

IPhO 2021

Theory problems and solutions · Vilnius, Lithuania (held online) · 3 theory problems with complete solutions

eulersolve.org/ipho/2021/

Problem 1: Planetary Physics

This problem consists of two independent problems related to the interior of planets. The effects of the surface curvature of the planets can be neglected. You might need to use the formula

$$(1 + x)^\varepsilon \approx 1 + \varepsilon x, \quad \text{when } |x| \ll 1. \quad (1)$$

Part A. Mid-ocean ridge (5.0 points)

Consider a large vessel of water that is situated in a uniform gravitational field with free-fall acceleration g . Two vertical rectangular plates parallel to each other are fitted into the vessel so that the vertical edges of the plates are in a tight gap-less contact with the vertical walls of the vessel. Length h of each plate is immersed in water (Fig. 1). The width of the plates along the y -axis is w , water density is ρ_0 .

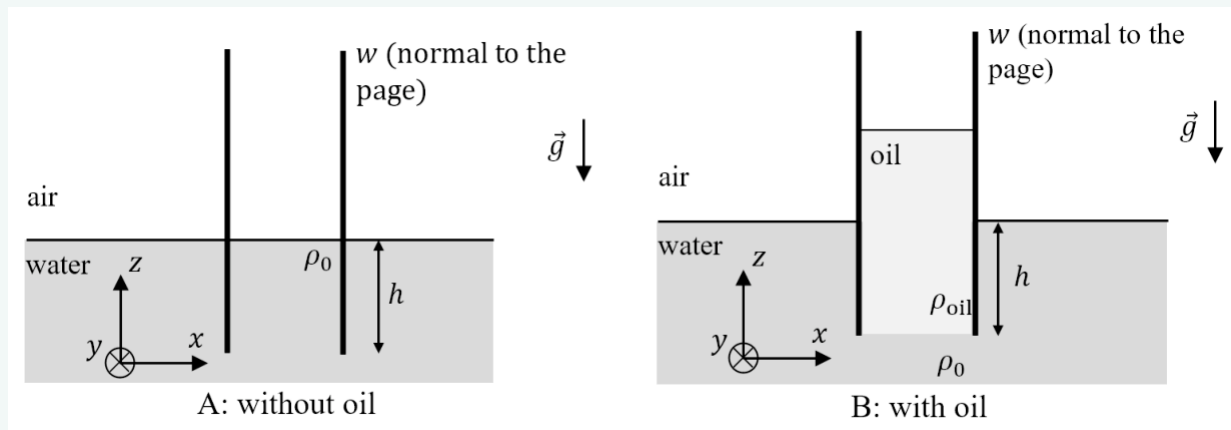


Figure 1. Parallel plates in water.

Oil of density ρ_{oil} ($\rho_{\text{oil}} < \rho_0$) is poured into the space between the plates until the lower level of the oil has reached the lower edges of the plates. Assume that plates and vessel edges are high enough for oil not to overflow them. Surface tension and mixing of fluids can be neglected.

A.1 — 0.8pt

What is the x -component of the net force F_x acting on the right plate (magnitude and direction)?

Fig. 2 shows a cross-section of a mid-ocean ridge. It consists of overlaying layers of mantle, crust and ocean water. The mantle is composed of rocks that we assume can flow in geological timescales and therefore, in this problem will be treated as a fluid. The thickness of the crust is much smaller than the characteristic length scale in the x -direction, hence, the crust behaves as a freely bendable plate. To high accuracy, such a ridge can be modeled as a two-dimensional

system, without any variation of variables along the y -axis, which is perpendicular to the plane of Fig. 2. Assume that the ridge length L along the y -axis is much larger than any other length introduced in this problem.

At the centre of the ridge the thickness of the crust is assumed to be zero. As the horizontal distance x from the centre increases, the crust gets thicker and approaches a constant thickness D as $x \rightarrow \infty$. Correspondingly, the ocean floor subsides by a vertical height h below the top of the ridge O , which we define as the origin of our coordinate system (see Fig. 2). Water density ρ_0 and temperature T_0 can be assumed to be constant in space and time. The same can be assumed for mantle density ρ_1 and its temperature T_1 . The temperature of the crust T is also constant in time but can depend on position.

