

IPhO 2022

Theory problems and solutions · Switzerland (held online) · 3 theory problems with complete solutions

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Problem 1: Permanent magnets

Strong permanent magnets are made from NdFeB alloy which obeys a very wide hysteresis loop so that the magnetization J can be assumed to be constant over a wide range of applications; in what follows, we assume that $J \equiv 1.5 \text{ T}/\mu_0$, where $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$, and the magnetization of all the permanent magnets is homogeneous. *Magnetization is defined as the volume density of the magnetic dipole moment of the matter.*

Hint 1. The following equality might be useful:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Hint 2. The magnetic field created by a spherical magnet is identical to that of a point dipole. The magnetic fields created by magnets of other shapes become equivalent to a point dipole fields only at distances much larger than their diameter.

Hint 3. Electric and magnetic fields of electric and magnetic point dipoles as functions of coordinates and of the corresponding dipole moment are similar, i.e. one can be obtained from the other by multiplying it by a constant factor.

Hint 4. The induced field due to a boundary condition can always be replaced by some configuration of field sources outside the given boundaries.

Part A. Interaction of magnets (4.5 points)

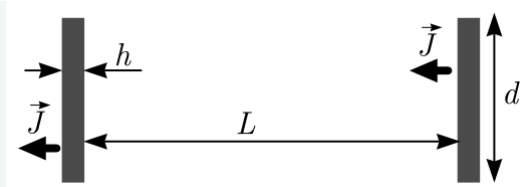
When the distance to a magnet is much larger than its size, the magnetic field created by it can be approximated with the magnetic field of its dipole moment $\vec{m}$,

$$\vec{B} = \frac{\mu_0}{4\pi r^3} (2\vec{m}_{\parallel} - \vec{m}_{\perp}).$$

Here $r = |\vec{r}|$, and we have decomposed the dipole moment into components parallel and perpendicular to the radius vector $\vec{r}$ drawn from the dipole to the observation point, $\vec{m} = \vec{m}_{\perp} + \vec{m}_{\parallel}$.

A.1 — 0.6pt

Find the magnitude of the interaction force between two coaxially placed cylindrical magnets of diameter $d = 20 \text{ mm}$ and thickness $h = 2 \text{ mm}$, magnetized parallel to their axis, if the distance between the centres of the magnets is $L = 20 \text{ cm}$. You may assume that $L \gg d, h$.



A.2 — 0.4pt

At distances much larger than $\frac{h}{2}$, the field created by the magnet from task A.1 is the same as that created by a circular current I . Find I .

A.3 — 1.0pt

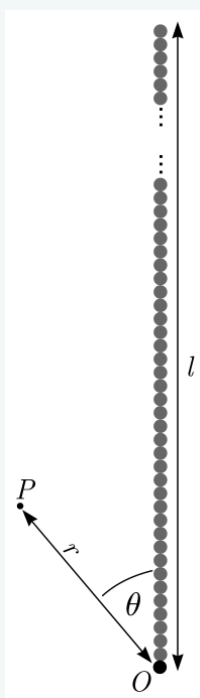
Find the interaction force between the magnets for the setup of task A.1 if instead $L = 5 \text{ mm}$. You may assume that $d \gg L \gg h$.

A.4 — 1.0pt

Identical spherical magnets of diameter $\delta = 5 \text{ mm}$, bound together by magnetic attraction, form a chain. What is the maximal length l for such a chain which does not break under its own weight when hanging beneath the topmost magnet? The density of NdFeB magnets $\rho = 7500 \text{ kg/m}^3$.

A.5 — 1.5pt

Consider the chain from part A.4. Obtain an expression for the magnitude of the magnetic B -field at such a point P which is at distance r from one of the chain's endpoint O , and the angle between the chain and the line OP is θ (cf. figure below), assuming that $l \gg r$ and $r \sin \theta \gg \delta$



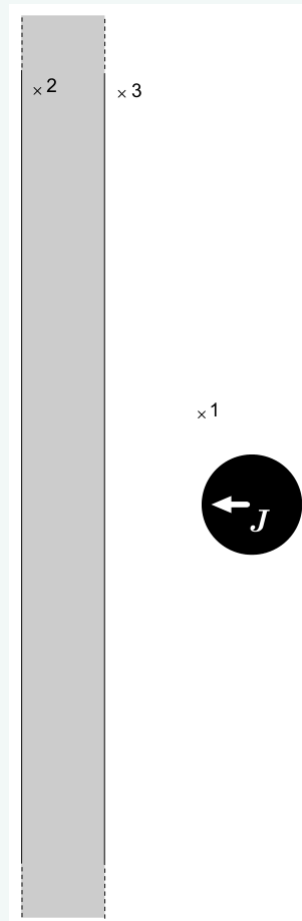
Part B. Interaction with ferromagnets (3.5 points)

Now we assume that in addition to the permanent magnets we have also plates made from a ferromagnetic material, similar to what is used in transformer cores. In the situations we're concerned with, it can be considered to have a constant but very large relative permeability $\mu_r \sim 10^5$.

Hint 5. Large permeability means that magnetic field lines near the outside surface of an object made of the material are nearly perpendicular to the surface. This is similar to the behavior of electric field lines near the outside surface of a conductor.

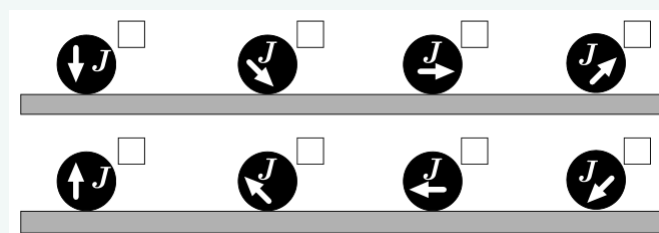
B.1 — 1.0pt

A spherical magnet from part A.4 is at a distance $s = \delta$ from a thick infinite ferromagnetic plate (see the answer sheet). The magnetization of the sphere is oriented perpendicular to the plate. Sketch the field lines in the cross-section shown in the answer sheet. In that figure, three points (denoted as 1, 2, and 3) are marked; you need to show field lines passing through each of these points in their full length, i.e. as much as fits into the figure.



B.2 — 1.0pt

Now the spherical magnet is brought into direct contact with the plate. Which direction is taken by the magnetization vector of the spherical magnet at a stable equilibrium and what is the normal force between the plate and the magnet? Mark the correct direction(s) with a tick in the corresponding box in the answer sheet. Incorrect ticks will reduce your score.



B.3 — 1.5pt

Now a magnet from part A.1 is placed between two thick circular ferromagnetic plates of diameter $D = 2d$ so that the flat faces of the magnet are pressed against the plates and all three discs are coaxial. Find the magnetic force F acting on each plate. *Hint:* You may neglect the magnetic field **both** outside the ferromagnetic plates and outside the gap between them.

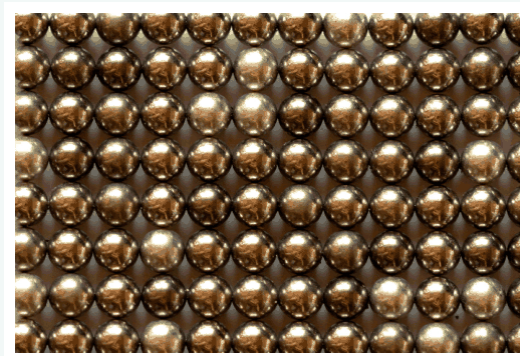
Part C. (Anti)ferromagnetic order (2 points)

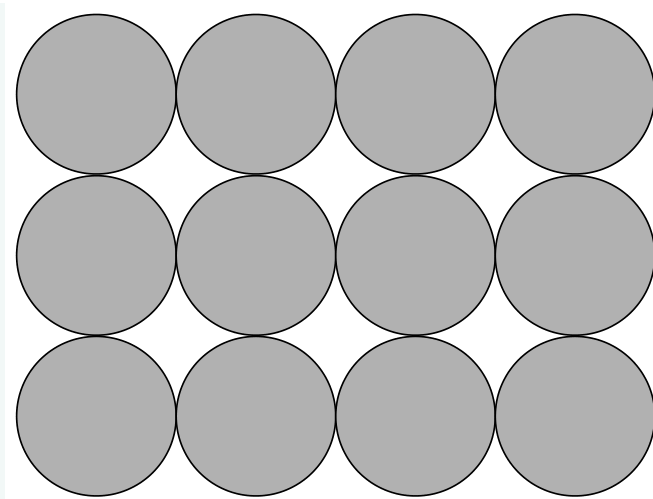
The magnetic properties of materials are due to the magnetic dipole moments of electrons and atomic nuclei. If the dipole moments orient themselves parallel to each other, the field created by them is magnified — these are ferromagnetic materials. On the other hand, if for each dipole moment there is another antiparallel dipole moment nearby, the fields cancel out — these are anti-ferromagnetic materials.

In what follows, we consider a very large number of spherical magnets of part A.4, arranged at the nodes of a two-dimensional lattice; **see real photos of stable equilibrium configurations below**. Assume that all the magnetization vectors lie in the plane of the figure. Consider in your calculations only nearest-neighbour interactions (on the figure of C.1, each magnet has four nearest neighbours, and on the figure of C.2 — six).

C.1 — 0.8pt

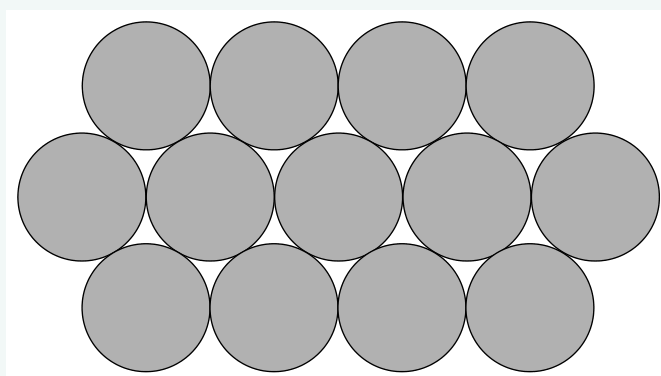
Show the magnetization directions of the magnets in the figure below. You are not required to prove that the configuration you suggested is the only possibility. You still need to justify that the configuration you suggested is indeed stable. Find the energy needed to pull one magnet out of this lattice from somewhere in the middle of the lattice, assuming the other magnets are kept stationary. Does this configuration correspond to the order of ferromagnetic or antiferromagnetic materials?





C.2 — 1.2pt

Answer the same questions as in task C.1 for the configuration shown in the figure below.



