c_{\text{crit}} = \frac{8B^2 \epsilon^3 (kT)^5}{\exp(4) N_A A^2 q^6},$$

where q is the elementary charge magnitude for NaCl. Coagulation from large separation occurs for $c > c_{\text{crit}}$. Exactly at $c = c_{\text{crit}}$, the stated noiseless, overdamped dynamics are marginal: the particles approach a finite separation asymptotically.

Solution

A.1

Integrating the equation of motion over the very short collision gives the impulse–momentum relation. The frictional impulse vanishes as the collision duration tends to zero, so

$$M[v(t_0^+) - v(t_0^-)] = I_0.$$

Since the particle was initially at rest,

$$v_0 = v(t_0^+) = \frac{I_0}{M}.$$

After the collision, both forces F and F_{ext} vanish. Thus

$$M \frac{dv}{dt} = -\gamma v,$$

whose solution is

$$v(t) = \frac{I_0}{M} \exp \left[-\frac{\gamma}{M} (t - t_0) \right].$$

Comparison with the given form yields

$$\boxed{v_0 = \frac{I_0}{M}, \quad \tau = \frac{M}{\gamma}.}$$

A.2

The equation of motion is linear. Each collision produces an instantaneous velocity increment I_i/M , which subsequently decays with relaxation time τ . Superposition therefore gives

$$\boxed{v(t) = \frac{1}{M} \sum_{0 < t_i < t} I_i \exp \left[-\frac{t - t_i}{\tau} \right].}$$

Only collisions after the initial time and before the observation contribute:

$$\boxed{0 < t_i < t.}$$

At a collision instant, the velocity has a jump I_i/M . If $v(t)$ denotes the velocity immediately after that collision, the upper endpoint is included.

B.1

During interval n , the displacement is $v_n \delta$. Hence, at $t = N\delta$,

$$\Delta x(t) = \delta \sum_{n=1}^N v_n.$$

Its expectation is

$$\langle \Delta x(t) \rangle = \delta \sum_{n=1}^N \langle v_n \rangle = 0.$$

For the second moment,

$$\begin{aligned} \langle \Delta x(t)^2 \rangle &= \delta^2 \sum_{n=1}^N \sum_{m=1}^N \langle v_n v_m \rangle \\ &= \delta^2 N C \\ &= C \delta t. \end{aligned}$$

The cross terms vanish because different intervals have uncorrelated velocities. Thus

$$\boxed{\langle \Delta x(t) \rangle = 0, \quad \langle \Delta x(t)^2 \rangle = C \delta t.}$$

B.2

At a fixed positive observation time, Brownian motion must have a finite, nonzero MSD in the diffusion limit. Since

$$\langle \Delta x(t)^2 \rangle = C\delta t,$$

the product $C\delta$ must tend to a finite, nonzero constant as $\delta \rightarrow 0$. Therefore

$$C \propto \delta^{-1}.$$

The remaining time dependence is linear:

$$\langle \Delta x(t)^2 \rangle \propto t.$$

Consequently,

$$\boxed{\alpha = -1, \quad \beta = 1.}$$

This is the continuum limit of the effective model, describing motion on time scales much longer than the microscopic relaxation time τ .

C.1

Particles moving rightward cross x_0 during the next interval δ if they initially lie between $x_0 - v\delta$ and x_0 . Half of the particles in this region move rightward. Dividing their number per unit area by δ ,

$$N_+(x_0) = \frac{1}{2\delta} \int_{x_0 - v\delta}^{x_0} n(x) dx.$$

Using the prescribed linear approximation,

$$n(x) \simeq n(x_0) + (x - x_0)n'(x_0),$$

we obtain

$$N_+(x_0) \simeq \frac{1}{2\delta} \left[v\delta n(x_0) - \frac{(v\delta)^2}{2} n'(x_0) \right].$$

Therefore,

$$\boxed{N_+(x_0) \simeq \frac{v}{2} n(x_0) - \frac{v^2\delta}{4} n'(x_0).}$$

This approximation requires the concentration to vary slowly over a distance $v\delta$.

C.2

Similarly, particles moving leftward originate in $x_0 < x < x_0 + v\delta$, giving

$$\begin{aligned} N_-(x_0) &= \frac{1}{2\delta} \int_{x_0}^{x_0 + v\delta} n(x) dx \\ &\simeq \frac{v}{2} n(x_0) + \frac{v^2\delta}{4} n'(x_0). \end{aligned}$$

Subtracting and averaging,

$$\begin{aligned}
J_D(x_0) &= \langle N_+(x_0) - N_-(x_0) \rangle \\
&= -\frac{\delta}{2} \langle v^2 \rangle n'(x_0) \\
&= -\frac{C\delta}{2} n'(x_0).
\end{aligned}$$

Comparison with Fick's law, $J_D = -Dn'$, gives

$$D = \frac{C\delta}{2}.$$

Substituting into B.1,

$$\langle \Delta x(t)^2 \rangle = 2Dt.$$

C.3

Consider the slab of cross-sectional area A and thickness Δx . Its electric force in the positive x -direction is

$$F_E = n(x)A\Delta x QE.$$

The net osmotic-pressure force is

$$F_{\Pi} = A\Pi(x) - A\Pi(x + \Delta x) \simeq -A\Delta x \frac{d\Pi}{dx}.$$

Force balance gives

$$n(x)QE = \frac{d\Pi}{dx}.$$

At uniform temperature, $\Pi = nkT$, so

$$\frac{d\Pi}{dx} = kT \frac{dn}{dx}.$$

Hence

$$\frac{dn}{dx}(x) = \frac{QE}{kT} n(x).$$

The positive gradient is consistent with positively charged particles accumulating in the direction of the electric field.

C.4

Taking the expectation of Eq.(1), using the constant applied force and $\langle F(t) \rangle = 0$, yields

$$M \frac{d\langle v \rangle}{dt} = -\gamma \langle v \rangle + QE.$$

The solution satisfying $\langle v(0) \rangle = 0$ is

$$\langle v(t) \rangle = \frac{QE}{\gamma} \left(1 - e^{-\gamma t/M} \right) = \frac{QE}{\gamma} \left(1 - e^{-t/\tau} \right).$$

Therefore the terminal velocity is

$$u = \lim_{t \rightarrow \infty} \langle v(t) \rangle = \frac{QE}{\gamma}.$$

C.5

At equilibrium, the diffusive flux balances the electric drift flux:

$$-D \frac{dn}{dx} + nu = 0.$$

Substituting the results of C.3 and C.4,

$$-D \frac{nQE}{kT} + n \frac{QE}{\gamma} = 0.$$

Thus

$$D = \frac{kT}{\gamma}.$$

This is the Einstein relation between diffusion and friction.

D.1

Using the tabulated displacements as the histogram bin representatives, the total number of measurements is

$$N_{\text{tot}} = \sum_j N_j = 400.$$

The weighted sum of squared displacements is

$$\sum_j N_j (\Delta x_j)^2 = 2536 \mu\text{m}^2.$$

Therefore the estimated MSD over $\Delta t = 60$ s is

$$\langle \widehat{\Delta x^2} \rangle = \frac{2536}{400} \mu\text{m}^2 = 6.34 \times 10^{-12} \text{m}^2.$$

From C.2,

$$D = \frac{\langle \widehat{\Delta x^2} \rangle}{2\Delta t} = \frac{6.34 \times 10^{-12}}{120} = 5.2833 \times 10^{-14} \text{m}^2 \text{s}^{-1}.$$

The friction coefficient is

$$\begin{aligned} \gamma &= 6\pi a \eta \\ &= 6\pi(5.0 \times 10^{-6})(8.9 \times 10^{-4}) \\ &= 8.3881 \times 10^{-8} \text{kg s}^{-1}. \end{aligned}$$

Using $k = R/N_A$ in the Einstein relation gives

$$D = \frac{RT}{N_A \gamma},$$

so

$$N_A = \frac{RT}{\gamma D} = \frac{2RT\Delta t}{\gamma \langle \widehat{\Delta x^2} \rangle}.$$

With $T = 298.15$ K,

$$N_A = \frac{8.31(298.15)}{(8.3881 \times 10^{-8})(5.2833 \times 10^{-14})} = 5.5907 \times 10^{23} \text{mol}^{-1}.$$

To two significant digits,

$$N_A \simeq 5.6 \times 10^{23} \text{ mol}^{-1}.$$

Neither an accepted value of N_A nor a numerical value of k was used.