It is known that, to high accuracy, the crustal material expands linearly with temperature T . Since water and mantle temperatures are assumed to be constant, it is convenient to use a rescaled version of the thermal expansion coefficient. Then $l(T) = l_1[1 - k_l(T_1 - T)/(T_1 - T_0)]$, where l is the length of a piece of crustal material, l_1 is its length at temperature T_1 , and k_l is the rescaled thermal expansion coefficient, which can be assumed to be constant.

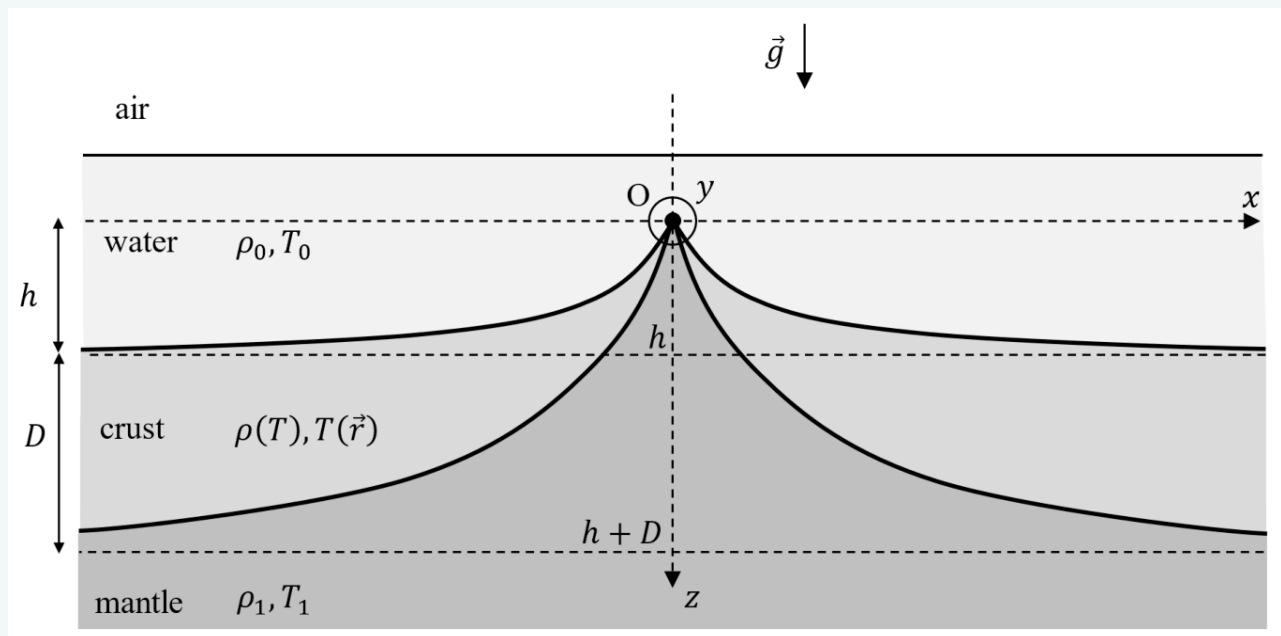


Figure 2. Mid-ocean ridge. Note that the z -axis is pointing downwards.

A.2 — 0.6pt

Assuming that the crust is isotropic, find how its density ρ depends on its temperature T .

Assuming that $|k_l| \ll 1$, write your answer in the approximate form

$$\rho(T) \approx \rho_1 \left[1 + k \frac{T_1 - T}{T_1 - T_0} \right], \quad (2)$$

where terms of order k_l^2 and higher are neglected. Then, identify constant k .

It is known that $k > 0$. Also, thermal conductivity of the crust κ can be assumed to be constant. As a consequence, very far away from the ridge axis the temperature of the crust depends linearly with depth.

A.3 — 1.1pt

By assuming that mantle and water each behave like an incompressible fluid at hydrostatic equilibrium, express the far-distance crust thickness D in terms of h , ρ_0 , ρ_1 , and k . Any motion of the material can be neglected.