Physical Constants

Below is the list of physical constants you might use in your solution in addition to the quantities given in the problem text.

Physical constant	Notation	Numerical value
Speed of light in vacuum	c	299 792 458 m/s
Vacuum permeability (magnetic constant)	μ_0	$4\pi \cdot 10^{-7} \text{ N/A}^2$; $[1 \text{ N/A}^2 = 1 \text{ Vs/Am}]$
Vacuum permittivity (electric constant)	ϵ_0	$8.8541878128(13) \cdot 10^{-12} \text{ F/m}$; $[1 \text{ F/m} = 1 \text{ As/Vm}]$
Elementary charge	e	$1.602176634 \cdot 10^{-19} \text{ C}$; $[1 \text{ C} = 1 \text{ A} \cdot \text{s}]$

Physical constant	Notation	Numerical value
Electron rest mass	m_e	$9.1093837015(28) \cdot 10^{-31} \text{ kg}$
Proton rest mass	m_p	$1.67262192369(51) \cdot 10^{-27} \text{ kg}$
Neutron rest mass	m_n	$1.67492749804(95) \cdot 10^{-27} \text{ kg}$
Atomic mass unit	m_{amu}	$1.66053906660(50) \cdot 10^{-27} \text{ kg}$
Gravitational constant	G	$6.67430(15) \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$
Avogadro constant	N_A	$6.02214076 \cdot 10^{23} \text{ mol}^{-1}$
Molar gas constant	R	$8.31446261815324 \dots \text{ J}/(\text{K} \cdot \text{mol})$
Boltzmann constant	k_B	$1.380649 \cdot 10^{-23} \text{ J/K}$
Stefan–Boltzmann constant	σ	$5.670374419 \dots \cdot 10^{-8} \text{ W}/(\text{m}^2 \cdot \text{K}^4)$
Planck constant	h	$6.62607015 \cdot 10^{-34} \text{ J} \cdot \text{s}$
Reduced Planck constant	$\hbar = h/(2\pi)$	$1.0545718176 \dots \cdot 10^{-34} \text{ J} \cdot \text{s}$

Official English text of the problem.

Answers

All force values below are magnitudes.

- **A.1:**

$$F = \frac{3\pi\mu_0 J^2 d^4 h^2}{32L^4} \simeq 2.1 \times 10^{-4} \text{ N}$$

The force is attractive for the directions shown.

- **A.2:** A circular current of diameter d , with

$$I = Jh \simeq 2.4 \times 10^3 \text{ A}.$$

- **A.3:**

$$F \simeq \frac{\mu_0 J^2 h^2 d}{2L} \simeq 14 \text{ N}$$

The force is attractive.

- **A.4:** In the long-chain approximation,

$$l_{\text{max}} \simeq \frac{\pi^4 \mu_0 J^2}{360 \rho g} \simeq 6.6 \text{ m}.$$

- **A.5:**

$$B \simeq \frac{\mu_0 J \delta^2}{24r^2}$$

This magnitude is independent of θ within the stated approximations.

- **B.1:** The exterior field is that of the real dipole together with an equally strong mirror dipole pointing in the same direction. Field lines meet the plate normally. The line through **1** forms a short return loop in air; the lines through **2** and **3** belong to larger loops carrying flux through the plate. Their detailed directions are described in the Solution.
- **B.2:** Tick the **first drawing in each row**: magnetization perpendicular to the plate, either toward it or away from it.

$$F = \frac{\pi\mu_0 J^2 \delta^2}{24} \simeq 5.9 \text{ N}$$

The magnetic force is attractive.

- **B.3:** Each plate is pulled toward the magnet and the other plate, with

$$F = \frac{3\pi\mu_0 J^2 d^2}{32} \simeq 2.1 \times 10^2 \text{ N}.$$

- **C.1:** All moments in a horizontal row point in the same direction; successive rows point oppositely. This is **antiferromagnetic order**.

$$W = \frac{\pi\mu_0 J^2 \delta^3}{24} \simeq 2.9 \times 10^{-2} \text{ J}.$$

- **C.2:** All moments point in the same horizontal direction. This is **ferromagnetic order**, passing the single-magnet rotational stability test described below.

$$W = \frac{\pi\mu_0 J^2 \delta^3}{48} \simeq 1.5 \times 10^{-2} \text{ J}.$$

This shows stability against rotating a single magnet only; in the nearest-neighbour model, some collective rotations of many magnets lower the energy.

Solution

The magnetization and the two relevant dipole moments are

$$J = \frac{1.5}{\mu_0} = 1.19366 \times 10^6 \text{ A/m},$$

$$m_c = J \frac{\pi d^2 h}{4} = 0.750 \text{ A m}^2, \quad m_s = J \frac{\pi \delta^3}{6} = 0.078125 \text{ A m}^2.$$

For two permanent dipoles, the interaction energy is

$$U_{12} = \frac{\mu_0}{4\pi r^3} [\vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \hat{r})(\vec{m}_2 \cdot \hat{r})].$$

Equivalently, $U_{12} = -\vec{m}_2 \cdot \vec{B}_1$. Forces follow by differentiating this energy with respect to position, holding the permanent moments fixed.

A.1

Since $L \gg d, h$, both cylinders can be replaced by point dipoles. Their moments are parallel to each other and to their common axis, so

$$U(L) = -\frac{\mu_0 m_c^2}{2\pi L^3}.$$

The force tends to decrease L , and its magnitude is

$$F = \frac{dU}{dL} = \frac{3\mu_0 m_c^2}{2\pi L^4}.$$

Substituting $m_c = J\pi d^2 h/4$,

$$F = \frac{3\pi\mu_0 J^2 d^4 h^2}{32L^4}.$$

With $L = 0.20$ m,

$$F = 2.109375 \times 10^{-4} \text{ N} \simeq \boxed{2.1 \times 10^{-4} \text{ N}}.$$

A.2

A uniformly magnetized body can be represented by bound currents. Its bound surface-current density is

$$\vec{K}_b = \vec{J} \times \hat{n}.$$

For an axially magnetized cylinder, this current flows around the curved side, with magnitude $K_b = J$. Integrating over the cylinder thickness gives an equivalent circular current

$$\boxed{I = Jh}.$$

At distances large compared with the thickness, the thin cylindrical current sheet can be replaced by a single loop of diameter d .

The same result follows by equating the magnetic moments:

$$I \frac{\pi d^2}{4} = J \frac{\pi d^2 h}{4}.$$

Numerically,

$$I = (1.19366 \times 10^6)(2.0 \times 10^{-3}) = 2.3873 \times 10^3 \text{ A} \simeq \boxed{2.4 \text{ kA}}.$$

Its circulation is chosen by the right-hand rule to produce the specified magnetization direction.

A.3

The condition $L \gg h$ permits the current-loop representation from A.2. Because $L \ll d$, corresponding nearby portions of the two loops can be treated as parallel straight wires separated by L .

The attractive force per unit length between parallel currents is

$$\frac{dF}{d\ell} = \frac{\mu_0 I^2}{2\pi L}.$$

To leading order, this force acts along the common axis. Multiplying by the circumference πd gives

$$F \simeq \frac{\mu_0 I^2}{2\pi L} \pi d = \frac{\mu_0 I^2 d}{2L}.$$

Thus

$$F \simeq \frac{\mu_0 J^2 h^2 d}{2L}.$$

For $L = 5.0 \times 10^{-3} \text{ m}$,

$$F \simeq 14.3239 \text{ N} \simeq \boxed{14 \text{ N}}.$$

This is the leading result under the stated hierarchy $d \gg L \gg h$.

A.4

The spheres align with their moments along the chain, all pointing in the same direction. Each sphere has mass

$$M = \rho \frac{\pi \delta^3}{6}.$$

Two spheres whose centres are $n\delta$ apart attract with force

$$F_n = \frac{3\mu_0 m_s^2}{2\pi(n\delta)^4} = \frac{F_1}{n^4},$$

where

$$F_1 = \frac{3\mu_0 m_s^2}{2\pi\delta^4} = \frac{\pi\mu_0 J^2 \delta^2}{24}.$$

For a chain containing N spheres, the held top sphere must support the other $N - 1$ spheres. The largest upward magnetic force it can exert on them, while all spheres remain touching, is

$$F_{\text{top}} = F_1 \sum_{n=1}^{N-1} \frac{1}{n^4}.$$

Consequently,

$$(N - 1)Mg \leq F_1 \sum_{n=1}^{N-1} \frac{1}{n^4}.$$

The uppermost junction is limiting: a cut farther down supports less weight, while its magnetic attraction includes at least one pair at every separation present across the uppermost junction.

For a long chain,

$$F_{\text{top}} \simeq F_1 \frac{\pi^4}{90}.$$

Also, to leading order in $1/N$, the suspended weight is

$$Mg \frac{l}{\delta} = \frac{\rho \pi \delta^2 g}{6} l.$$

Equating this to the limiting magnetic force,

$$\frac{\rho \pi \delta^2 g}{6} l_{\text{max}} = \frac{\pi \mu_0 J^2 \delta^2}{24} \frac{\pi^4}{90}.$$

Therefore,

$$l_{\text{max}} \simeq \frac{\pi^4 \mu_0 J^2}{360 \rho g}.$$

The sphere diameter cancels: the magnetic holding force and the chain's mass per unit length both scale as δ^2 .

Taking $g = 9.81 \text{ m/s}^2$,

$$F_1 = 5.8594 \text{ N}, \quad F_{\text{top}} \simeq 6.3417 \text{ N},$$

$$l_{\text{max}} \simeq 6.5848 \text{ m} \simeq \boxed{6.6 \text{ m}}.$$

Counting the supported top sphere and rounding to a whole number of spheres changes the length only by an amount of order δ .