D.2

With the terminal drift included, the displacement is

$$\Delta x(t) = ut + \delta \sum_{n=1}^N v_n.$$

The random part has zero mean and second moment $2Dt$. Thus the cross term between drift and random displacement averages to zero:

$$\begin{aligned} \langle \Delta x(t)^2 \rangle &= u^2 t^2 + 2ut \delta \sum_n \langle v_n \rangle + \delta^2 \sum_{n,m} \langle v_n v_m \rangle \\ &= u^2 t^2 + 2Dt. \end{aligned}$$

Therefore,

$$\langle \Delta x(t)^2 \rangle = 2Dt + u^2 t^2.$$

The crossover occurs when the two contributions are equal:

$$2Dt_* = u^2 t_*^2,$$

giving

$$t_* = \frac{2D}{u^2}.$$

The limiting power laws are

$$\begin{aligned} t \ll t_* : \quad & \langle \Delta x(t)^2 \rangle \simeq 2Dt, \\ t \gg t_* : \quad & \langle \Delta x(t)^2 \rangle \simeq u^2 t^2. \end{aligned}$$

Here "small t " refers to the effective diffusive model, with the microscopic relaxation already neglected and the terminal drift established.

For a precise description of the requested log-log sketch, define

$$s = \frac{t}{t_*}, \quad Y = \frac{\langle \Delta x(t)^2 \rangle}{2Dt_*}.$$

Then

$$Y = s + s^2.$$

The graph follows the straight asymptote $Y = s$, of slope 1, at small s , bends smoothly upward near $s = 1$, and approaches $Y = s^2$, of slope 2, at large s . The two asymptotes intersect at $(s, Y) = (1, 1)$, while the actual curve passes through $(1, 2)$. Thus mark $t = t_*$ at the bend. At that point,

$$\langle \Delta x(t_*)^2 \rangle = \frac{8D^2}{u^2}.$$

D.3

Write the displacement as the sum of thermal and swimming contributions:

$$\Delta x(t) = \Delta x_B(t) + \Delta x_S(t).$$

The thermal motion and the swimming direction choices are taken to be independent, and both contributions have zero mean. Hence

$$\langle \Delta x(t)^2 \rangle = 2Dt + \langle \Delta x_S(t)^2 \rangle.$$

For $t \ll \delta_0$, the swimming direction is effectively unchanged during the observation interval. Therefore

$$\Delta x_S(t) \simeq \pm u_0 t, \quad \langle \Delta x_S(t)^2 \rangle \simeq u_0^2 t^2.$$

Thus

$$\langle \Delta x(t)^2 \rangle \simeq 2Dt + u_0^2 t^2 \quad (t \ll \delta_0).$$

The two terms become comparable at

$$t_s = \frac{2D}{u_0^2}.$$

Consequently, thermal diffusion dominates for $t \ll t_s$, and swimming produces a ballistic regime for $t_s \ll t \ll \delta_0$.

For long times, consider N complete swimming intervals, with $t = N\delta_0$. Let $s_j = \pm 1$ denote the independently chosen sign in interval j . Then

$$\Delta x_S(t) = u_0 \delta_0 \sum_{j=1}^N s_j.$$

Because $\langle s_j \rangle = 0$ and $\langle s_j s_\ell \rangle = 0$ for $j \neq \ell$,

$$\begin{aligned} \langle \Delta x_S(t)^2 \rangle &= u_0^2 \delta_0^2 N \\ &= u_0^2 \delta_0 t. \end{aligned}$$

An incomplete final interval does not alter this leading behavior at $t \gg \delta_0$. Therefore the long-time MSD is

$$\langle \Delta x(t)^2 \rangle \simeq (2D + u_0^2 \delta_0) t.$$

Equivalently, the long-time diffusion coefficient is

$$D_{\text{eff}} = D + \frac{u_0^2 \delta_0}{2}.$$

For a well-developed intermediate regime, as in the figure, the time scales must satisfy

$$\frac{2D}{u_0^2} \ll \delta_0.$$

This also implies $u_0^2 \delta_0 \gg 2D$. The three observed asymptotes are therefore

Time range	$\langle \Delta x(t)^2 \rangle$	Log-log slope
$t \ll 2D/u_0^2$	$2Dt$	1
$2D/u_0^2 \ll t \ll \delta_0$	$u_0^2 t^2$	2
$t \gg \delta_0$	$u_0^2 \delta_0 t$	1

with the full long-time coefficient $2D + u_0^2 \delta_0$ retained when thermal diffusion is appreciable.

E.1

Define the positive energy scale

$$K = \frac{B\epsilon(kT)^2}{q^2}.$$

Then

$$U(d) = -\frac{A}{d} + Ke^{-d/\lambda},$$

and

$$U'(d) = \frac{A}{d^2} - \frac{K}{\lambda}e^{-d/\lambda}.$$

The force tending to increase the separation is $-U'(d)$. Under the instantaneous-terminal-velocity assumption, each particle's velocity is proportional to its force. For two identical particles with friction coefficient γ ,

$$\dot{d} = -\frac{2}{\gamma}U'(d).$$

Thus the particles approach each other wherever $U'(d) > 0$.

For particles approaching from large separation to reach contact without thermal fluctuations, there must be no intervening repulsive region or stable finite-separation equilibrium. The critical condition is therefore the disappearance of the region where $U'(d) < 0$.

Rewrite the derivative as

$$U'(d) = \frac{A}{d^2} \left[1 - \frac{K\lambda}{A} \left(\frac{d}{\lambda} \right)^2 e^{-d/\lambda} \right].$$

Set $z = d/\lambda > 0$. The function controlling the sign is

$$f(z) = z^2 e^{-z}.$$

Its derivative is

$$f'(z) = ze^{-z}(2 - z),$$

so its global maximum occurs at $z = 2$, with

$$f_{\max} = 4e^{-2}.$$

Accordingly, $U'(d) \geq 0$ for every separation precisely when

$$\frac{K\lambda}{A} 4e^{-2} \leq 1.$$

The critical screening length is

$$\lambda_{\text{crit}} = \frac{\exp(2)A}{4K} = \frac{\exp(2)Aq^2}{4B\epsilon(kT)^2}.$$

At this threshold, the two stationary points of the potential merge at

$$d_{\text{crit}} = 2\lambda_{\text{crit}},$$

where both $U'(d_{\text{crit}})$ and $U''(d_{\text{crit}})$ vanish.

Increasing the salt concentration decreases λ . Substituting λ_{crit} into Eq.(7),

$$\begin{aligned} c_{\text{crit}} &= \frac{\epsilon kT}{2N_A q^2 \lambda_{\text{crit}}^2} \\ &= \frac{\epsilon kT}{2N_A q^2} \left[\frac{4B\epsilon(kT)^2}{\exp(2)Aq^2} \right]^2. \end{aligned}$$

Therefore,

$$c_{\text{crit}} = \frac{8B^2\epsilon^3(kT)^5}{\exp(4) N_A A^2 q^6}.$$

For NaCl, the ions are monovalent, so q is the elementary charge magnitude. The concentration of each ionic species equals the NaCl molar concentration; the factor 2 in Eq.(7) already accounts for both species. With SI parameters, the result has units mol m^{-3} .

For $c > c_{\text{crit}}$, $U'(d) > 0$ at every positive separation, and the deterministic motion brings an initially separated pair into contact. For $c < c_{\text{crit}}$, a pair approaching from sufficiently large separation becomes trapped at the outer local minimum of U .

There is a precise qualification at equality. At $c = c_{\text{crit}}$,

$$U'(d) \propto (d - d_{\text{crit}})^2$$

near d_{crit} . The overdamped approach from above therefore takes infinite time to reach that separation. Thus c_{crit} is the critical threshold, or infimum of concentrations producing coagulation from large separation; the idealized dynamics require $c > c_{\text{crit}}$ for actual passage to contact.

The solution was checked against the official solution.

Problem 2: Neutron Stars

We discuss the stability of large nuclei and estimate the mass of neutron stars theoretically and experimentally.