A.4 — 1.6pt

Find, to the leading order in k , the net horizontal force F acting on the right half ($x > 0$) of the crust in terms of ρ_0 , ρ_1 , h , L , k and g .

Suppose that crust is thermally isolated from the rest of the Earth. As a result of heat conduction, the temperatures of the upper and lower surfaces of the crust are going to get closer to each other until the crust reaches thermal equilibrium. Specific heat of the crust is c and can be assumed to be constant.

A.5 — 0.9pt

By using dimensional analysis or order-of-magnitude analysis, estimate the characteristic time τ in which the difference between the upper and lower surface temperatures of the crust far away from the ridge axis is going to approach zero. You can assume that τ does not depend on the two initial surface temperatures of the crust.

Part B. Seismic waves in a stratified medium (5.0 points)

Suppose that a short earthquake happens at the surface of some planet. The seismic waves can be assumed to originate from a line source situated at $z = x = 0$, where x is the horizontal coordinate and z is the depth below the surface (Fig. 3). The seismic wave source can be assumed to be much longer than any other length considered in this question.

As a result of the earthquake, a uniform flux of the so-called longitudinal P waves is emitted along all the directions in the x - z plane that have positive component along the z -axis. Since the wave theory in a solid is generally complicated, in this problem we neglect all the other waves emitted by the earthquake. The crust of the planet is stratified so that the P-wave speed v depends on depth z according to $v = v_0(1 + z/z_0)$, where v_0 is the speed at the surface and z_0 is a known positive constant.

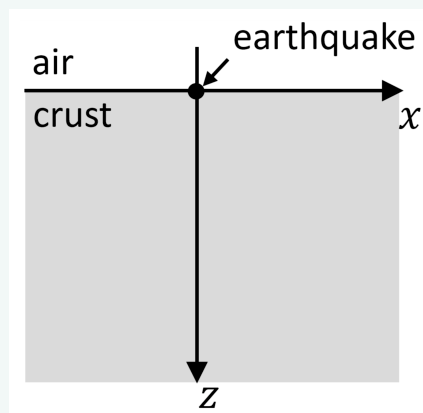


Figure 3. Coordinate system used in part B.

B.1 — 1.5pt

Consider a single ray emitted by the earthquake that makes an initial angle $0 < \theta_0 < \pi/2$ with the z -axis and travels in the x - z plane. What is the horizontal coordinate $x_1(\theta_0) \neq 0$ at which this ray can be detected at the surface of the planet? It is known that the ray path is an arc of a circle. Write your answer in the form $x_1(\theta_0) = A \cot(b\theta_0)$, where A and b are constants to be found.

If you were unable to find A and b , in the following questions you can use the result $x_1(\theta_0) = A \cot(b\theta_0)$ as given. Suppose that total energy per unit length of the source released as P waves into the crust during the earthquake is E . Assume that waves are completely absorbed when they reach the surface of the planet from below.

B.2 — 1.5pt

Find how the energy density per unit area $\varepsilon(x)$ absorbed by the surface depends on the distance along the surface x . Sketch the plot of $\varepsilon(x)$.

From now on, assume that the waves are instead fully reflected when reaching the surface. Imagine a device positioned at $z = x = 0$ that has the same geometry as the previously considered earthquake source. The device is capable of emitting P waves in a freely chosen angular distribution. We make the device emit a signal with a narrow range of emission angles. In particular, the initial angle the signal makes with the vertical belongs to the interval $[\theta_0 - \frac{1}{2}\delta\theta_0, \theta_0 + \frac{1}{2}\delta\theta_0]$, where $0 < \theta_0 < \pi/2$, $\delta\theta_0 \ll 1$ and $\delta\theta_0 \ll \theta_0$.

B.3 — 2.0pt

At what distance $x_{\max}$ along the surface from the source is the furthest point that the signal does not reach? Write your answer in terms of θ_0 , $\delta\theta_0$ and other constants given above.

Official English text of the problem.