A.5

Because $r \sin \theta \gg \delta$, the discrete chain can be replaced by a continuous line of dipoles. Its dipole moment per unit length is

$$\lambda_m = \frac{m_s}{\delta} = \frac{\pi J \delta^2}{6}.$$

Place O at the origin and let the chain extend along $+z$, from 0 to l . Choose all moments along $+z$. In the exterior region, define a magnetic scalar potential by

$$\vec{B} = -\mu_0 \nabla \Phi.$$

The potential of the line is

$$\Phi(\vec{r}) = \frac{\lambda_m}{4\pi} \int_0^l \frac{z - z'}{|\vec{r} - z' \hat{z}|^3} dz'.$$

Since

$$\frac{\partial}{\partial z'} \frac{1}{|\vec{r} - z' \hat{z}|} = \frac{z - z'}{|\vec{r} - z' \hat{z}|^3},$$

integration gives

$$\Phi(\vec{r}) = \frac{\lambda_m}{4\pi} \left(\frac{1}{|\vec{r} - l \hat{z}|} - \frac{1}{r} \right).$$

Thus the exterior field is equivalent to that of two opposite effective magnetic poles at the endpoints:

$$\vec{B} = \frac{\mu_0 \lambda_m}{4\pi} \left[\frac{\vec{r} - l \hat{z}}{|\vec{r} - l \hat{z}|^3} - \frac{\vec{r}}{r^3} \right].$$

For $l \gg r$, the distant endpoint contributes negligibly, leaving

$$\vec{B} \simeq -\frac{\mu_0 \lambda_m}{4\pi r^2} \hat{r}.$$

Hence

$$\boxed{B \simeq \frac{\mu_0 J \delta^2}{24 r^2}}.$$

The field is radial about the nearby endpoint, so its magnitude is independent of θ . Reversing all moments reverses the field direction.

B.1

Let the plate occupy $x < 0$, with its surface at $x = 0$, and let the sphere's centre be at $x = a > 0$. In the answer-sheet drawing,

$$\vec{m} = -m_s \hat{x}.$$

For a highly permeable plate, the exterior tangential field must nearly vanish. This boundary condition is produced by placing a mirror dipole at $x = -a$, with the **same normal orientation**:

$$\vec{m}_{\text{image}} \simeq -m_s \hat{x}.$$

More precisely, its magnitude is multiplied by

$$\frac{\mu_r - 1}{\mu_r + 1} \simeq 1.$$

At the surface, the tangential components of the two dipole fields cancel. If y denotes height above the sphere's centre, the remaining normal component is

$$B_x(0, y) = \frac{\mu_0 m_s}{2\pi} \frac{y^2 - 2a^2}{(a^2 + y^2)^{5/2}}.$$

Thus flux enters the plate near the sphere's axis, where $|y| < \sqrt{2}a$, and emerges farther away, where $|y| > \sqrt{2}a$.

The requested sketch is described as follows:

- The sphere's **left side is its north side**, and its right side is its south side. Exterior arrows run from the left side toward the right side along curved return paths. Inside the sphere, arrows run approximately right to left.
- **Through point 1:** draw a short upper return loop entirely in the air outside the plate. It leaves the sphere's upper-left surface, rises through point 1, bends over toward the right, and enters the sphere's upper-right surface.
- **Through point 2:** draw a larger loop carrying flux upward through the plate. At point 2 it points mainly upward and somewhat toward the surface. It bends rightward and emerges from the surface above point 3, perpendicular to the surface, before curving back to the sphere's south side. Traced backward, its segment inside the plate descends and connects to flux entering the plate nearer the magnet.
- **Through point 3:** draw a line emerging approximately horizontally to the right from the plate. It then curves downward and rightward toward the sphere's south side. Continue it backward through the plate as a curved path returning to a lower entry point, supplied by flux from the sphere's north side.
- The lower-half pattern is the reflection of the upper-half pattern about the horizontal line through the sphere's centre. Every line continues through the material and magnet to form a closed magnetic-field line.

For the ideal half-space limit, the field inside the plate is approximately twice the field that the real dipole alone would create there. This fixes the curved continuations through point 2 as well as their directions.

B.2

Let $\hat{n}$ point normally out of the plate, and decompose the sphere's moment as

$$\vec{m} = m_n \hat{n} + \vec{m}_t.$$

Its image has moment

$$\vec{m}' = m_n \hat{n} - \vec{m}_t.$$

If the sphere's centre is distance a from the plate, the image is $2a$ away. Its field at the real dipole is

$$\vec{B}_{\text{image}} = \frac{\mu_0}{4\pi(2a)^3} (2m_n \hat{n} + \vec{m}_t).$$

Because the plate magnetization is induced, the interaction energy is

$$U = -\frac{1}{2} \vec{m} \cdot \vec{B}_{\text{image}}.$$

Writing θ for the angle between $\vec{m}$ and the plate normal,

$$U(a, \theta) = -\frac{\mu_0 m_s^2}{8\pi(2a)^3} (1 + \cos^2 \theta).$$

This is minimized at

$$\theta = 0 \quad \text{or} \quad \theta = \pi.$$

Therefore, both directions perpendicular to the plate are stable: toward the plate and away from it. These are the first drawings in the upper and lower rows.

At either stable orientation, the attractive force has magnitude

$$F = \frac{\partial U}{\partial a} = \frac{3\mu_0 m_s^2}{2\pi(2a)^4}.$$

At contact, $a = \delta/2$, so

$$F = \frac{3\mu_0 m_s^2}{2\pi\delta^4} = \boxed{\frac{\pi\mu_0 J^2 \delta^2}{24}}.$$

Numerically,

$$F = 5.8594 \text{ N} \simeq \boxed{5.9 \text{ N}}.$$

B.3

Let

$$A = \frac{\pi d^2}{4}$$

be the magnet's face area, and

$$A_p = \frac{\pi D^2}{4} = 4A$$

the area of each plate.

The permeability of the plates is very large, so their internal H -field is negligible. Each plate is therefore approximately an equipotential surface for the magnetic scalar potential. Neglecting fringing, the axial H -field has the same value in the magnet and in the surrounding annular air gap.

Let B_m be the signed axial field in the magnet and B_a that in the annular gap. Since the magnetization is fixed,

$$B_m = \mu_0(H + J), \quad B_a = \mu_0 H.$$

Hence

$$B_m - B_a = \mu_0 J.$$

All flux passing through the magnet must return through the annular gap, because exterior flux is neglected. Therefore,

$$AB_m + (A_p - A)B_a = 0.$$

Solving,

$$B_m = \mu_0 J \left(1 - \frac{A}{A_p} \right), \quad B_a = -\mu_0 J \frac{A}{A_p}.$$

For $A_p = 4A$,

$$B_m = \frac{3}{4}\mu_0 J = 1.125 \text{ T}, \quad B_a = -\frac{1}{4}\mu_0 J = -0.375 \text{ T}.$$

A magnetic field normal to a highly permeable surface produces an attractive pressure

$$p = \frac{B^2}{2\mu_0}.$$

At the magnet–plate contact this can be evaluated by imagining an infinitesimal vacuum slit: normal B is continuous across it. Both the central region and the annular return-flux region pull the plate inward, irrespective of their opposite field directions.

Thus

$$F = \frac{AB_m^2 + (A_p - A)B_a^2}{2\mu_0}.$$

Substitution gives

$$F = \frac{\mu_0 J^2 A}{2} \left(1 - \frac{A}{A_p} \right).$$

With $A_p = 4A$,

$$\boxed{F = \frac{3\pi\mu_0 J^2 d^2}{32}}.$$

Numerically,

$$F = 210.9375 \text{ N} \simeq \boxed{2.1 \times 10^2 \text{ N}}$$

on each plate.

C.1

Define the nearest-neighbour dipolar energy scale

$$\varepsilon_d = \frac{\mu_0 m_s^2}{4\pi\delta^3} = \frac{\pi\mu_0 J^2 \delta^3}{144} = 4.8828125 \times 10^{-3} \text{ J}.$$

Choose moments along horizontal rows, reversing their direction in successive rows. On the answer sheet, the top row can point right, the middle row left, and the bottom row right:

$$\begin{array}{cccc} \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \leftarrow & \leftarrow & \leftarrow & \leftarrow \\ \rightarrow & \rightarrow & \rightarrow & \rightarrow \end{array}$$

Consider a central magnet pointing right.

Its two horizontal neighbours also point right and lie on its dipole axis. Each contributes interaction energy

$$U_{\text{horizontal}} = -2\varepsilon_d.$$

Its two vertical neighbours point left and lie perpendicular to its moment. Each contributes

$$U_{\text{vertical}} = -\varepsilon_d.$$

The central magnet's total interaction energy with its neighbours is therefore

$$U_{\text{central}} = 2(-2\varepsilon_d) + 2(-\varepsilon_d) = -6\varepsilon_d.$$

All four neighbours produce fields parallel to the central moment, with total magnitude

$$B_{\text{nn}} = \frac{6\varepsilon_d}{m_s}.$$

Rotating that magnet through an angle α , while keeping its neighbours fixed, increases the energy by

$$\Delta U = 6\varepsilon_d(1 - \cos \alpha) > 0.$$

Thus it experiences a restoring torque.

For fixed square-lattice sites, the pattern also minimizes the full nearest-neighbour orientational energy. To see this, write a unit moment as (u_i, v_i) . Then

$$\frac{U}{\varepsilon_d} = \sum_{\langle ij \rangle_x} (-2u_i u_j + v_i v_j) + \sum_{\langle ij \rangle_y} (u_i u_j - 2v_i v_j) \geq -3 \sum_i (u_i^2 + v_i^2) = -3N.$$

The inequality follows from $2|ab| \leq a^2 + b^2$, and the proposed stripe pattern attains the bound in the bulk.

Removing the central magnet to infinity eliminates its four bonds, so the required work is

$$W = 0 - U_{\text{central}} = 6\varepsilon_d.$$

There is no factor of $1/2$: each bond involving the removed magnet is counted once. Therefore,

$$W = \frac{\pi\mu_0 J^2 \delta^3}{24} \simeq 2.9 \times 10^{-2} \text{ J}.$$

Successive rows have opposite magnetization, giving **antiferromagnetic order**.

C.2

Choose all moments horizontal and pointing right, including every sphere in the answer-sheet diagram.

A central sphere has two horizontal neighbours and four diagonal neighbours whose displacement vectors make angles of 60° or 120° with the moments.

Each horizontal pair contributes

$$U_{\text{horizontal}} = -2\varepsilon_d.$$

For parallel moments at angle β to their separation,

$$U_{\text{pair}} = \varepsilon_d(1 - 3\cos^2 \beta).$$

Thus each diagonal neighbour contributes

$$U_{\text{diagonal}} = \varepsilon_d \left(1 - \frac{3}{4} \right) = \frac{1}{4} \varepsilon_d.$$

Summing the six bonds,

$$U_{\text{central}} = 2(-2\varepsilon_d) + 4\left(\frac{1}{4}\varepsilon_d\right) = -3\varepsilon_d.$$

The transverse components of the neighbours' fields cancel. The remaining field points along the central moment and has magnitude

$$B_{\text{nn}} = \frac{3\varepsilon_d}{m_s}.$$

Consequently, rotating one magnet through α with all its neighbours fixed costs

$$\Delta U = 3\varepsilon_d(1 - \cos \alpha) > 0.$$

This establishes a restoring torque against an individual rotation. It does not establish stability against collective rotations: in the nearest-neighbour model, rotating all moments together through the same angle leaves the energy unchanged, because the six bond directions are 60° apart, and some non-uniform collective rotations lower the energy.