Part A. Mass and stability of nuclei (2.5 points)

The rest-energy of a nucleus $m(Z, N)c^2$ consisting of Z protons and N neutrons is smaller than the sum of rest-energies of protons and neutrons, hereafter called nucleons, by the binding energy $B(Z, N)$, where c is the speed of light in vacuum. Ignoring minor corrections, we can approximate the binding energy consisting of the volume term with a_V , the surface term with a_S , the Coulomb energy term with a_C , and the symmetry energy term with a_{sym} in the following way.

$$m(Z, N)c^2 = Am_Nc^2 - B(Z, N), \quad B(Z, N) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N - Z)^2}{A}, \quad (1)$$

where $A = Z + N$ is the mass number and m_N is the nucleon mass. In the calculation, use $a_V \approx 15.8 \text{ MeV}$, $a_S \approx 17.8 \text{ MeV}$, $a_C \approx 0.711 \text{ MeV}$, and $a_{\text{sym}} \approx 23.7 \text{ MeV}$ ($\text{MeV} = 10^6$ electron volts).

A.1 — 0.9pt

Under the condition of $Z = N$, determine A for maximizing the binding energy per nucleon, B/A .

A.2 — 0.9pt

Under the condition of fixed A , the atomic number of the most stable nucleus Z^* is determined by maximizing $B(Z, A - Z)$. For $A = 197$, calculate Z^* using Eq. (1).

A.3 — 0.7pt

A nucleus having large A breaks up into lighter nuclei through fission in order to minimize the total rest-mass energy. For simplicity, we consider one of multiple ways to break a nucleus with (Z, N) into two equal nuclei, which occurs when the following energy relation holds,

$$m(Z, N)c^2 > 2m(Z/2, N/2)c^2.$$

When this relation is written as

$$Z^2/A > C_{\text{fission}} \frac{a_S}{a_C},$$

obtain C_{fission} up to two significant digits.

Part B. Neutron star as a gigantic nucleus (1.5 points)

For large nuclei with a large enough mass number $A > A_c$ with a threshold A_c , these nuclei stay stable against nuclear fission because of the sufficiently large binding energy due to gravity.

B.1 — 1.5pt

We assume that $N = A$ and $Z = 0$ is realized for sufficiently large A and Eq. (1) continues to hold with the addition of the gravitational binding energy. The binding energy due to gravity is

$$B_{\text{grav}} = \frac{3}{5} \frac{GM^2}{R},$$

where $M = m_N A$ and $R = R_0 A^{1/3}$ with $R_0 \simeq 1.1 \times 10^{-15} \text{ m} = 1.1 \text{ fm}$ are the mass and the radius of the nucleus, respectively.

For $B_{\text{grav}} = a_{\text{grav}} A^{5/3}$, obtain a_{grav} in the MeV unit up to the first significant digit. Then, ignoring the surface term, estimate A_c up to the first significant digit. In the calculation, use $m_N c^2 \simeq 939 \text{ MeV}$ and $G = \hbar c / M_P^2$ where $M_P c^2 \simeq 1.22 \times 10^{22} \text{ MeV}$ and $\hbar c \simeq 197 \text{ MeV} \cdot \text{fm}$.

Part C. Neutron star in a binary system (6.0 points)

Some neutron stars are pulsars regularly emitting electromagnetic waves, which we call "light" for simplicity here, at a constant period. Neutron stars often make binary systems with a White Dwarf. Let us consider the star configuration shown in Fig. 1, where a light pulse from a neutron star **N** to the Earth **E** passes near a White Dwarf **W** of the binary system. Measuring these pulses influenced by the star's gravity leads to an accurate estimation of the mass of **W** as explained below, resulting in the estimation of the mass of **N**.

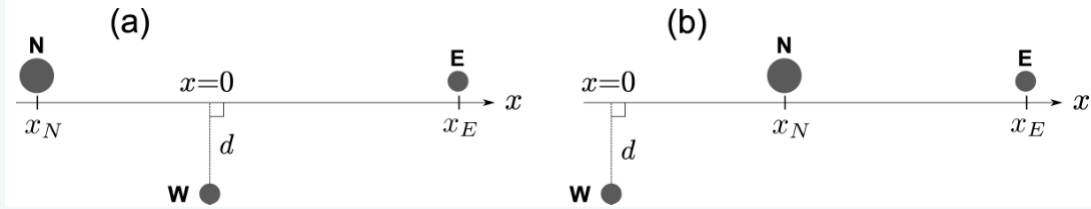
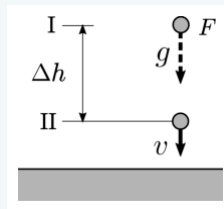


Fig. 1: Configurations with the x -axis along the line connecting **N** and **E**. (a) for $x_N < 0$ and (b) for $x_N > 0$.

C.1 — 1.0pt

As shown in the figure below, under the constant gravitational acceleration g we place two levels I and II with the height difference $\Delta h(> 0)$. Set the identical clocks at I, II, and F , the free-falling system, denoted by clock-I, clock-II, and clock- F , respectively.



Set-up of the thought experiment.

We assume that an observer sits with clock- F , and initially F is placed at the same height as that of clock-I and its velocity is zero. Since the clocks are identical, they register equal time intervals, $\Delta\tau_F = \Delta\tau_I$. Then, we let F fall freely, and work in the frame of F , which is considered to be inertial. In this frame, clock-II passes by clock- F with velocity v , so that the time dilation of clock-II can be determined by the Lorentz transformation. When time $\Delta\tau_I$ elapses on clock- F , time $\Delta\tau_{II}$ elapses on clock-II.

Determine $\Delta\tau_{II}$ in terms of $\Delta\tau_I$ up to the first order in $\Delta\phi/c^2$, where $\Delta\phi = g\Delta h$ is a difference of the gravitational potential, i.e., the gravitational potential energy per unit mass.

C.2 — 1.8pt

Under the gravitational potential ϕ , time delays change the effective speed of light, c_{eff} , observed at the infinity, though the local speed of light is c . When $\phi(r = \infty) = 0$, c_{eff} can be given up to the first order in ϕ/c^2 as

$$c_{\text{eff}} \approx \left(1 + \frac{2\phi}{c^2}\right)c$$

including the effect of space distortion, which was not featured in **C.1**. We note that the light path can be approximated as a straight line.

As shown in Fig. 1 (a), we take the x -axis along the light path from the neutron star **N** to the Earth **E** and place $x = 0$ at the point where the White Dwarf **W** is the closest to the light path. Let $x_N(< 0)$ be the x -coordinate of **N**, $x_E(> 0)$ be that of **E**, and d be the distance between **W** and the light path.

Estimate the changes of the arrival time Δt of the light from **N** to **E** caused by the White Dwarf with mass M_{WD} and evaluate the answer in a simple form disregarding higher order terms of the following small quantities: $d/|x_N| \ll 1$, $d/x_E \ll 1$, and $GM_{\text{WD}}/(c^2 d) \ll 1$. If necessary, use

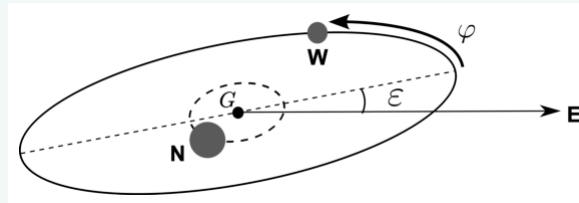
the following formula.

$$\int \frac{dx}{\sqrt{x^2 + d^2}} = \frac{1}{2} \log \left(\frac{\sqrt{x^2 + d^2} + x}{\sqrt{x^2 + d^2} - x} \right) + C.$$

(log is the natural logarithm)

C.3 – 1.8pt

As shown below, in a binary star system **N** and **W** are assumed to be moving in circular orbits with zero eccentricity around the center of mass G on the orbit plane. Let ε be the orbital inclination angle measured from the orbit plane to the line directed toward **E** from G , and let L be the length between **N** and **W** and M_{WD} be the mass of the White Dwarf. In the following, we assume $\varepsilon \ll 1$.



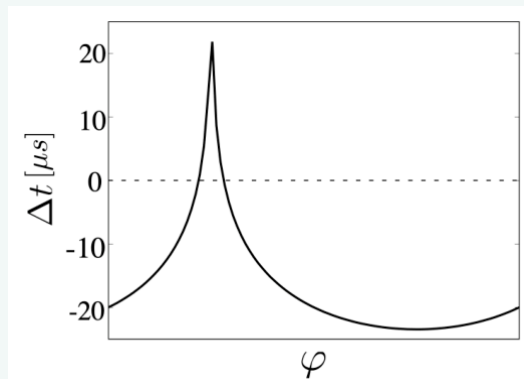
Binary star system.