Answers• **A.1**

$$F_x = \frac{\rho_0 g w h^2}{2} \left(\frac{\rho_0}{\rho_{\text{oil}}} - 1 \right)$$

The force points in the $+x$ direction, away from the oil.

• **A.2**

$$\rho(T) = \rho_1 \left[1 - k_l \frac{T_1 - T}{T_1 - T_0} \right]^{-3} \approx \rho_1 \left[1 + 3k_l \frac{T_1 - T}{T_1 - T_0} \right], \quad k = 3k_l.$$

• **A.3**

$$D = \frac{2(\rho_1 - \rho_0)h}{\rho_1 k}$$

within the first-order density approximation.

- **A.4** The total horizontal force exerted by the fluids is, to leading order,

$$F_x \simeq \frac{2gLh^2(\rho_1 - \rho_0)^2}{\rho_1 k^2},$$

directed away from the ridge, along $+x$.

- **A.5**

$$\tau \sim \frac{\rho_1 c D^2}{\kappa} = \frac{4c(\rho_1 - \rho_0)^2 h^2}{\kappa \rho_1 k^2}.$$

This is an order-of-magnitude estimate.

- **B.1**

$$x_1(\theta_0) = 2z_0 \cot \theta_0, \quad A = 2z_0, \quad b = 1.$$

- **B.2**

$$\varepsilon(x) = \frac{2Ez_0}{\pi(x^2 + 4z_0^2)}.$$

The graph is even, peaks at $\varepsilon(0) = E/(2\pi z_0)$, reaches half its maximum at $x = \pm 2z_0$, and approaches zero as x^{-2} .

- **B.3** In the narrow-band limit,

$$x_{\max} \simeq \frac{2z_0 \cos^2 \theta_0}{\delta \theta_0}.$$

This is the outer boundary of the last unreached interval along the beam's propagation direction. An exact expression accounting for the integer number of reflections is derived below.

Solution

A.1

Let H be the height of the oil column above the lower edges of the plates. At those edges, the oil and water pressures must agree. With atmospheric pressure subtracted,

$$\rho_{\text{oil}} g H = \rho_0 g h, \quad H = \frac{\rho_0}{\rho_{\text{oil}}} h.$$

A fluid column of height a and density ρ exerts a horizontal force

$$w \int_0^a \rho g(a - u) du = \frac{1}{2} \rho g w a^2$$

on a vertical plate, where u measures height above the column's bottom.

The oil pushes the right plate towards $+x$; the water pushes it towards $-x$. Atmospheric pressure cancels between the two sides. Including the oil above the exterior water level gives

$$F_x = \frac{gw}{2} (\rho_{\text{oil}} H^2 - \rho_0 h^2).$$

Substituting H ,

$$F_x = \frac{\rho_0 g w h^2}{2} \left(\frac{\rho_0}{\rho_{\text{oil}}} - 1 \right) > 0.$$

Thus the plate is pushed outwards.

A.2

Define the dimensionless temperature difference

$$q = \frac{T_1 - T}{T_1 - T_0}.$$

Isotropic expansion changes all three linear dimensions by the same factor. Consequently,

$$\frac{V(T)}{V(T_1)} = (1 - k_l q)^3.$$

Mass is conserved, and the reference density at T_1 is ρ_1 , so

$$\rho(T) = \rho_1 (1 - k_l q)^{-3}.$$

Using the given expansion and neglecting terms of order k_l^2 ,

$$\rho(T) \approx \rho_1 (1 + 3k_l q).$$

Comparison with equation (2) gives

$$k = 3k_l.$$

A.3

Use the downward-pointing z -axis of Fig. 2. Far from the ridge, the crust extends from $z = h$ to $z = h + D$. Its temperature is

$$T(z) = T_0 + (T_1 - T_0) \frac{z - h}{D},$$

and therefore its density is

$$\rho(z) \approx \rho_1 \left[1 + k \left(1 - \frac{z - h}{D} \right) \right].$$

Its mean density is

$$\bar{\rho} = \frac{1}{D} \int_h^{h+D} \rho(z) dz = \rho_1 \left(1 + \frac{k}{2} \right).$$