The work required to remove the central magnet is

$$W = 0 - U_{\text{central}} = 3\varepsilon_d,$$

so

$$W = \frac{\pi\mu_0 J^2 \delta^3}{48} \simeq 1.5 \times 10^{-2} \text{ J}.$$

All moments point in the same direction, so this configuration has **ferromagnetic order**.

The solution was checked against the official solution.

Problem 2: James Webb Space Telescope

This is a question on the physics of the James Webb Space Telescope. Light from a star strikes the primary mirror, with an area of $A_{\text{mirror}} = 25 \text{ m}^2$, and reflects off of a secondary mirror. The focal length of the system is $f = 130 \text{ m}$. The light is focused into the ISIM (Integrated Science Instrument Module), which contains the CCD (charged-coupled device) cameras.

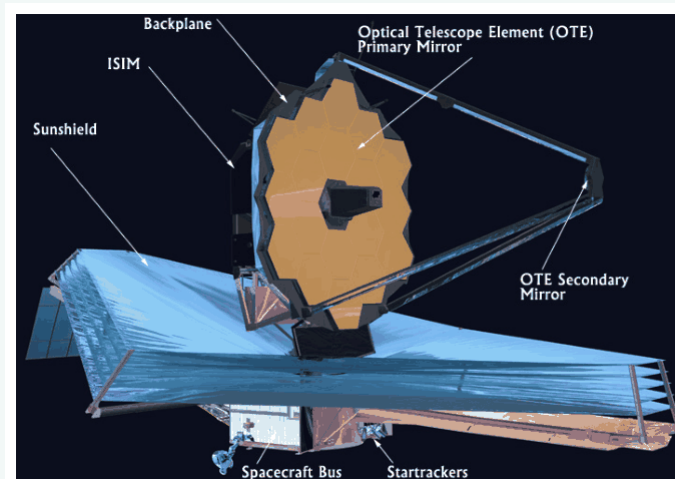


Image Credit: NASA

Part A. Imaging a Star (1.8 points)

The nearest Red Giant is 89 light-years distant, has a temperature of $T_{\text{star}} = 3600 \text{ K}$, and a diameter of $d_o = 1.7 \times 10^{11} \text{ m}$.

A.1 — 0.4pt

Calculate the diameter of a focused image of the star on the CCD camera imaging surface.

A.2 — 0.4pt

Estimate the diameter of a diffraction central maximum on the CCD camera imaging surface. Assume a wavelength of $\lambda = 800 \text{ nm}$, which is the strongest intensity wavelength from the red giant star.

A.3 — 1.0pt

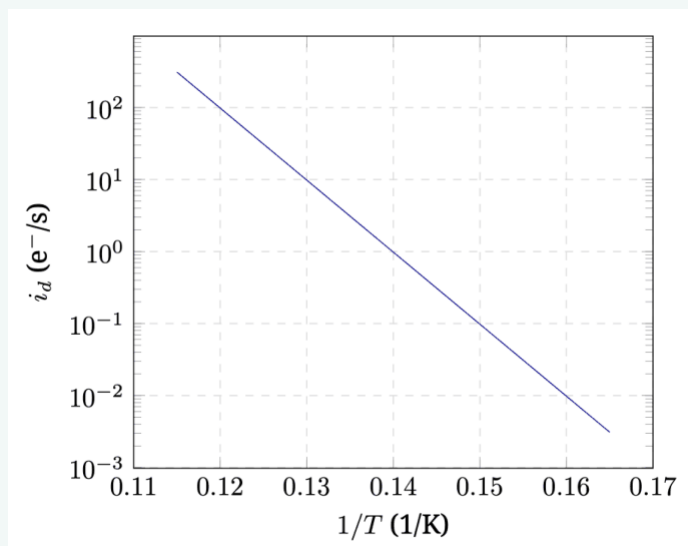
If the CCD is not cooled and can lose heat only by radiating from the top of the imaging surface, what would be the equilibrium temperature of the CCD at the location of the image of the red giant star? Assume the CCD surface is a blackbody. Provide a formula and a numerical estimate.

Part B. Counting Photons (1.8 points)

The absorption of a photon by the CCD camera leads to the emission of an electron within the apparatus. This occurs only if the photon has sufficient energy to excite an electron across an energy gap ΔE_g . Assume that every photon with sufficient energy succeeds. There is also leakage of electrons across the gap caused by the temperature of the CCD camera; this is the dark current i_d and is measured in the number of electrons per second. It is a function of temperature according to

$$i_d = i_0 e^{-|\Delta E_g|/(6k_B T)}. \quad (1)$$

where i_0 is a constant.



The graph shows how dark current varies with temperature. The units for dark current, e^-/s should be thought of as counting a number of electrons per second.

B.1 — 0.4pt

From the dark current graph, provide an order of magnitude estimate for the temperature of a distant source of thermal photons that would just be capable of exciting an electron on the pixel.

The electrons are collected in a capacitor, and after an exposure time τ , the electrons are counted. There are three main sources of uncertainty in the process: a fixed uncertainty in the counting process called read out noise; a Poisson distribution error associated with the dark current, and a Poisson distribution error associated with the detected incoming photons. Poisson distribution errors are equal to the square root of the number of counts associated with a process. The measured photon count is equal to the number of electrons in the capacitor, minus the number of electrons associated with the dark current.

B.2 — 0.4pt

Write an expression for the total count uncertainty σ_t , if there is a readout noise σ_r , a dark current i_d , an incoming photon rate p , and an exposure time τ .

For remaining questions in this part assume the exposure time is $\tau = 10^4$ s and the read out noise is a fixed $\sigma_r = 14$.

B.3 — 0.5pt

Assume an operating temperature of $T_p = 7.5$ K. Calculate the minimum photon rate p so that the photon count is ten times the count uncertainty.

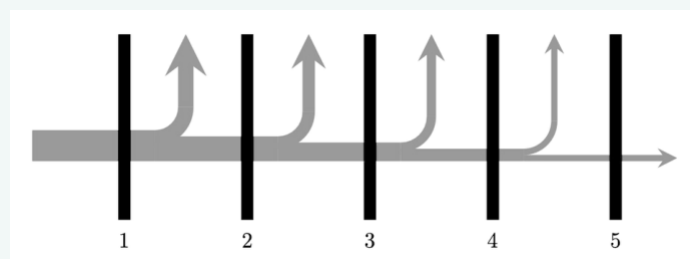
B.4 — 0.5pt

Assuming all photons are just capable of exciting an electron across the band gap, what is the intensity of the source of photons found in B.3 on the primary mirror? Express your answer in W/m^2

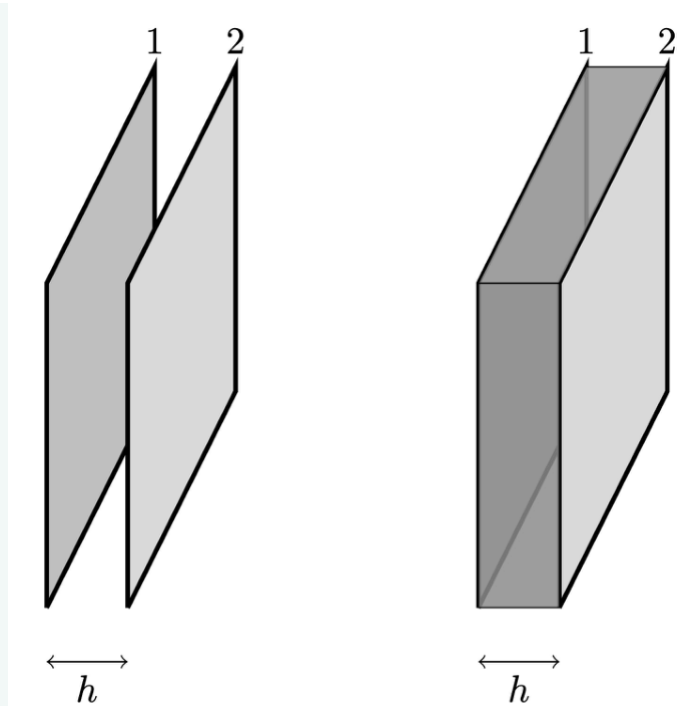
Part C. Passive Cooling (4.4 points)

An infrared CCD camera must be kept at a low temperature. The first tool is a shield to protect from the sun's radiation.

The sun-shield consists of five separated reflective layers in thin sheets (black); radiant energy (gray) from the sun is incident on the first sheet on left, and some energy escapes between every pair of sheets.



Schematic of energy flow: the vertical lines (black) are the sheets, the flow of energy (gray) is from the left to the right, however, between sheets, some energy flows up and out into space.



On the left is a simple model of two adjacent sheets 1 and 2 separated by a distance h . The sheets are not connected, and the perimeter is open to space. Assume the sheets are parallel. Thermal radiation can be exchanged between the sheets, and thermal radiation can escape through the perimeter gap. On the right, the perimeter gap has been shaded to help visualize.

Assume the following simplifications:

- Sheets are square, each with area $A_{\text{sheet}} = 200 \text{ m}^2$.
- Sheets are parallel and separated by $h = 25 \text{ cm}$ along the perimeter.
- Sheets have constant emissivity $\epsilon \ll 1$. Assume that all reflections off of sheet surfaces are diffuse.
- Sheets are thin with temperature on the front and back surfaces equal and uniform.
- The fraction of radiant flux emitted by a sheet that is absorbed by the adjacent sheet is $\alpha \leq 1$. This means that if sheet 1 in the figure above emits an amount of heat Q_1 toward sheet 2 then sheet 2 will absorb an amount αQ_1 from sheet 1.
- The amount of radiant flux ejected out of the perimeter gap between two sheets is approximated as βQ_{12} where αQ_{12} is the net flux between the two sheets. The fraction $\beta < 1$. This is equivalent to saying that the heat loss to space between two sheets is proportional to the net heat exchange between the sheets. This is a rough approximation for this problem.
- Background temperature of space is negligible.

C.1 – 2.4pt

Derive expressions for the equilibrium temperatures of the first sheet and fifth sheet in terms of the incident solar radiation intensity I_0 , the constants α and β , and any necessary physical constants. To simplify your expression, you may define additional constants in terms of α and β , etc.