We observe light pulses from **N** on **E** far away from **N**. The light path to **E** varies with time depending on the configuration of **N** and **W**. The delay in the time interval of arriving pulses on **E** has the maximum value $\Delta t_{\max}$ for $x_N \simeq -L$ and the minimum value $\Delta t_{\min}$ for $x_N \simeq L$ (see Fig. 1 (b) for the configuration).

Calculate $\Delta t_{\max} - \Delta t_{\min}$ in a simple form disregarding higher order terms of small quantities as done in **C.2**. We note that the delays due to gravity from stellar objects other than **W** are assumed to cancel out in $\Delta t_{\max} - \Delta t_{\min}$.

C.4 – 0.8pt

The below figure shows the observed time delays as a function of the orbital phase φ for the binary star system with $L \approx 6 \times 10^6$ km and $\cos \varepsilon \approx 0.99989$. Estimate M_{WD} in terms of the solar mass $M_{\odot}$ and show the results for $M_{WD}/M_{\odot}$ up to the first significant digit. Here the approximate relation, $GM_{\odot}/c^3 \approx 5 \mu s$, can be used.



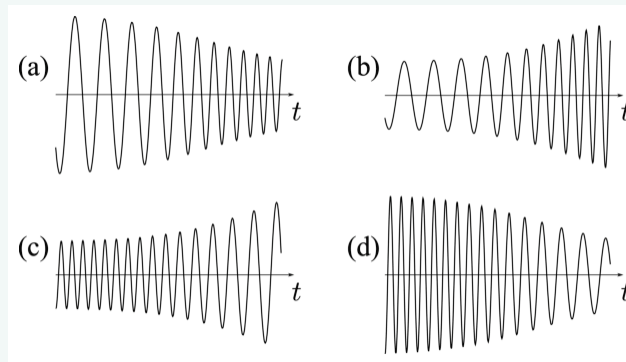
Observed time delays Δt as a function of the orbital phase φ (see the figure in **C.3**) to locate **N** and **W** on the orbits.

C.5 — 0.4pt

In the binary system of neutron stars, two stars release energy and angular momentum by emitting gravitational waves and eventually collide to merge. For simplicity, let us consider only a circular motion with the radius R and the angular velocity ω and then $\omega = \chi R^p$ holds with the constant χ depending on neither ω nor R if relativistic effects are ignored. Determine the value for p .

C.6 — 0.2pt

The amplitude of the emitted gravitational wave from the binary system in **C.5** is proportional to $R^2\omega^2$. Figure below qualitatively shows four different temporal profiles of the observed gravitational waves before the two-star collision. Select the most appropriate profile from (a) to (d).



Observed data profiles of gravitational waves.

Official English text of the problem.

Answers

- **A.1:** $A = \frac{2a_s}{a_c} \approx 50$.
- **A.2:** $Z^* = \frac{A}{2 + \frac{a_c}{2a_{\text{sym}}} A^{2/3}} = 78.55 \dots$, giving the integer atomic number $\boxed{79}$.
- **A.3:** $C_{\text{fission}} = \frac{2^{1/3} - 1}{1 - 2^{-2/3}} \approx \boxed{0.70}$.
- **B.1:** $a_{\text{grav}} \approx \boxed{6 \times 10^{-37} \text{ MeV}}$, $A_c \approx \boxed{4 \times 10^{55}}$, using the positive-total-binding-energy criterion discussed below.
- **C.1:** $\Delta\tau_{\text{II}} \approx \Delta\tau_{\text{I}} \left(1 - \frac{\Delta\phi}{c^2} \right)$.
- **C.2:** $\Delta t \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{4x_E |x_N|}{d^2} \right)$.
- **C.3:** $\Delta t_{\text{max}} - \Delta t_{\text{min}} \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{1 + \cos \varepsilon}{1 - \cos \varepsilon} \right) \approx \frac{4GM_{\text{WD}}}{c^3} \ln \left(\frac{2}{\varepsilon} \right)$.
- **C.4:** $\frac{M_{\text{WD}}}{M_{\odot}} \approx 0.5$, or $M_{\text{WD}} \approx 0.5M_{\odot}$.

- C.5: $p = -\frac{3}{2}$.

- C.6: (b): both amplitude and frequency increase as the stars approach collision.

Solution

A.1

For $Z = N = A/2$, the symmetry term vanishes. The binding energy per nucleon becomes

$$\frac{B}{A} = a_V - a_S A^{-1/3} - \frac{a_C}{4} A^{2/3}.$$

Treating A as continuous,

$$\frac{d}{dA} \left(\frac{B}{A} \right) = \frac{a_S}{3} A^{-4/3} - \frac{a_C}{6} A^{-1/3} = \frac{2a_S - a_C A}{6A^{4/3}}.$$

This derivative is positive below $A = 2a_S/a_C$ and negative above it, so this stationary point is a maximum:

$$A = \frac{2a_S}{a_C} = \frac{2(17.8)}{0.711} = 50.07 \dots$$

Thus

$$A \approx 50.$$

The nearest allowed even mass number, consistent with integer $Z = N$, is $A = 50$.

A.2

At fixed A , $N = A - Z$, hence $N - Z = A - 2Z$. Only the Coulomb and symmetry terms depend on Z :

$$B(Z, A - Z) = \text{constant} - a_C Z^2 A^{-1/3} - \frac{a_{\text{sym}}}{A} (A - 2Z)^2.$$

The maximizing condition is

$$\frac{\partial B}{\partial Z} = -2a_C Z A^{-1/3} + \frac{4a_{\text{sym}}}{A} (A - 2Z) = 0.$$

Rearranging,

$$4a_{\text{sym}} A = Z (8a_{\text{sym}} + 2a_C A^{2/3}),$$

so

$$Z^* = \frac{A}{2 + \frac{a_C}{2a_{\text{sym}}} A^{2/3}}.$$

It is a maximum because

$$\frac{\partial^2 B}{\partial Z^2} = -2a_C A^{-1/3} - \frac{8a_{\text{sym}}}{A} < 0.$$

For $A = 197$,

$$Z^* = \frac{197}{2 + \frac{0.711}{2(23.7)} 197^{2/3}} = 78.553 \dots$$

Since the dependence on Z is a concave quadratic, the optimal integer is the nearest integer:

$$\boxed{Z^* = 79}.$$

A.3

The total number of nucleons is conserved, so the free-nucleon rest-energy terms cancel. Fission is energetically favorable when

$$2B(Z/2, N/2) - B(Z, N) > 0.$$

The volume and symmetry terms are unchanged in total. The daughters' total surface term is

$$-2a_S(A/2)^{2/3} = -a_S 2^{1/3} A^{2/3},$$

and their total Coulomb term is

$$-2a_C \frac{(Z/2)^2}{(A/2)^{1/3}} = -a_C 2^{-2/3} \frac{Z^2}{A^{1/3}}.$$

Therefore,

$$2B(Z/2, N/2) - B(Z, N) = -a_S(2^{1/3} - 1)A^{2/3} + a_C(1 - 2^{-2/3})\frac{Z^2}{A^{1/3}}.$$

The condition for positive energy release becomes

$$\frac{Z^2}{A} > \frac{2^{1/3} - 1}{1 - 2^{-2/3}} \frac{a_S}{a_C}.$$

Hence

$$\boxed{C_{\text{fission}} = \frac{2^{1/3} - 1}{1 - 2^{-2/3}} = 0.702414 \dots \approx 0.70}.$$

Physically, fission becomes favorable when the reduction in Coulomb repulsion exceeds the cost of the additional surface area.