Let p_O be the common fluid pressure at the ridge crest, where the crust has zero thickness. Hydrostatic equilibrium gives the pressures immediately above and below the distant crust:

$$\begin{aligned} p_{\text{top}} &= p_O + \rho_0 g h, \\ p_{\text{bottom}} &= p_O + \rho_1 g (h + D). \end{aligned}$$

Vertical force balance on a distant horizontal column of crust requires

$$p_{\text{bottom}} - p_{\text{top}} = \bar{\rho} g D.$$

Thus

$$\rho_1 (h + D) - \rho_0 h = \rho_1 D \left(1 + \frac{k}{2} \right).$$

After cancelling the common term $\rho_1 D$,

$$(\rho_1 - \rho_0)h = \frac{1}{2}\rho_1 k D,$$

so

$$D = \frac{2(\rho_1 - \rho_0)h}{\rho_1 k}.$$

A.4

We calculate the pressure forces on the actual water–crust and mantle–crust interfaces.

Write the upper and lower crust surfaces as $z = z_t(x)$ and $z = z_b(x)$. Their endpoint values are

$$z_t(0) = z_b(0) = 0, \quad z_t(\infty) = h, \quad z_b(\infty) = h + D.$$

The fluid pressure fields are

$$p_w(z) = p_O + \rho_0 g z, \quad p_m(z) = p_O + \rho_1 g z.$$

For an interface element of horizontal length dx and width L , the horizontal component of pressure force contains the surface slope. The upper fluid pushes towards negative x , while the lower fluid pushes towards positive x :

$$dF_x = L \left[p_m(z_b) \frac{dz_b}{dx} - p_w(z_t) \frac{dz_t}{dx} \right] dx.$$

The factors from the inclined surface area and the unit normal cancel, so this expression does not require an additional small-slope approximation. Integrating,

$$\begin{aligned} F_x &= L \int_0^{h+D} (p_O + \rho_1 g z) dz - L \int_0^h (p_O + \rho_0 g z) dz \\ &= p_O L D + \frac{gL}{2} [\rho_1 (h + D)^2 - \rho_0 h^2]. \end{aligned}$$

Equivalently,

$$F_x = \frac{1}{2} \rho_1 g L D^2 + (p_O + \rho_1 g h) L D + \frac{1}{2} (\rho_1 - \rho_0) g L h^2.$$

From A.3, $D \propto k^{-1}$ at fixed h . With the background pressure fixed, these three contributions are respectively of orders k^{-2} , k^{-1} , and k^0 . Hence the leading term is

$$F_x \simeq \frac{1}{2} \rho_1 g L D^2.$$

Substituting the expression for D ,

$$F_x \simeq \frac{2g L h^2 (\rho_1 - \rho_0)^2}{\rho_1 k^2}.$$

It points along $+x$, away from the ridge. This is the total force exerted by the fluids; no lithostatic force on an artificial cut through the distant crust has been subtracted.

A.5

To leading order in k , the crust has volumetric heat capacity $\rho_1 c$, giving thermal diffusivity

$$\chi = \frac{\kappa}{\rho_1 c}.$$

Far from the ridge, the relevant distance over which heat must diffuse is D . The heat equation,

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial z^2},$$

therefore gives the scale estimate

$$\frac{\Delta T}{\tau} \sim \chi \frac{\Delta T}{D^2}.$$

The temperature difference cancels, as required:

$$\tau \sim \frac{D^2}{\chi} = \frac{\rho_1 c D^2}{\kappa}.$$

Using A.3,

$$\tau \sim \frac{4c(\rho_1 - \rho_0)^2 h^2}{\kappa \rho_1 k^2}.$$

Dimensional analysis does not determine a numerical prefactor. The surface temperature difference approaches zero asymptotically rather than vanishing at a finite time.