C.2 — 1.6pt

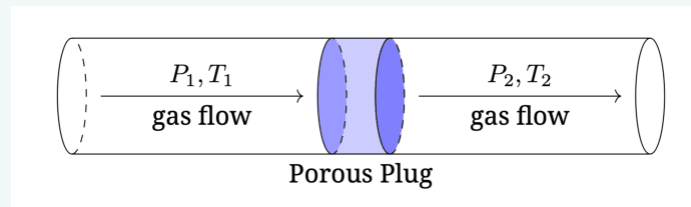
Derive numerical estimates for α and β from the information about the sheet geometry assuming an emissivity $\epsilon = 0.05$. You are encouraged to consider the rectangular box model of the sheets above, where the perimeter area effectively acts as a perfect absorber of radiant energy.

C.3 — 0.4pt

Numerically determine the temperatures of sheet 1 and sheet 5. The solar intensity is $I_0 = 1360 \text{ W/m}^2$.

Part D. Cryo-cooler (4 points)

The last stage of the cooling system directly cools the CCD camera. A closed cycle refrigeration system has a supply pipe line feeding helium gas at constant pressure P_1 moving through a sponge like porous plug into a pipe with constant pressure P_2 . The pipe carries the gas to cool the CCD. The helium gas then passes through a pump before returning to the supply line.



Helium gas supplied on the left at well defined pressure P_1 and temperature T_1 is forced through the plug to well defined pressure P_2 and temperature T_2 , where it is carried away on the right.

As the gas moves through the porous plug, viscous friction with the narrow walls of the channels in the sponge becomes an important effect; however, no heat is transferred to or from the gas during the process. The bulk speed of the gas in region 2 is only marginally greater than the bulk speed in region 1.

Helium is not an ideal gas, but does remain in a gaseous state throughout this process.

D.1 — 1.0pt

Consider a mole of gas that passes from left to right through the plug. Complete the table in your answer sheet by writing '>' or '<' to identify the quantity that must be greater, '=' to identify quantities that must be equal, or '?' if it is not possible to know which is greater or equal without more information.

Quantity Name	State 1	Compare using ">", "<", "=", or "?"	State 2
Internal Energy	U_1		U_2
Temperature	T_1		T_2
Entropy	S_1		S_2
Pressure	P_1		P_2
Volume	V_1		V_2

D.2 — 0.6pt

Identify a conserved quantity constructed from U (internal energy), P (pressure), and V (volume) as a mole of gas moves through the plug; show work on how you derived this conserved quantity.

Your answer sheets have graphs of internal energy per mass against volume per mass for helium with isotherms and lines of constant entropy.

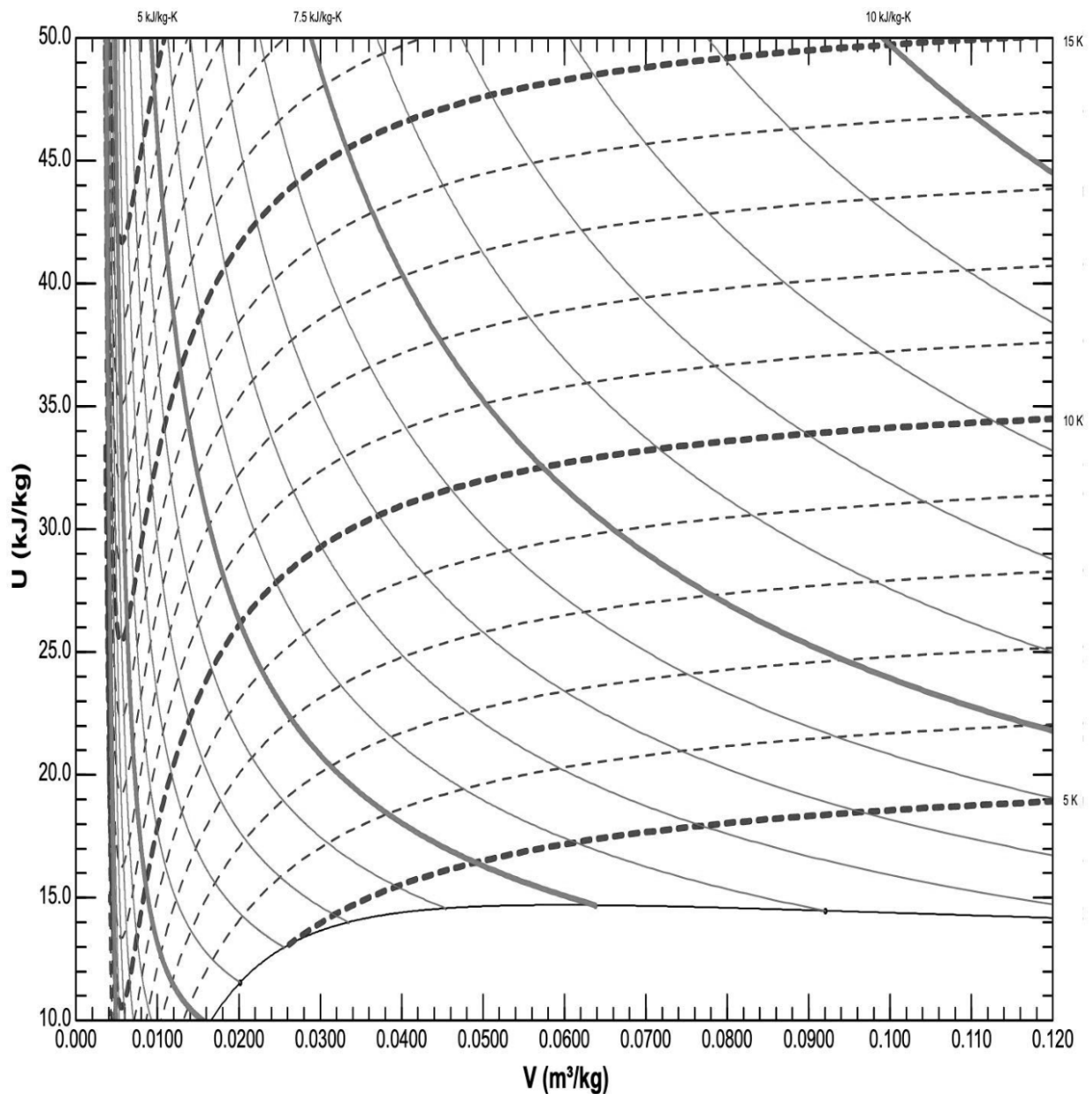
D.3 — 1.4pt

Assuming that $V_2 = 0.100 \text{ m}^3/\text{kg}$ and $T_2 = 7.5 \text{ K}$, use the graph to find a numerical value for the conserved quantity that you found in Part D.2. Show the construction on the graph!

Solid curves (trending down with increasing volume) are constant entropy S ; values for the bold curves are on top.

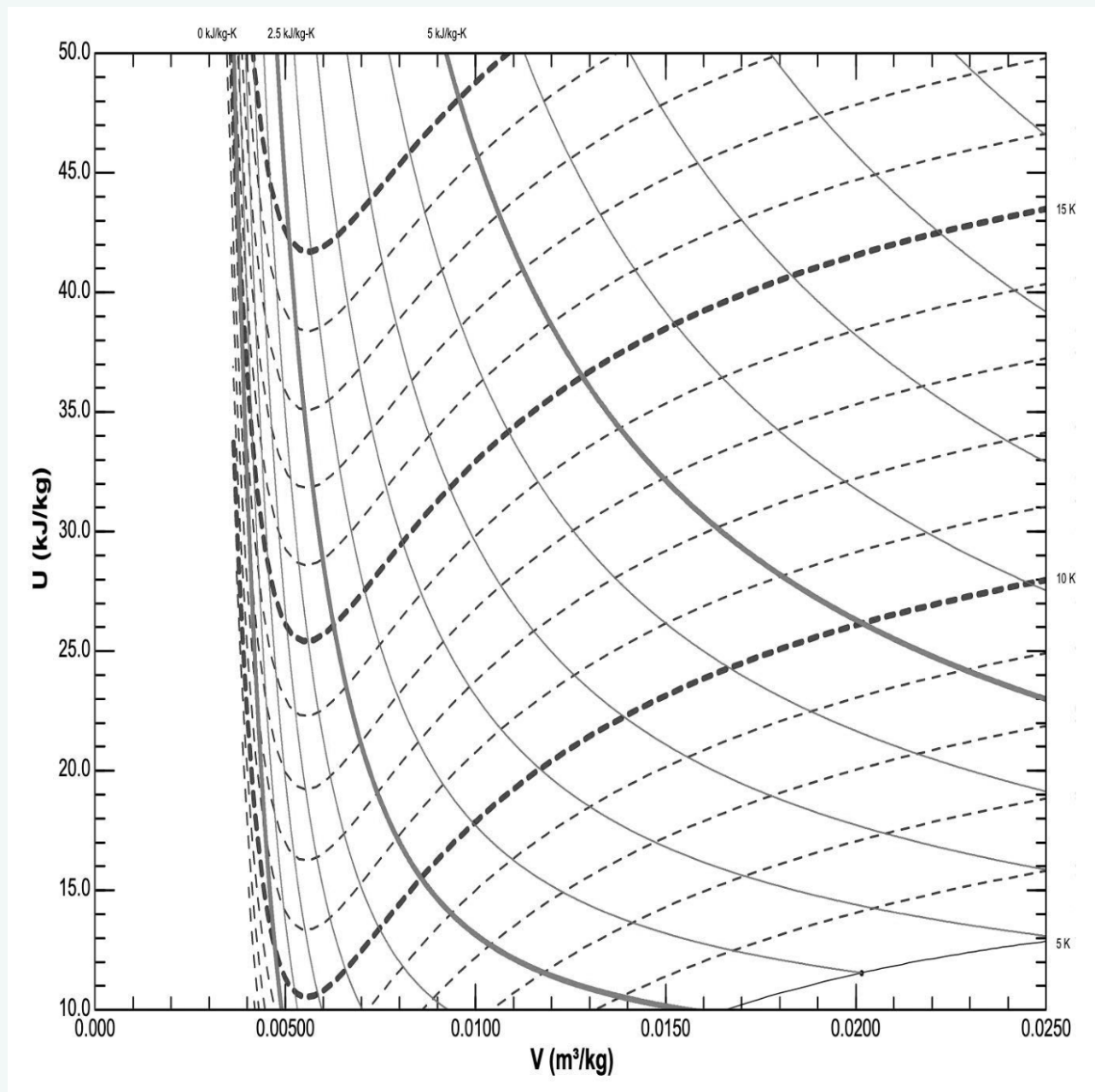
Dashed curves (trending up with increasing volume) are constant temperature T ; values for the bold curves are on right.