B.1

Substituting the prescribed mass and radius into the gravitational binding energy gives

$$B_{\text{grav}} = \frac{3}{5} \frac{G(m_N A)^2}{R_0 A^{1/3}} = \frac{3Gm_N^2}{5R_0} A^{5/3}.$$

Thus

$$a_{\text{grav}} = \frac{3Gm_N^2}{5R_0} = \frac{3}{5} \frac{\hbar c}{R_0} \left(\frac{m_N c^2}{M_{\text{Pl}} c^2} \right)^2.$$

Using the supplied constants,

$$a_{\text{grav}} = \frac{3}{5} \frac{197 \text{ MeV fm}}{1.1 \text{ fm}} \left(\frac{939}{1.22 \times 10^{22}} \right)^2 = 6.3656 \times 10^{-37} \text{ MeV}.$$

To one significant digit,

$$a_{\text{grav}} \approx 6 \times 10^{-37} \text{ MeV}.$$

For $Z = 0$ and $N = A$, the Coulomb term vanishes and the symmetry term is $-a_{\text{sym}}A$. Ignoring the surface term as instructed,

$$B_{\text{tot}}(A) = (a_V - a_{\text{sym}})A + a_{\text{grav}}A^{5/3},$$

or

$$\frac{B_{\text{tot}}}{A} = -7.9 \text{ MeV} + a_{\text{grav}}A^{2/3}.$$

Taking the finite threshold to mean the onset of binding relative to A free neutrons, the condition is

$$B_{\text{tot}} > 0.$$

Consequently,

$$A > A_c, \quad A_c = \left(\frac{a_{\text{sym}} - a_V}{a_{\text{grav}}} \right)^{3/2}.$$

Retaining the unrounded value of a_{grav} during this calculation,

$$A_c = \left(\frac{7.9}{6.3656 \times 10^{-37}} \right)^{3/2} = 4.3721 \times 10^{55}.$$

Thus

$$A_c \approx 4 \times 10^{55}.$$

The distinction between this binding threshold and a strict two-drop fission comparison is essential here: for $Z = 0$ and with the surface term ignored, $B_{\text{tot}}(A) - 2B_{\text{tot}}(A/2)$ equals $(1 - 2^{-2/3}) a_{\text{grav}}A^{5/3}$, which is positive for every A , so the two-drop comparison gives no finite threshold.

C.1

During free fall through height Δh , Newtonian energy conservation gives, to the required accuracy,

$$\frac{1}{2}v^2 = g\Delta h = \Delta\phi,$$

so

$$v^2 = 2\Delta\phi.$$

In the instantaneous inertial frame of the freely falling observer, clock-II moves with speed v . Special-relativistic time dilation gives

$$\Delta\tau_{\text{II}} = \Delta\tau_F \sqrt{1 - \frac{v^2}{c^2}}.$$

Using the clock comparison prescribed in the problem, $\Delta\tau_F = \Delta\tau_{\text{I}}$, and expanding to first order,

$$\begin{aligned} \Delta\tau_{\text{II}} &= \Delta\tau_{\text{I}} \sqrt{1 - \frac{2\Delta\phi}{c^2}} \\ &\approx \Delta\tau_{\text{I}} \left(1 - \frac{\Delta\phi}{c^2} \right). \end{aligned}$$

Therefore,

$$\Delta\tau_{\text{II}} \approx \Delta\tau_{\text{I}} \left(1 - \frac{\Delta\phi}{c^2} \right).$$

The lower clock runs more slowly.

C.2

At position x , the distance from the white dwarf is

$$r(x) = \sqrt{x^2 + d^2},$$

and its gravitational potential is

$$\phi(x) = -\frac{GM_{\text{WD}}}{\sqrt{x^2 + d^2}}.$$

The given weak-field expression implies

$$\frac{1}{c_{\text{eff}}} \approx \frac{1}{c} \left(1 - \frac{2\phi}{c^2} \right).$$

The travel time without the white dwarf is $(x_E - x_N)/c$. Subtracting it from the actual travel time along the same straight path gives

$$\begin{aligned} \Delta t &= \int_{x_N}^{x_E} \left(\frac{1}{c_{\text{eff}}} - \frac{1}{c} \right) dx \\ &\approx -\frac{2}{c^3} \int_{x_N}^{x_E} \phi(x) dx \\ &= \frac{2GM_{\text{WD}}}{c^3} \int_{x_N}^{x_E} \frac{dx}{\sqrt{x^2 + d^2}}. \end{aligned}$$

An equivalent form of the supplied antiderivative is $\ln(x + \sqrt{x^2 + d^2})$. Thus, at first order in the gravitational potential,

$$\Delta t \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{x_E + \sqrt{x_E^2 + d^2}}{x_N + \sqrt{x_N^2 + d^2}} \right).$$

For $x_E \gg d$,

$$x_E + \sqrt{x_E^2 + d^2} \approx 2x_E.$$

For $x_N < 0$ and $|x_N| \gg d$, rationalizing avoids cancellation:

$$x_N + \sqrt{x_N^2 + d^2} = \frac{d^2}{\sqrt{x_N^2 + d^2} + |x_N|} \approx \frac{d^2}{2|x_N|}.$$

Therefore,

$$\Delta t \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{4x_E|x_N|}{d^2} \right).$$

This is a positive gravitational propagation delay. The logarithm is dimensionless, and GM_{WD}/c^3 has units of time.

C.3

At the two conjunctions, projection of the separation L onto the line of sight gives

$$x_N = -L \cos \varepsilon \quad \text{at maximum delay,}$$

and

$$x_N = +L \cos \varepsilon \quad \text{at minimum delay.}$$

At either conjunction,

$$d = L \sin \varepsilon, \quad \sqrt{x_N^2 + d^2} = L.$$

Since Earth is far from the binary, we may use the same x_E at both configurations to the retained accuracy. The integral result from C.2, before assuming a negative x_N , gives

$$\Delta t \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{2x_E}{L + x_N} \right).$$

Hence

$$\Delta t_{\text{max}} \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{2x_E}{L(1 - \cos \varepsilon)} \right),$$

whereas

$$\Delta t_{\text{min}} \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{2x_E}{L(1 + \cos \varepsilon)} \right).$$

Subtracting eliminates both x_E and L :

$$\Delta t_{\text{max}} - \Delta t_{\text{min}} \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{1 + \cos \varepsilon}{1 - \cos \varepsilon} \right).$$

For $\varepsilon \ll 1$,

$$1 - \cos \varepsilon \approx \frac{\varepsilon^2}{2}, \quad 1 + \cos \varepsilon \approx 2.$$

Thus the requested small-angle form is

$$\Delta t_{\text{max}} - \Delta t_{\text{min}} \approx \frac{2GM_{\text{WD}}}{c^3} \ln \left(\frac{4}{\varepsilon^2} \right) = \frac{4GM_{\text{WD}}}{c^3} \ln \left(\frac{2}{\varepsilon} \right).$$

Here ε is in radians. The logarithmic growth at small inclination corresponds to the light passing increasingly close to the white dwarf, within the stated approximations.

C.4

Reading the graph gives approximately

$$\Delta t_{\text{max}} \approx +22 \mu\text{s}, \quad \Delta t_{\text{min}} \approx -23 \mu\text{s}.$$

The measurable range is therefore approximately

$$\Delta t_{\text{max}} - \Delta t_{\text{min}} \approx 45 \mu\text{s}.$$

An arbitrary constant offset in the plotted delays does not affect this difference.

Using the supplied inclination,

$$\ln \left(\frac{1 + \cos \varepsilon}{1 - \cos \varepsilon} \right) = \ln \left(\frac{1.99989}{0.00011} \right) \approx 9.808.$$

C.3 then gives

$$45 \mu\text{s} \approx 2 (5 \mu\text{s}) \frac{M_{\text{WD}}}{M_{\odot}} (9.808).$$

Therefore,

$$\frac{M_{\text{WD}}}{M_{\odot}} \approx \frac{45}{98.08} \approx 0.46.$$

To one significant digit,

$$\boxed{\frac{M_{\text{WD}}}{M_{\odot}} \approx 0.5}, \quad \boxed{M_{\text{WD}} \approx 0.5 M_{\odot}}.$$

The supplied separation L cancels from the maximum-minus-minimum delay.

C.5

Let the stellar masses be M_1, M_2 , and let their separation be r . Newtonian circular two-body motion obeys

$$\omega^2 r^3 = G(M_1 + M_2).$$

For fixed masses, either star's orbital radius R is a constant multiple of r . Consequently,

$$\omega \propto r^{-3/2} \propto R^{-3/2}.$$

Comparing with $\omega = \chi R^p$,

$$\boxed{p = -\frac{3}{2}}.$$

C.6

As the binary loses orbital energy, its orbital radius decreases. From C.5,

$$\omega \propto R^{-3/2},$$

so the orbital frequency, and hence the gravitational-wave frequency, increases.

The given amplitude dependence yields

$$\text{amplitude} \propto R^2 \omega^2 \propto R^2 R^{-3} = R^{-1}.$$

Thus the amplitude also increases as R decreases. The required waveform has an expanding envelope and progressively closer oscillations:

$$\boxed{\text{Profile (b)}}.$$

The solution was checked against the official solution.

Problem 3: Water and Objects

In this problem, we consider the phenomena caused by the interaction between water and objects, related to surface tension. Part A treats motion, while Parts B and C are regarding static situations.