B.1

In a medium whose wave speed depends only on depth, Snell's law preserves the horizontal slowness:

$$\frac{\sin \theta(z)}{v(z)} = \frac{\sin \theta_0}{v_0},$$

where $\theta(z)$ is the local angle with the vertical. Since

$$v(z) = v_0 \frac{z + z_0}{z_0},$$

we obtain

$$\sin \theta(z) = \frac{z + z_0}{z_0} \sin \theta_0.$$

Define

$$R = \frac{z_0}{\sin \theta_0}.$$

On the descending part of the ray,

$$\frac{dx}{dz} = \tan \theta = \frac{z + z_0}{\sqrt{R^2 - (z + z_0)^2}}.$$

Integration with $x = 0$ at $z = 0$ gives

$$x = \sqrt{R^2 - z_0^2} - \sqrt{R^2 - (z + z_0)^2}.$$

Thus the trajectory belongs to the circle

$$(x - x_c)^2 + (z + z_0)^2 = R^2,$$

whose centre has coordinates

$$x_c = \sqrt{R^2 - z_0^2} = z_0 \cot \theta_0, \quad z_c = -z_0.$$

The ray turns at the deepest point of this circle and returns symmetrically to the surface. Its two intersections with $z = 0$ are $x = 0$ and $x = 2x_c$. Therefore

$$x_1(\theta_0) = 2z_0 \cot \theta_0,$$

and

$$A = 2z_0, \quad b = 1.$$

The surface speed v_0 affects travel times but cancels from the ray geometry.

B.2

The downward half-plane spans an angular range π . Uniform emission therefore assigns energy per unit source length

$$dE = \frac{E}{\pi} d\theta$$

to an angular interval $d\theta$.

For rays returning at $x > 0$,

$$x = 2z_0 \cot \theta, \quad \left| \frac{dx}{d\theta} \right| = 2z_0 \csc^2 \theta.$$

Energy conservation between the emitted angular interval and its receiving surface strip gives

$$\varepsilon(x) dx = \frac{E}{\pi} |d\theta|.$$

Hence

$$\varepsilon(x) = \frac{E}{2\pi z_0} \sin^2 \theta.$$

Using

$$\cot \theta = \frac{x}{2z_0}, \quad \sin^2 \theta = \frac{1}{1 + x^2/(4z_0^2)},$$

we find

$$\varepsilon(x) = \frac{2Ez_0}{\pi(x^2 + 4z_0^2)}.$$

Left-going emission gives the same expression for $x < 0$. Its normalization is

$$\int_{-\infty}^{\infty} \varepsilon(x) dx = \frac{E}{\pi} \left[\arctan \frac{x}{2z_0} \right]_{-\infty}^{\infty} = E.$$

For the requested sketch: the curve is smooth and symmetric about $x = 0$, with a single maximum $E/(2\pi z_0)$. At $x = \pm 2z_0$ its height is $E/(4\pi z_0)$, so its full width at half maximum is $4z_0$. It remains positive at every finite x , and its tails satisfy

$$\varepsilon(x) \sim \frac{2Ez_0}{\pi x^2} \quad (|x| \gg z_0).$$

Since E is energy per unit source length, ε has units of energy per unit area.

B.3

At a specular surface reflection, the vertical component of the ray velocity reverses while its horizontal component is preserved. Each ray therefore repeats the same arc, translated horizontally. Its surface encounters occur at

$$x_n(\theta) = n\ell(\theta), \quad \ell(\theta) = 2z_0 \cot \theta, \quad n = 1, 2, \dots$$

Let

$$\theta_{\pm} = \theta_0 \pm \frac{\delta\theta_0}{2}.$$

For a band lying entirely within $0 < \theta < \pi/2$, the shortest and longest distances between successive surface encounters are

$$\ell_{\min} = 2z_0 \cot \theta_+, \quad \ell_{\max} = 2z_0 \cot \theta_-.$$

Because the beam fills the angular interval, the surface interval reached at the n -th encounter is

$$I_n = [n\ell_{\min}, n\ell_{\max}].$$

A gap exists between I_n and I_{n+1} precisely when

$$n\ell_{\max} < (n+1)\ell_{\min},$$

or

$$n(\ell_{\max} - \ell_{\min}) < \ell_{\min}.$$

Put $\Delta\ell = \ell_{\max} - \ell_{\min}$. Once $n\Delta\ell \geq \ell_{\min}$, adjacent intervals overlap, and they continue to overlap for every larger n . Consequently, the exact outer boundary of the last gap is

$$x_{\max} = \ell_{\min} \left\lceil \frac{\ell_{\min}}{\Delta\ell} \right\rceil.$$

The ceiling accounts for the integer number of surface encounters. The boundary itself is reached; unreached points approach it from below.