Internal energy (per kg) U is on vertical axis; volume (per kg) V is on horizontal axis



D.4 — 0.8pt

Find the maximum possible temperature for T_1 . Show the construction on the graph!



D.5 — 0.2pt

Assuming your value for the maximum T_1 found in D.4, find a numerical value for P_1 .

Physical Constants

Below is the list of physical constants you might use in your solution in addition to the quantities given in the problem text.

Physical constant	Notation	Numerical value
Speed of light in vacuum	c	299 792 458 m/s
Vacuum permeability (magnetic constant)	μ_0	$4\pi \cdot 10^{-7} \text{ N/A}^2$; $[1 \text{ N/A}^2 = 1 \text{ V s}/(\text{A m})]$

Physical constant	Notation	Numerical value
Vacuum permittivity (electric constant)	ϵ_0	$8.8541878128(13) \cdot 10^{-12} \text{ F/m}; \quad [1 \text{ F/m} = 1 \text{ A s/(V m)}]$
Elementary charge	e	$1.602176634 \cdot 10^{-19} \text{ C}; \quad [1 \text{ C} = 1 \text{ A} \cdot \text{s}]$
Electron rest mass	m_e	$9.1093837015(28) \cdot 10^{-31} \text{ kg}$
Proton rest mass	m_p	$1.67262192369(51) \cdot 10^{-27} \text{ kg}$
Neutron rest mass	m_n	$1.67492749804(95) \cdot 10^{-27} \text{ kg}$
Atomic mass unit	m_{amu}	$1.66053906660(50) \cdot 10^{-27} \text{ kg}$
Gravitational constant	G	$6.67430(15) \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$
Avogadro constant	N_A	$6.02214076 \cdot 10^{23} \text{ mol}^{-1}$
Molar gas constant	R	$8.31446261815324 \dots \text{ J/(K} \cdot \text{mol)}$
Boltzmann constant	k_B	$1.380649 \cdot 10^{-23} \text{ J/K}$
Stefan–Boltzmann constant	σ	$5.670374419 \dots \cdot 10^{-8} \text{ W/(m}^2 \cdot \text{K}^4)$
Planck constant	h	$6.62607015 \cdot 10^{-34} \text{ J} \cdot \text{s}$
Reduced Planck constant	$\hbar = h/(2\pi)$	$1.0545718176 \dots \cdot 10^{-34} \text{ J} \cdot \text{s}$

Official English text of the problem.

Answers

- **A.1:** $d_{\text{image}} \simeq 2.6 \times 10^{-5} \text{ m} = 26 \mu\text{m}$.
- **A.2:** $d_{\text{diff}} \simeq 4.5 \times 10^{-5} \text{ m} = 45 \mu\text{m}$, using an equivalent circular aperture.
- **A.3:** With distance L and a uniformly heated diffraction spot,

$$T_{\text{image}} \simeq T_{\text{star}} \left(\frac{A_{\text{mirror}} d_o^2}{\pi L^2 d_{\text{diff}}^2} \right)^{1/4} \simeq 4.1 \times 10^2 \text{ K}.$$

- **B.1:** The characteristic-energy estimate is $T_{\text{source}} \sim 10^3 \text{ K}$, since $|\Delta E_g|/k_B \simeq 1.4 \times 10^3 \text{ K}$. A wavelength-peak criterion instead gives approximately $2.8 \times 10^2 \text{ K}$.
- **B.2:** $\sigma_t = \sqrt{\sigma_r^2 + i_d \tau + p \tau}$.
- **B.3:** $p_{\text{min}} \simeq 0.22 \text{ s}^{-1}$.
- **B.4:** $I \simeq 1.7 \times 10^{-22} \text{ W/m}^2$.
- **C.1:** Define

$$r = 1 + \frac{\beta}{\alpha}, \quad K = 1 + \frac{1 + r + r^2 + r^3}{\alpha}.$$

Then

$$T_1 = \left[\frac{I_0}{\sigma} \frac{K}{K + r^4} \right]^{1/4}, \quad T_5 = \left[\frac{I_0}{\sigma(K + r^4)} \right]^{1/4}.$$

- **C.2:** $\alpha \simeq 0.293$, $\beta \simeq 0.414$.

- **C.3:** $T_1 \simeq 360 \text{ K}$, $T_5 \simeq 120 \text{ K}$.
- **D.1:** In table order: ?, ?, <, >, <.
- **D.2:** $H = U + PV$, neglecting the small change in bulk kinetic energy.
- **D.3:** $H/m \simeq 4.0 \times 10^4 \text{ J/kg}$, equivalent to approximately 160 J/mol for helium.
- **D.4:** $T_{1,\text{max}} \simeq 11 \text{ K}$.
- **D.5:** $P_1 \simeq 2 \times 10^6 \text{ Pa}$.

Solution

A.1

Let the star's distance be L . Using $1 \text{ light-year} \simeq 9.46 \times 10^{15} \text{ m}$,

$$L \simeq 89(9.46 \times 10^{15}) = 8.42 \times 10^{17} \text{ m}.$$

Since $L \gg f$, the image lies essentially in the focal plane. Its angular diameter is d_o/L , so

$$d_{\text{image}} = f \frac{d_o}{L} = \frac{130(1.7 \times 10^{11})}{8.42 \times 10^{17}} = 2.62 \times 10^{-5} \text{ m}.$$

Thus

$$\boxed{d_{\text{image}} \simeq 26 \mu\text{m}.$$

A.2

Approximate the primary mirror by a circular aperture of the same area:

$$D = 2\sqrt{\frac{A_{\text{mirror}}}{\pi}} = 5.64 \text{ m}.$$

For a circular aperture, the first diffraction minimum has angular radius $1.22\lambda/D$. Consequently, the central maximum has diameter

$$d_{\text{diff}} = 2.44 \frac{\lambda f}{D}.$$

Numerically,

$$d_{\text{diff}} = 2.44 \frac{(800 \times 10^{-9})(130)}{5.64} = 4.50 \times 10^{-5} \text{ m}.$$

Therefore,

$$\boxed{d_{\text{diff}} \simeq 45 \mu\text{m}.$$

The diffraction diameter exceeds the geometrical image diameter.

A.3

Treat the star as a blackbody and neglect optical losses. Its luminosity is

$$P_{\text{star}} = 4\pi \left(\frac{d_o}{2} \right)^2 \sigma T_{\text{star}}^4.$$

The power collected by the mirror is therefore

$$P_{\text{collected}} = A_{\text{mirror}} \frac{P_{\text{star}}}{4\pi L^2} = A_{\text{mirror}} \sigma T_{\text{star}}^4 \frac{d_o^2}{4L^2} \simeq 2.43 \times 10^{-6} \text{ W}.$$

For the requested estimate, take the illuminated spot diameter to be the larger of the geometrical and diffraction diameters:

$$d_{\text{spot}} \simeq d_{\text{diff}}, \quad A_{\text{spot}} \simeq \frac{\pi d_{\text{diff}}^2}{4}.$$

Approximate the heating as uniform over this spot. Since only the top surface radiates, equilibrium requires

$$P_{\text{collected}} = \sigma A_{\text{spot}} T_{\text{image}}^4.$$

Thus

$$T_{\text{image}} \simeq T_{\text{star}} \left(\frac{A_{\text{mirror}} d_o^2}{\pi L^2 d_{\text{diff}}^2} \right)^{1/4}$$

and

$$T_{\text{image}} \simeq 405 \text{ K} \approx 4.1 \times 10^2 \text{ K}.$$

This estimate neglects the detailed diffraction profile and the energy in its outer rings. In the geometrical-optics limit, substituting $d_{\text{spot}} = f d_o / L$ would instead give

$$T_{\text{image,geom}} = T_{\text{star}} \left(\frac{A_{\text{mirror}}}{\pi f^2} \right)^{1/4} \simeq 530 \text{ K}.$$

B.1

Taking logarithms of the dark-current law gives

$$\ln i_d = \ln i_0 - \frac{|\Delta E_g|}{6k_B} \frac{1}{T}.$$

From the graph, increasing $1/T$ from 0.12 K^{-1} to 0.16 K^{-1} reduces the current from 10^2 to 10^{-2} s^{-1} . Hence

$$\frac{\Delta \ln i_d}{\Delta(1/T)} = \frac{\ln(10^{-4})}{0.04 \text{ K}^{-1}} = -100 \ln 10 \text{ K}.$$

It follows that

$$\frac{|\Delta E_g|}{k_B} = 600 \ln 10 \text{ K} \simeq 1.38 \times 10^3 \text{ K},$$

or

$$|\Delta E_g| \simeq 1.91 \times 10^{-20} \text{ J} \simeq 0.119 \text{ eV}.$$

Using the characteristic thermal-energy estimate $E_\gamma \sim k_B T$,

$$T_{\text{source}} \sim 10^3 \text{ K}.$$

There is a convention-dependent numerical factor here. The threshold wavelength is

$$\lambda_g = \frac{hc}{|\Delta E_g|} \simeq 10.4 \mu\text{m}.$$

If “just capable” means that the wavelength peak of the thermal spectrum reaches this threshold, Wien scaling and the star’s given peak yield

$$T_{\text{source}} \simeq 3600 \text{ K} \frac{0.800 \mu\text{m}}{10.4 \mu\text{m}} \simeq 2.8 \times 10^2 \text{ K}.$$

Both calculations identify the relevant thermal scale; a thermal spectrum has no sharp upper photon-energy cutoff.

B.2

The mean photon count is $p\tau$, and the mean dark count is $i_d\tau$. Their Poisson variances equal their respective means.

Assuming independent noise sources, variances add:

$$\sigma_t^2 = \sigma_r^2 + i_d\tau + p\tau.$$

Therefore,

$$\sigma_t = \sqrt{\sigma_r^2 + i_d\tau + p\tau}.$$

Subtracting the known mean dark count removes its mean contribution but leaves its statistical fluctuations.