If necessary, you can use the fact that if the function $y(x)$ satisfies the differential equation $y''(x) = ay(x)$ (a is a positive constant), then its general solution is $y(x) = Ae^{\sqrt{a}x} + Be^{-\sqrt{a}x}$, where A and B are arbitrary constants.

Part A. Merger of water drops (2.0 points)

As shown in Fig.1, we consider two stationary, spherical water drops on the surface of a super-hydrophobic material, i.e., very strong repulsive force exists between material and water.

Initially neighboring two identical spherical water drops are placed on the surface; then these two drops are merged after touching each other and form a larger spherical water drop, which suddenly jumps up.

A.1 2.0pt

The radius a of both water drops before the merger is $100\ \mu\text{m}$. The density of water ρ is $1.00 \times 10^3\ \text{kg/m}^3$. The surface tension γ is $7.27 \times 10^{-2}\ \text{J/m}^2$.

A portion k of the difference of the surface energy before and after the merger, ΔE , is transformed into the kinetic energy of the jumped water drop. Then, determine the initial jump-up velocity, v , of the merged water drop in two significant digits under the following assumptions:

- $k = 0.06$
- Before and after merger, the total volume of water is conserved.

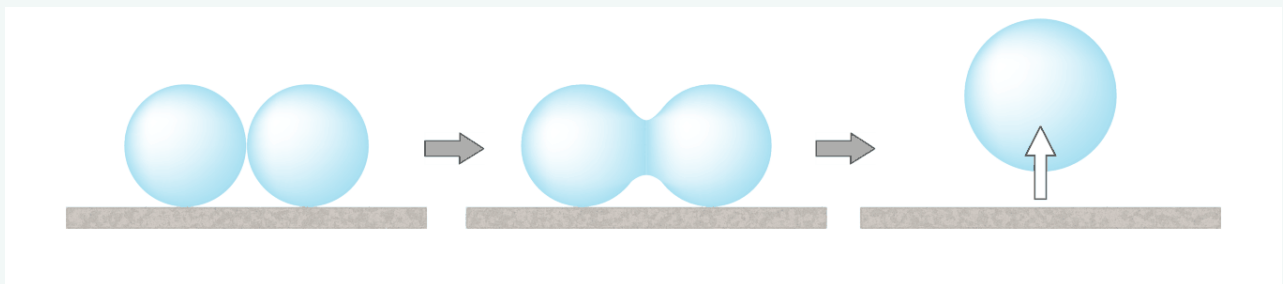


Fig. 1: Merger of two water drops and jump of the merged water drop.

Part B. A vertically placed board (4.5 points)

A flat board is immersed vertically in water. Figures 2(a) and 2(b) respectively show water surface forms for the hydrophilic (attractive) and hydrophobic board materials. We neglect the thickness of the board.

The board surface is on the yz plane, and the horizontal water surface far away from the board is on the xy plane with $z = 0$. The surface shape does not depend on the y -coordinate. Let $\theta(x)$ be the angle between the water surface and the horizontal plane at a point (x, z) on the water surface in the xz plane. Here $\theta(x)$ is measured with respect to the positive x axis and the

counterclockwise rotation is taken as positive. Let $\theta(x)$ be θ_0 at the point of contact between the board and the water surface ($x = 0$). In the following, θ_0 is fixed by the properties of the board material.

Water density ρ is constant and water surface tension γ is uniform. The gravitational acceleration constant is given by g . The atmospheric pressure, P_0 , is assumed to be always uniform. Let us determine the water surface form in the following steps. Note that the unit of surface tension is J/m^2 as well as N/m .

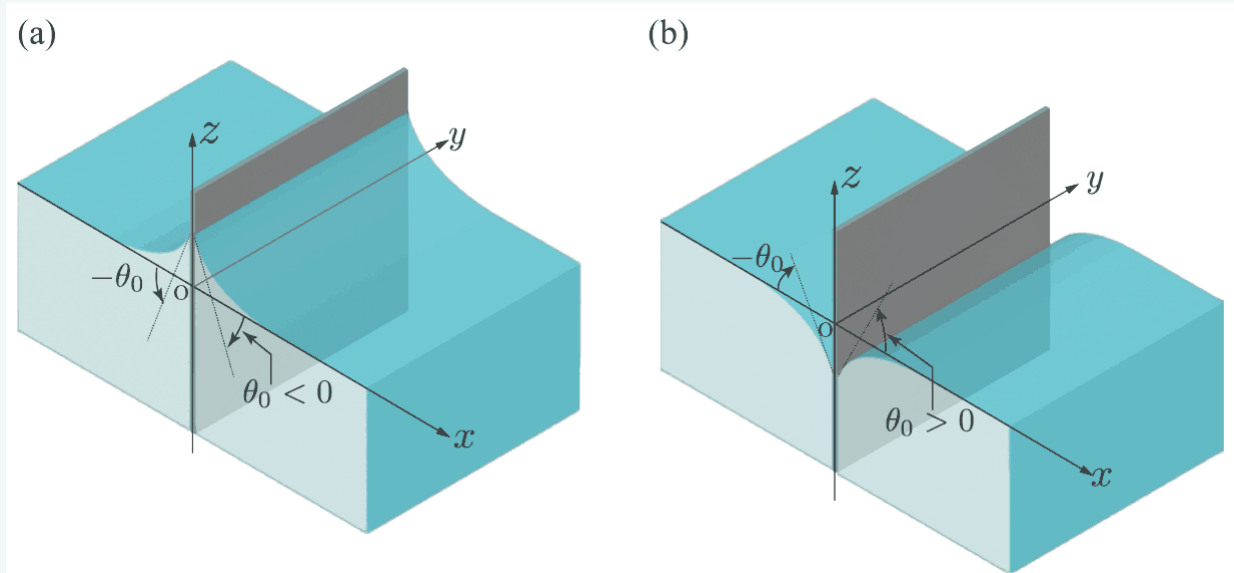


Fig. 2: Boards vertically immersed in the water. (a) hydrophilic board case; (b) hydrophobic board case.

B.1 0.6pt

We consider a hydrophilic board case, as shown in Fig.2(a). We note that the water pressure, P , satisfies the conditions $P < P_0$ for $z > 0$ and $P = P_0$ for $z = 0$. Then, express P at z in terms of ρ , g , z , and P_0 .

B.2 0.8pt

We consider a water block whose cutout is shown as shaded in Fig.3(a). Its xz plane cross-section is shown in a hatched area in Fig.3(b). Let z_1 and z_2 respectively be the left and right edge coordinates of the boundary (water surface) between the water block and the air.

Obtain a horizontal component (x component) of the net force per unit length along the y -axis, f_x , which is exerted on the water block due to the pressure, in terms of ρ , g , z_1 , and z_2 . Note that P_0 results in no net horizontal force on the water block.

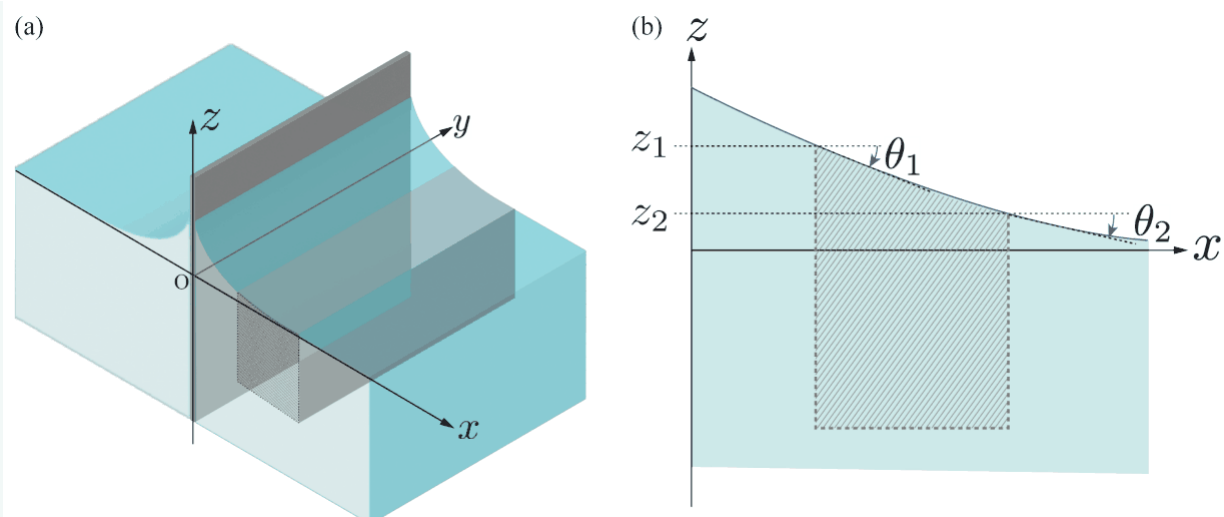


Fig. 3: Cutout form of water block on the water surface. (a) Bird's eye view and (b) cross-sectional view.