For the requested narrow-band approximation,

$$\ell_0 = 2z_0 \cot \theta_0,$$

and

$$\Delta\ell \simeq \left| \frac{d\ell}{d\theta} \right|_{\theta_0} \delta\theta_0 = 2z_0 \csc^2 \theta_0 \delta\theta_0.$$

At fixed θ_0 , a sufficiently narrow band requires many reflections before the intervals overlap. To leading order, the integer rounding can then be neglected:

$$x_{\max} \simeq \frac{\ell_0^2}{\Delta\ell}.$$

Therefore

$$x_{\max} \simeq \frac{(2z_0 \cot \theta_0)^2}{2z_0 \csc^2 \theta_0 \delta\theta_0} = \frac{2z_0 \cos^2 \theta_0}{\delta\theta_0}.$$

The solution was checked against the official solution.

Problem 2: Electrostatic lens

Consider a uniformly charged metallic ring of radius R and total charge q . The ring is a hollow toroid of thickness $2a \ll R$. This thickness can be neglected in parts A, B, C, and E. The xy plane coincides with the plane of the ring, while the z -axis is perpendicular to it, as shown in Figure 1. In parts A and B you might need to use the formula (Taylor expansion)

$$(1 + x)^\varepsilon \approx 1 + \varepsilon x + \frac{1}{2}\varepsilon(\varepsilon - 1)x^2, \quad \text{when } |x| \ll 1.$$

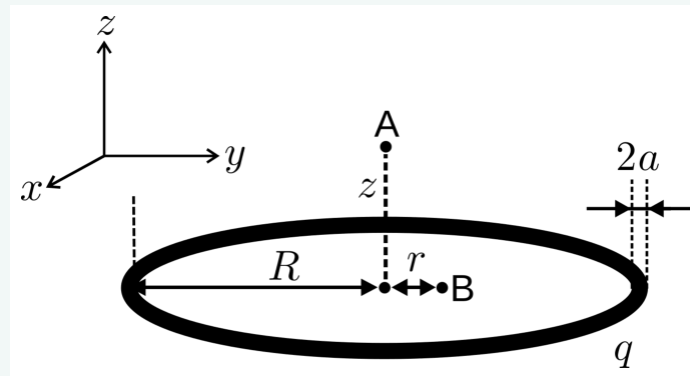


Figure 1. A charged ring of radius R .

Part A. Electrostatic potential on the axis of the ring (1 point)

A.1	Calculate the electrostatic potential $\Phi(z)$ along the axis of the ring at a z distance from its center (point A in Figure 1).	0.3pt
A.2	Calculate the electrostatic potential $\Phi(z)$ to the lowest non-zero power of z , assuming $z \ll R$.	0.4pt
A.3	An electron (mass m and charge $-e$) is placed at point A (Figure 1, $z \ll R$). What is the force acting on the electron? Looking at the expression of the force, determine the sign of q so that the resulting motion would correspond to oscillations. The moving electron does not influence the charge distribution on the ring.	0.2pt
A.4	What is the angular frequency ω of such harmonic oscillations?	0.1pt

Part B. Electrostatic potential in the plane of the ring (1.7 points)

In this part of the problem you will have to analyze the potential $\Phi(r)$ in the plane of the ring ($z = 0$) for $r \ll R$ (point B in Figure 1). To the lowest non-zero power of r the electrostatic potential is given by $\Phi(r) \approx q(\alpha + \beta r^2)$.