B.3

The straight line on the graph corresponds approximately to

$$i_d(T) = 10^{14-100 \text{ K}/T} \text{ s}^{-1}.$$

At $T_p = 7.5 \text{ K}$,

$$i_d = 10^{14-100/7.5} = 10^{2/3} \simeq 4.64 \text{ s}^{-1}.$$

Write $N = p\tau$ for the expected photon count. The limiting condition is

$$N = 10\sqrt{\sigma_r^2 + i_d\tau + N}.$$

Squaring and solving the resulting quadratic gives the positive root

$$N = 50 + \sqrt{2500 + 100(\sigma_r^2 + i_d\tau)}.$$

With $\sigma_r = 14$ and $\tau = 10^4 \text{ s}$,

$$N \simeq 2.21 \times 10^3.$$

Thus

$$p_{\text{min}} = \frac{N}{\tau} \simeq 0.221 \text{ s}^{-1} \approx 0.22 \text{ s}^{-1}.$$

B.4

Each threshold photon has energy $|\Delta E_g|$. With unit detection efficiency and lossless collection, the required power at the mirror is $p|\Delta E_g|$. Hence

$$I = \frac{p|\Delta E_g|}{A_{\text{mirror}}}.$$

Substitution gives

$$I = \frac{(0.221)(1.91 \times 10^{-20})}{25} \simeq 1.69 \times 10^{-22} \text{ W/m}^2.$$

Therefore,

$$I \simeq 1.7 \times 10^{-22} \text{ W/m}^2.$$

C.1

Let $A = A_{\text{sheet}}$, and define the power emitted by one face of sheet i as

$$q_i = \epsilon \sigma A T_i^4.$$

By Kirchhoff's law, the constant emissivity also gives the absorptivity, so sheet 1 absorbs solar power

$$P_{\odot} = \epsilon I_0 A.$$

Following the stipulated left-to-right leakage model, define the net power received by sheet $i + 1$ from the preceding gap as

$$F_i = \alpha(q_i - q_{i+1}).$$

The accompanying power escaping through that gap is

$$L_i = \beta(q_i - q_{i+1}) = \frac{\beta}{\alpha} F_i.$$

Thus the total power entering a gap from its hotter side is

$$F_i + L_i = r F_i, \quad r = 1 + \frac{\beta}{\alpha}.$$

For each intermediate sheet, equilibrium requires

$$F_{i-1} = r F_i, \quad i = 2, 3, 4.$$

The final sheet radiates q_5 from its exposed face, so

$$F_4 = q_5.$$

Working back through the gaps,

$$F_3 = r q_5, \quad F_2 = r^2 q_5, \quad F_1 = r^3 q_5.$$

The temperature differences satisfy

$$q_i - q_{i+1} = \frac{F_i}{\alpha}.$$

Adding all four differences gives

$$q_1 - q_5 = \frac{q_5}{\alpha} (1 + r + r^2 + r^3).$$

Define

$$K = 1 + \frac{1 + r + r^2 + r^3}{\alpha};$$

then $q_1 = K q_5$.

Finally, the first sheet loses q_1 through its Sun-facing surface and $r F_1$ into the first gap:

$$\epsilon I_0 A = q_1 + r F_1 = (K + r^4) q_5.$$

Using $q_5 = \epsilon \sigma A T_5^4$, we obtain

$$T_5 = \left[\frac{I_0}{\sigma(K + r^4)} \right]^{1/4}$$

and, since $T_1^4 = KT_5^4$,

$$T_1 = \left[\frac{I_0}{\sigma} \frac{K}{K + r^4} \right]^{1/4}.$$

The explicit factors of ϵA cancel; the emissivity still influences α and β .

C.2

The side length of a sheet is

$$a = \sqrt{A} = \sqrt{200} = 14.14 \text{ m.}$$

The perimeter opening has area

$$A_{\text{gap}} = 4ah = 4(14.14)(0.25) = 14.14 \text{ m}^2.$$

Because the sheets are highly reflective and reflections are diffuse, approximate the radiation inside the box as well mixed after repeated reflections. Absorption probabilities are then proportional to absorbing area.

Each sheet has effective absorbing area ϵA , while the opening acts as a black absorber with effective area A_{gap} . The total effective absorbing area is therefore

$$2\epsilon A + A_{\text{gap}}.$$

The fraction ultimately absorbed by the other sheet is approximately

$$\alpha \simeq \frac{\epsilon A}{2\epsilon A + A_{\text{gap}}}.$$

The fraction escaping is approximately

$$\beta \simeq \frac{A_{\text{gap}}}{2\epsilon A + A_{\text{gap}}}.$$

For $\epsilon A = 0.05(200) = 10 \text{ m}^2$,

$$\alpha \simeq \frac{10}{20 + 14.14} = 0.293,$$

$$\beta \simeq \frac{14.14}{20 + 14.14} = 0.414.$$

The remaining fraction, approximately α , returns to and is absorbed by the emitting sheet; accordingly, $2\alpha + \beta = 1$ in this estimate.

C.3

The preceding estimates give

$$r = 1 + \frac{\beta}{\alpha} = 2.414,$$

$$r^4 \simeq 33.97, \quad K \simeq 80.60, \quad K + r^4 \simeq 114.57.$$

Therefore,

$$T_5 = \left[\frac{1360}{(5.670374419 \times 10^{-8})(114.57)} \right]^{1/4} \simeq 120.3 \text{ K},$$

and

$$T_1 = K^{1/4} T_5 \simeq 360.4 \text{ K}.$$

Thus

$$T_1 \simeq 360 \text{ K}, \quad T_5 \simeq 120 \text{ K}.$$

D.1

The required entries are:

Quantity	Comparison
Internal energy	$U_1 ? U_2$
Temperature	$T_1 ? T_2$
Entropy	$S_1 < S_2$
Pressure	$P_1 > P_2$
Volume	$V_1 < V_2$

A pressure drop drives the flow through the resistive plug, so $P_1 > P_2$. Viscous dissipation makes the adiabatic process irreversible, giving $S_2 > S_1$.

Using the equal pipe cross sections shown, steady mass conservation gives

$$\rho_1 w_1 = \rho_2 w_2,$$

where w is bulk speed. Since $w_2 > w_1$, the density decreases and the volume occupied by one mole increases.

For a real gas, neither internal energy nor temperature has a universal direction of change during throttling. Their comparisons require further information about the states.

D.2

Follow one mole through the plug. The upstream gas does work $P_1 V_1$ on this mole, while it does work $P_2 V_2$ on the downstream gas.

There is no heat transfer. Neglecting the small change in bulk kinetic energy, the first law gives

$$U_2 - U_1 = P_1 V_1 - P_2 V_2.$$

Rearranging,

$$U_1 + P_1 V_1 = U_2 + P_2 V_2.$$

Thus the conserved quantity is the enthalpy,

$$H = U + PV.$$

More precisely, steady adiabatic flow conserves $h + w^2/2$ per unit mass. The stated approximation permits neglecting the kinetic-energy difference.

D.3

For clarity, write u and v for the specific internal energy and specific volume labelled U and V on the graphs.

The thermodynamic identity is

$$du = T ds - P dv.$$

Along a constant-entropy curve,

$$\left(\frac{\partial u}{\partial v} \right)_s = -P.$$

Therefore, the tangent to an isentrope at (v, u) has equation

$$u_{\text{tan}}(x) = u - P(x - v) = (u + Pv) - Px.$$

Its intercept on the vertical axis is precisely the specific enthalpy:

$$\boxed{h = u + Pv.}$$

The required construction on image 9 is:

1. Locate $v_2 = 0.100 \text{ m}^3/\text{kg}$.
2. Interpolate to the 7.5 K isotherm.
3. At this state, draw the tangent to the constant-entropy curve passing through it.
4. Extend that tangent to $v = 0$; read its vertical intercept.

The state has approximately

$$u_2 \simeq 26 \text{ kJ/kg}.$$

The tangent slope is approximately

$$-P_2 \simeq -140 \text{ kPa}.$$

Consequently,

$$h_2 = u_2 + P_2 v_2 \simeq 26 + (140)(0.100) = 40 \text{ kJ/kg}.$$

Thus

$$\boxed{h_1 = h_2 \simeq 4.0 \times 10^4 \text{ J/kg}.$$

For helium with molar mass approximately $4.0 \times 10^{-3} \text{ kg/mol}$, this corresponds to

$$H_{\text{molar}} \simeq 160 \text{ J/mol}.$$

D.4

All admissible inlet states must have the same specific enthalpy $h_0 \simeq 40 \text{ kJ/kg}$.

The graphical construction on image 10 is:

1. Mark the point $(0, h_0)$ on the vertical axis.
2. Draw tangents from this point to several solid constant-entropy curves.
3. Mark their points of tangency. Every such point has $u + Pv = h_0$, so these points trace the constant-enthalpy curve.
4. Find the highest-temperature dashed isotherm reached by this curve. At the maximum temperature, the constant-enthalpy curve touches that isotherm.

The limiting contact is close to the 11 K isotherm, with graphical readings approximately

$$v_1 \sim 1.0 \times 10^{-2} \text{ m}^3/\text{kg}, \quad u_1 \sim 2.1 \times 10^4 \text{ J/kg}.$$

The graph supports a maximum of roughly $10.5\text{--}11 \text{ K}$, so

$$T_{1,\max} \simeq 11 \text{ K.}$$

D.5

The pressure is the magnitude of the isentropic tangent slope at the selected inlet state. Equivalently,

$$P_1 = \frac{h_0 - u_1}{v_1}.$$

Using the approximate graphical coordinates,

$$P_1 \simeq \frac{(40 - 21) \times 10^3}{1.0 \times 10^{-2}} = 1.9 \times 10^6 \text{ Pa.}$$

To the precision justified by the graph,

$$P_1 \simeq 2 \times 10^6 \text{ Pa.}$$

The solution was checked against the official solution.

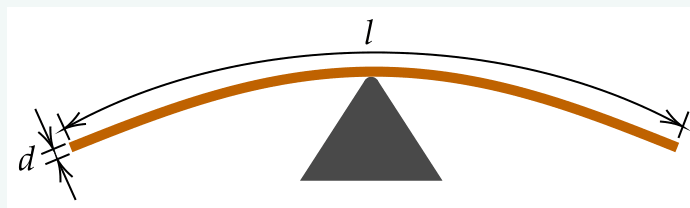
Problem 3: Scaling laws

Scaling laws describe the functional relationship between two physical quantities that scale with each other over a significant interval. This functional relationship can be a power law, but there are other possibilities, too. Oftentimes, exact expressions are beyond reach, but scaling laws can still be derived.

Part A. Spaghetti (2.0 points)

A.1 2.0pt

A spaghetti straw of diameter d is being balanced horizontally from its middle. If $d = 1 \text{ mm}$, the straw breaks under its own weight once its length reaches $l = 50 \text{ cm}$. What is the maximum length l' of the straw of diameter $d' = 1 \text{ cm}$ before it breaks under its own weight?