B.3 0.8pt

Surface tension acting on the water block is balanced with the force f_x discussed in B.2. We respectively define θ_1 and θ_2 as the angles between the water surface and the horizontal plane at the left and right edges. Express f_x in terms of γ , θ_1 , and θ_2 .

B.4 0.8pt

The following equation holds at an arbitrary point (x, z) on the water surface,

$$\frac{1}{2} \left(\frac{z}{\ell} \right)^a + \cos \theta(x) = \text{constant}. \quad (1)$$

Determine the exponent a and express the constant ℓ in terms of γ and ρ . Note that this equation holds regardless of hydrophilic or hydrophobic board materials.

B.5 1.5pt

In Eq. (1) in B.4, we assume that variation of the water surface is slow, i.e., $|z'(x)| \ll 1$, so that we can expand $\cos \theta(x)$ with respect to $z'(x)$ up to the second order. Then, differentiating the resultant equation with respect to x , we obtain the differential equation satisfied by $z(x)$. Solve this differential equation and determine $z(x)$ for $x \geq 0$ in terms of $\tan \theta_0$ and ℓ . Note that the vertical directions of Figs. 2 and 3 are exaggerated for better view and they do not satisfy the condition, $|z'(x)| \ll 1$.

Part C. Interaction between two rods (3.5 points)

The identical rods A and B made of the same material floating in parallel on the water surface are placed at the same distance away from the y -axis (Fig.4).

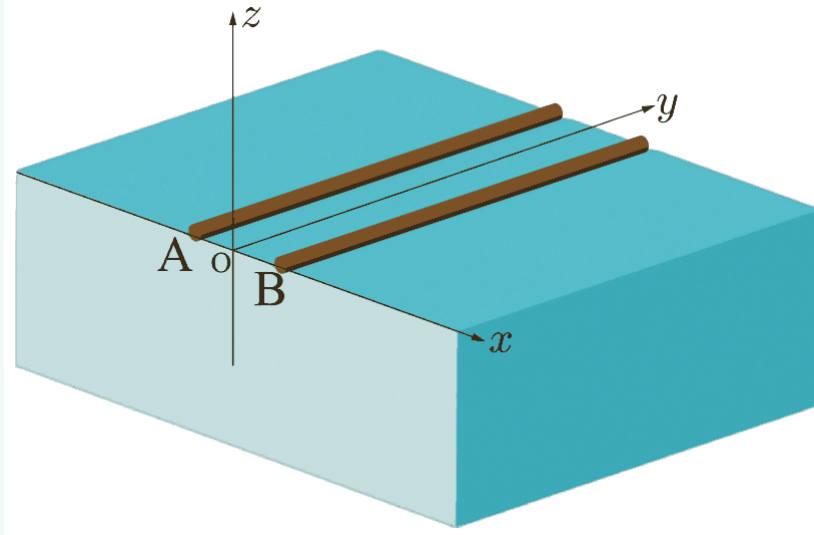


Fig. 4: Two rods A and B floating on the water surface.

C.1 1.0pt

At the contact points of the rod B and the water surface, we define the z -coordinates z_a and z_b , and the angles θ_a and θ_b , as shown in Fig.5. Determine the horizontal force component, F_x , on the rod B per unit length along the y -axis in terms of θ_a , θ_b , z_a , z_b , ρ , g , and γ .

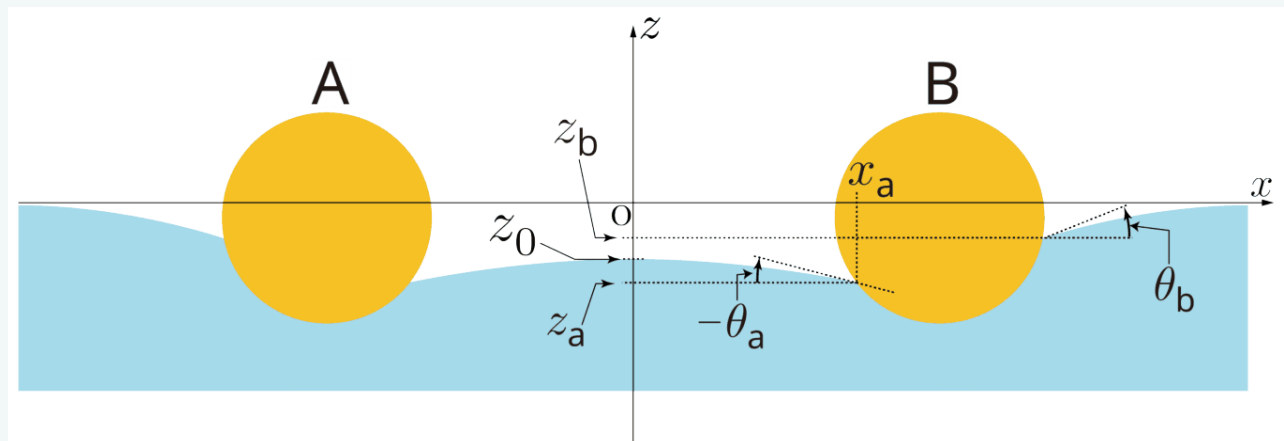


Fig. 5: Vertical cross-sectional view of two rods floating on the water surface.

C.2 1.5pt

We define the z -coordinate of the water surface, z_0 , at the midpoint of two rods in the xz plane. Express the force F_x obtained in C.1 without using θ_a , θ_b , z_a , and z_b .

C.3 1.0pt

Let x_a be the x -coordinate of the contact point between the water surface and the left side of the rod B. Using the differential equation obtained in B.5, express the water level coordinate z_0 of the midpoint of these two rods A and B in terms of x_a and z_a . You can use the constant ℓ introduced in B.4.

Official English text of the problem.

Answers

Positive horizontal forces point along $+x$. Both f_x and F_x are forces per unit length along y , with units N/m.

- A.1

$$v = \sqrt{\frac{3k\gamma}{\rho a} (2 - 2^{2/3})} = 0.23 \text{ m/s} \quad \text{upward.}$$

- B.1

$$P(z) = P_0 - \rho g z.$$

- B.2

$$f_x = \frac{\rho g}{2} (z_2^2 - z_1^2).$$

- B.3

$$f_x = \gamma (\cos \theta_1 - \cos \theta_2).$$

- B.4

$$a = 2, \quad \ell = \sqrt{\frac{\gamma}{\rho g}}.$$

The constant on the right of Eq. (1) is 1 for the board's meniscus.

- B.5, in the small-slope approximation:

$$z''(x) = \frac{z(x)}{\ell^2}, \quad z(x) = -\ell \tan \theta_0 e^{-x/\ell}, \quad x \geq 0.$$

- C.1

$$F_x = \gamma (\cos \theta_b - \cos \theta_a) + \frac{\rho g}{2} (z_b^2 - z_a^2).$$

- C.2

$$F_x = -\frac{\rho g}{2} z_0^2.$$

The force on B is toward A when $z_0 \neq 0$.

- C.3, in the small-slope approximation:

$$z_0 = \frac{z_a}{\cosh(x_a/\ell)} = \frac{2z_a}{e^{x_a/\ell} + e^{-x_a/\ell}}.$$

Solution

A.1

Let R be the radius of the merged drop. Conservation of volume gives

$$\frac{4\pi R^3}{3} = 2 \frac{4\pi a^3}{3}, \quad R = 2^{1/3}a.$$

The mass of the merged drop is therefore

$$m = \rho \frac{4\pi R^3}{3} = \frac{8\pi}{3} \rho a^3.$$

Surface energy equals surface tension multiplied by surface area. Thus the decrease in surface energy is

$$\begin{aligned} \Delta E &= \gamma (2 \cdot 4\pi a^2 - 4\pi R^2) \\ &= 4\pi \gamma a^2 (2 - 2^{2/3}). \end{aligned}$$

Using the stipulated conversion of a fraction k into translational kinetic energy,

$$\frac{1}{2}mv^2 = k\Delta E.$$

Consequently,

$$v^2 = \frac{3k\gamma}{\rho a} (2 - 2^{2/3}).$$

With $a = 1.00 \times 10^{-4} \text{ m}$,

$$m = 8.3776 \times 10^{-9} \text{ kg}, \quad \Delta E = 3.7694 \times 10^{-9} \text{ J},$$

and

$$v = 0.23236 \dots \text{ m/s}.$$

To two significant digits, the initial velocity is

$$\boxed{v = 0.23 \text{ m/s upward.}}$$

B.1

In hydrostatic equilibrium, the upward pressure force balances gravity, giving

$$\frac{dP}{dz} = -\rho g.$$

Since ρ is constant and $P(0) = P_0$, integration yields

$$\boxed{P(z) = P_0 - \rho g z.}$$

In particular, water above $z = 0$ has pressure below atmospheric pressure, as stated.

B.2

A uniform pressure P_0 applied over the entire closed boundary produces zero resultant force. We can therefore calculate using the gauge pressure

$$p(z) = P(z) - P_0 = -\rho g z.$$

The gauge pressure on the air-facing surface is zero. Let z_* be the common height of the block's bottom. Pressure on the left vertical face pushes toward $+x$, and pressure on the right face pushes toward $-x$. The bottom contributes no horizontal force.

Per unit length along y ,

$$\begin{aligned}
f_x &= \int_{z_*}^{z_1} p(z) dz - \int_{z_*}^{z_2} p(z) dz \\
&= -\rho g \int_{z_2}^{z_1} z dz \\
&= \frac{\rho g}{2} (z_2^2 - z_1^2).
\end{aligned}$$

Hence

$$f_x = \frac{\rho g}{2} (z_2^2 - z_1^2).$$

For the illustrated hydrophilic meniscus, $z_1 > z_2 > 0$, so this pressure force points toward $-x$.

B.3

Surface tension pulls tangentially away from the block at each edge. Its horizontal components per unit length are

$$-\gamma \cos \theta_1 \quad \text{at the left edge,} \quad +\gamma \cos \theta_2 \quad \text{at the right edge.}$$

Horizontal equilibrium requires

$$f_x - \gamma \cos \theta_1 + \gamma \cos \theta_2 = 0.$$

Therefore,

$$f_x = \gamma (\cos \theta_1 - \cos \theta_2).$$

B.4

Equating the expressions from B.2 and B.3,

$$\frac{\rho g}{2} (z_2^2 - z_1^2) = \gamma (\cos \theta_1 - \cos \theta_2).$$

Rearranging gives

$$\frac{\rho g}{2\gamma} z_1^2 + \cos \theta_1 = \frac{\rho g}{2\gamma} z_2^2 + \cos \theta_2.$$

Since the two edges can be chosen arbitrarily along the surface,

$$\frac{\rho g}{2\gamma} z^2 + \cos \theta(x) = C.$$

Comparison with Eq. (1) gives

$$a = 2, \quad \ell = \sqrt{\frac{\gamma}{\rho g}}.$$

The length ℓ is the capillary length. The derivation permits either sign of z , so it applies to both raised and depressed menisci.

Far from the board, $z \rightarrow 0$ and $\theta \rightarrow 0$. Thus $C = 1$, and the board's exact surface relation is

$$\frac{z^2}{2\ell^2} + \cos \theta(x) = 1.$$

B.5

The tangent to the surface satisfies

$$\tan \theta(x) = z'(x).$$

For the surface represented as a graph,

$$\cos \theta(x) = \frac{1}{\sqrt{1 + [z'(x)]^2}}.$$

Under the stated approximation $|z'(x)| \ll 1$, its expansion through second order is

$$\cos \theta(x) = 1 - \frac{1}{2}[z'(x)]^2 + O([z'(x)]^4).$$

Substitution into Eq. (1) gives, to this order,

$$\frac{z^2}{2\ell^2} + 1 - \frac{1}{2}(z')^2 = C.$$

Differentiating,

$$\frac{zz'}{\ell^2} - z'z'' = 0.$$

For a nonflat profile this yields

$$z'' = \frac{z}{\ell^2},$$

with the equation extending to isolated stationary points by continuity. The flat profile $z = 0$ also satisfies it.

The general solution is

$$z(x) = Ae^{x/\ell} + Be^{-x/\ell}.$$

The boundary condition $z(x) \rightarrow 0$ as $x \rightarrow +\infty$ requires $A = 0$. At the board,

$$z'(0) = \tan \theta_0 = -\frac{B}{\ell},$$

so $B = -\ell \tan \theta_0$. Therefore,

$$z(x) = -\ell \tan \theta_0 e^{-x/\ell}, \quad x \geq 0.$$

For a hydrophilic board, $\theta_0 < 0$, so $z > 0$. For a hydrophobic board, $\theta_0 > 0$, so $z < 0$. In both cases the surface approaches $z = 0$ exponentially over the length scale ℓ . Consistency requires $|\tan \theta_0| \ll 1$.

C.1

There are two contributions to the horizontal force on rod B: surface tension at the contact lines and pressure on the rod.

Surface tension pulls the rod toward the adjoining free surface. Its net horizontal component is therefore

$$F_{\gamma,x} = -\gamma \cos \theta_a + \gamma \cos \theta_b.$$

For the pressure force, subtract the uniform atmospheric pressure, whose resultant over the closed rod surface is zero. The remaining pressure on the wetted surface is $p(z) = -\rho g z$.

The horizontal pressure force on a curved surface is obtained by integrating pressure over its vertical projection. Let z_* be the height of the bottom of the rod. The left wetted side contributes toward $+x$, and the right wetted side toward $-x$:

$$\begin{aligned} F_{p,x} &= \int_{z_*}^{z_a} (-\rho g z) dz - \int_{z_*}^{z_b} (-\rho g z) dz \\ &= \frac{\rho g}{2} (z_b^2 - z_a^2). \end{aligned}$$

Gravity has no horizontal component. Adding the two contributions,

$$F_x = \gamma (\cos \theta_b - \cos \theta_a) + \frac{\rho g}{2} (z_b^2 - z_a^2).$$

C.2

The surface relation from B.4 applies separately to the surface between the rods and to the exterior surface. Their integration constants need not be equal.

On the exterior side of B, the surface becomes horizontal at $z = 0$ far away. Thus

$$\frac{z_b^2}{2\ell^2} + \cos \theta_b = 1.$$

Between the identical rods, reflection symmetry about $x = 0$ makes the surface horizontal at the midpoint. There $z = z_0$, so the interior relation is

$$\frac{z_a^2}{2\ell^2} + \cos \theta_a = 1 + \frac{z_0^2}{2\ell^2}.$$

Since $\rho g = \gamma/\ell^2$, the force from C.1 can be written as

$$F_x = \gamma \left[\left(\cos \theta_b + \frac{z_b^2}{2\ell^2} \right) - \left(\cos \theta_a + \frac{z_a^2}{2\ell^2} \right) \right].$$

Substitution gives

$$F_x = \gamma \left[1 - \left(1 + \frac{z_0^2}{2\ell^2} \right) \right].$$

Hence

$$F_x = -\frac{\gamma z_0^2}{2\ell^2} = -\frac{\rho g}{2} z_0^2.$$

Rod B lies on the positive- x side, so the negative sign means attraction toward rod A. The force vanishes when $z_0 = 0$. This result does not require the small-slope approximation.

C.3

Now apply the small-slope equation from B.5 to the surface between the rods:

$$z'' = \frac{z}{\ell^2}.$$

Writing its solution in hyperbolic form,

$$z(x) = C_1 \cosh(x/\ell) + C_2 \sinh(x/\ell).$$

Symmetry gives $z'(0) = 0$, which requires $C_2 = 0$. Since $z(0) = z_0$, we have $C_1 = z_0$, and therefore

$$z(x) = z_0 \cosh(x/\ell).$$

At the inner contact with rod B, $x = x_a$ and $z = z_a$. Consequently,

$$z_a = z_0 \cosh(x_a/\ell),$$

so

$$z_0 = \frac{z_a}{\cosh(x_a/\ell)} = \frac{2z_a}{e^{x_a/\ell} + e^{-x_a/\ell}}.$$

This height–separation relation is valid within the small-slope approximation.

The solution was checked against the official solution.