Part B. Sand castle (2.0 points)

B.1 2.0pt

The average grain volume of coarse-grained sand is 10 times as large as that of fine-grained sand. Wet fine-grained sand and wet coarse-grained sand have both optimal water content (i.e. assuming the maximal strength of the constructions from it) and are used to build two cylinders of exactly the same shape and size. The strength of each cylinder is tested by pressing it between two parallel plates. The cylinder made of coarse-grained sand gets destroyed once the force applied to press the plates reaches $F_c = 10 \text{ N}$. How large is the force F_f needed to destroy the cylinder made of fine-grained sand? You may ignore the effects of gravity.

Part C. Interstellar travel (2.0 points)

C.1 2.0pt

The spaceship of an interstellar expedition travels at a constant magnitude of the proper acceleration $g = 10 \text{ m/s}^2$, i.e., this is the acceleration of the spaceship in the inertial frame of reference where it is instantaneously at rest. The passengers must be able to return to Earth within their remaining expected lifetime of 50 years. The maximum distance from Earth reached by the spaceship is d . If the acceleration is increased to $g' = 15 \text{ m/s}^2$, the spaceship can reach a farther distance d' . What is the ratio d'/d ?

Hint 1. You may wish to use the relativistic velocity addition formula, however, there are also other approaches.

Hint 2. You may need to deal with hyperbolic functions defined as follows:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Hint 3. Depending on your approach, you may need one or more of these integrals:

$$\int \frac{dx}{1-x^2} = \operatorname{atanh} x + C, \quad \int \frac{dx}{\sqrt{1+x^2}} = \operatorname{asinh} x + C, \quad \int \sinh x \, dx = \cosh x + C,$$

where $\operatorname{asinh} x$ and $\operatorname{atanh} x$ are the inverse functions of the respective hyperbolic functions.

Part D. That sinking feeling (2.0 points)

D.1 2.0pt

A solid wooden ball of radius r_0 is floating in the water. Ignoring frictional effects, the frequency of small oscillations would be ω_0 , but because of viscous friction, after being displaced vertically, the frequency of decaying oscillations is actually $0.99 \omega_0$. What is the minimum radius $r_{\min}$ of a wooden ball floating in water that undergoes small oscillations when displaced? *Hint:* the viscous drag force acting on a given body is proportional to its speed relative to the bulk of the fluid, and to the viscosity η of the fluid it is moving in. The unit of the viscosity is $\text{kg}/(\text{m} \cdot \text{s})$.

Official English text of the problem.

Answers

- **A.1:** $l' = l \sqrt{\frac{d'}{d}} = 0.50 \sqrt{10} \text{ m} \approx 1.6 \text{ m}.$
- **B.1:** $F_f = 10^{1/3} F_c = 10^{4/3} \text{ N} \approx 22 \text{ N}.$
- **C.1:** With $T = 50$ years measured by the passengers,

$$\frac{d'}{d} = \frac{g}{g'} \frac{\cosh(g'T/(4c)) - 1}{\cosh(gT/(4c)) - 1} \approx 4.8 \times 10^2.$$

- **D.1:** $r_{\min} = (1 - 0.99^2)^{1/3} r_0 \approx 0.27 r_0.$ This is the critical radius; oscillations require $r > r_{\min}.$

Solution

A.1

Assume the spaghetti has the same density and breaking stress at both diameters, and use the slender-beam approximation. Its cross-sectional area scales as $A \propto d^2$, so its weight per unit length is

$$q = \rho g_E A \propto d^2,$$

where g_E is the gravitational acceleration.

Consider one half of the straw. Its weight is $ql/2$, acting a distance $l/4$ from the central support. Thus the largest bending moment, occurring at the middle, is

$$M_{\max} = \frac{ql}{2} \frac{l}{4} = \frac{ql^2}{8} \propto d^2 l^2.$$

In beam bending, the maximum longitudinal stress is

$$\sigma_{\max} = \frac{M_{\max}(d/2)}{I},$$

where the second moment of area satisfies $I \propto d^4$. Consequently,

$$\sigma_{\max} \propto \frac{l^2}{d}.$$

Breaking occurs at the same material stress, so

$$\frac{l'^2}{d'} = \frac{l^2}{d}, \quad l' = l \sqrt{\frac{d'}{d}}.$$

Since $d'/d = 10$,

$$l' = 0.50\sqrt{10} \text{ m} = 1.581 \dots \text{ m} \approx 1.6 \text{ m}.$$

B.1

Wet sand is held together by capillary forces from water bridges between grains. Let a be a characteristic grain size and γ the water surface tension. Assume similar grain shapes, packing and wetting properties. At optimal water content, the bridge geometry scales with a .

The capillary force at a contact scales as

$$f_{\text{cap}} \propto \gamma a.$$

This follows either from surface tension acting along a contact line of length proportional to a , or from a capillary pressure of order γ/a acting over an area of order a^2 .

The number of contacts transmitting force across a fixed macroscopic area S scales as S/a^2 . Hence the characteristic cohesive stress scales as

$$\sigma_{\text{coh}} \propto \frac{f_{\text{cap}}}{a^2} \propto \frac{\gamma}{a}.$$

For cylinders of the same shape and size, the compressive failure load has the same scaling:

$$F_{\text{break}} \propto \frac{1}{a}.$$

Because grain volume is proportional to a^3 ,

$$\frac{a_c}{a_f} = \left(\frac{V_c}{V_f} \right)^{1/3} = 10^{1/3}.$$

Therefore,

$$\frac{F_f}{F_c} = \frac{a_c}{a_f} = 10^{1/3},$$

giving

$$F_f = 10 \cdot 10^{1/3} \text{ N} = 21.54 \dots \text{ N} \approx 22 \text{ N}.$$

C.1

Treat Earth as an inertial reference frame, neglect gravitational fields, and consider a straight-line journey starting and ending at rest relative to Earth. The available 50 years is the passengers' **proper time**, denoted by T . Acceleration reversals are idealized as instantaneous.

First derive the displacement under constant proper acceleration. Define the rapidity

$$\theta = a \tanh(v/c).$$

During a proper-time interval $d\tau$, the velocity change in the instantaneous rest frame is $g d\tau$.

Relativistic velocity addition gives, to first order,

$$dv = g \left(1 - \frac{v^2}{c^2} \right) d\tau,$$

and therefore

$$\frac{d\theta}{d\tau} = \frac{g}{c}.$$

Starting from rest,

$$\theta = \frac{g\tau}{c}, \quad v = c \tanh \theta, \quad \frac{dt}{d\tau} = \cosh \theta.$$

It follows that

$$\frac{dx}{d\tau} = v \frac{dt}{d\tau} = c \sinh \left(\frac{g\tau}{c} \right).$$

Integrating,

$$x(\tau) = \frac{c^2}{g} \left[\cosh \left(\frac{g\tau}{c} \right) - 1 \right].$$

To maximize the outward distance in a given proper time while starting and finishing the outward leg at rest, the ship accelerates outward for half that time and decelerates for the other half. Indeed, for an outward leg of proper duration L , its rapidity cannot exceed

$$\theta(\tau) \leq \frac{g}{c} \min(\tau, L - \tau).$$

The accelerate-then-decelerate trajectory attains this bound, maximizing the displacement because $dx/d\tau = c \sinh \theta$ increases with θ .

The return journey over the same distance requires the same minimum proper time. Thus the optimal round trip consists of four equal intervals:

1. Accelerate away from Earth.
2. Decelerate to rest at the greatest distance.
3. Accelerate toward Earth.
4. Decelerate to rest at Earth.

Each interval lasts $T/4 = 12.5$ years. The two outward intervals cover equal distances, so

$$d(g) = \frac{2c^2}{g} \left[\cosh\left(\frac{gT}{4c}\right) - 1 \right].$$

Hence

$$\boxed{\frac{d'}{d} = \frac{g}{g'} \frac{\cosh(g'T/(4c)) - 1}{\cosh(gT/(4c)) - 1}}.$$

Using $c \simeq 3.00 \times 10^8$ m/s and $1 \text{ year} \simeq 3.156 \times 10^7$ s,

$$\frac{gT}{4c} \simeq 13.15, \quad \frac{g'T}{4c} \simeq 19.72.$$

Both arguments are large, so $\cosh x - 1 \simeq e^x/2$. Thus

$$\frac{d'}{d} \simeq \frac{g}{g'} \exp\left(\frac{(g' - g)T}{4c}\right) \simeq \frac{2}{3} e^{6.575}.$$

The result is

$$\boxed{\frac{d'}{d} \approx 4.8 \times 10^2}.$$

D.1

Assume all balls are made of the same wood, buoyancy determines their equilibrium immersion, and the stated linear viscous-drag model applies.

For a small downward displacement z , the additional submerged volume is $A_w z$, where A_w is the area of the sphere's cross-section at the water surface. The additional buoyancy provides a restoring force

$$F_{\text{rest}} = -kz, \quad k = \rho_w g_E A_w.$$

The equilibrium submerged fraction is fixed by the wood-to-water density ratio. The floating configurations are therefore geometrically similar, giving

$$A_w \propto r^2, \quad k \propto r^2, \quad m \propto r^3.$$

The undamped angular frequency consequently satisfies

$$\Omega^2 = \frac{k}{m} \propto r^{-1}.$$

Write the viscous force as

$$F_{\text{visc}} = -b\dot{z}.$$

Since it is proportional to viscosity and velocity, dimensional analysis for geometrically similar bodies gives

$$b = C\eta r \propto r,$$

where C is a dimensionless geometric factor.

The equation of motion is

$$m\ddot{z} + b\dot{z} + kz = 0.$$

Define the damping rate

$$\beta = \frac{b}{2m} \propto r^{-2}.$$

In the underdamped regime, the oscillation frequency is

$$\omega_d = \sqrt{\Omega^2 - \beta^2}.$$

At radius r_0 , the given frequencies imply

$$(0.99\omega_0)^2 = \omega_0^2 - \beta_0^2,$$

so

$$\frac{\beta_0^2}{\omega_0^2} = 1 - 0.99^2 = 0.0199.$$

Combining the radius scalings,

$$\frac{\beta^2(r)}{\Omega^2(r)} = 0.0199 \left(\frac{r_0}{r} \right)^3.$$

Oscillations occur when this ratio is less than one. The critical radius therefore satisfies

$$0.0199 \left(\frac{r_0}{r_{\min}} \right)^3 = 1,$$

which yields

$$r_{\min} = (0.0199)^{1/3} r_0 = 0.27099 \dots r_0 \approx 0.27r_0.$$

At equality the motion is critically damped and does not oscillate. Thus $r_{\min}$ is the limiting minimum radius, with oscillatory motion occurring for $r > r_{\min}$.

The solution was checked against the official solution.