B.1	Find the expression for β . You might need to use the Taylor expansion formula given above.	1.5pt
B.2	An electron is placed at point B (Figure 1, $r \ll R$). What is the force acting on the electron? Looking at the expression of the force, determine the sign of q so that the resulting motion would correspond to harmonic oscillations. The moving electron does not influence the charge distribution on the ring.	0.2pt

Part C. The focal length of the idealized electrostatic lens: instantaneous charging (2.3 points)

One wants to build a device to focus electrons—an electrostatic lens. Let us consider the following construction. The ring is situated perpendicularly to the z -axis, as shown in Figure 2. We have a source that produces on-demand packets of non-relativistic electrons. Kinetic energy of these electrons is $E = mv^2/2$ (v is velocity) and they leave the source at precisely controlled moments. The system is programmed so that the ring is charge-neutral most of the time, but its charge becomes q when electrons are closer than a distance $d/2$ ($d \ll R$) from the plane of the ring (shaded region in Figure 2, called “active region”). In part C assume that charging and de-charging processes are instantaneous and the electric field “fills the space” instantaneously as well. One can neglect magnetic fields and assume that the velocity of electrons in the z -direction is constant. Moving electrons do not perturb the charge distribution on the ring.

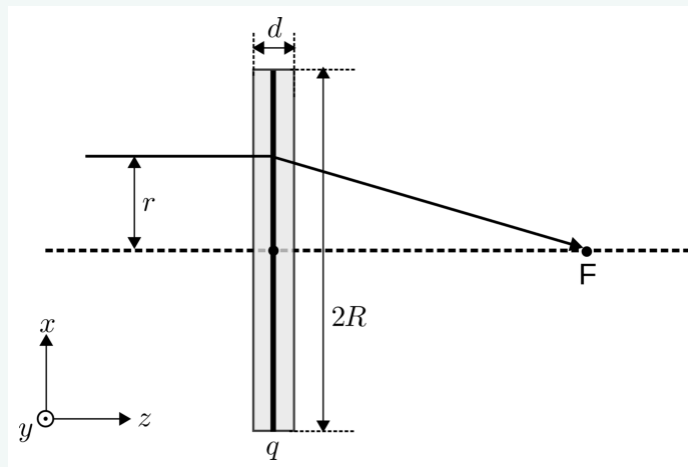


Figure 2. A model of an electrostatic lens.

C.1	Determine the focal length f of this lens. Assume that $f \gg d$. Express your answer in terms of the constant β from question B.1 and other known quantities. Assume that before reaching the “active region” the electron packet is parallel to the z -axis and $r \ll R$. The sign of q is such so that the lens is focusing.	1.3pt
-----	--	-------

In reality the electron source is placed on the z -axis at a distance $b > f$ from the center of the ring. Consider that electrons are no longer parallel to the z -axis before reaching the “active region”, but are emitted from a point source at a range of different angles $\gamma \ll 1$ rad to the z -axis. Electrons are focused in a point situated at a distance c from the center of the ring.

C.2	Find c . Express your answer in terms of the constant β from question B.1 and other known quantities.	0.8pt
C.3	Is the equation of a thin optical lens $\frac{1}{b} + \frac{1}{c} = \frac{1}{f}$ fulfilled for the electrostatic lens? Show it by explicitly calculating $1/b + 1/c$.	0.2pt

Part D. The ring as a capacitor (3 points)

The model considered above was idealized and we assumed that the ring charged instantaneously. In reality charging is non-instantaneous, as the ring is a capacitor with a finite capacitance C . In this part we will analyze the properties of this capacitor. You might need the following integrals:

$$\int \frac{dx}{\sin x} = -\ln \left| \frac{\cos x + 1}{\sin x} \right| + \text{const}$$

and

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln \left| x + \sqrt{1+x^2} \right| + \text{const.}$$

D.1	Calculate the capacitance C of the ring. Consider that the ring has a finite width $2a$, but remember that $a \ll R$.	2.0pt
-----	---	-------

When electrons reach the "active region", the ring is connected to a source of voltage V_0 (Figure 3). When electrons pass the "active region", the ring is connected to the ground. The resistance of contacts is R_0 and the resistance of the ring itself can be neglected.

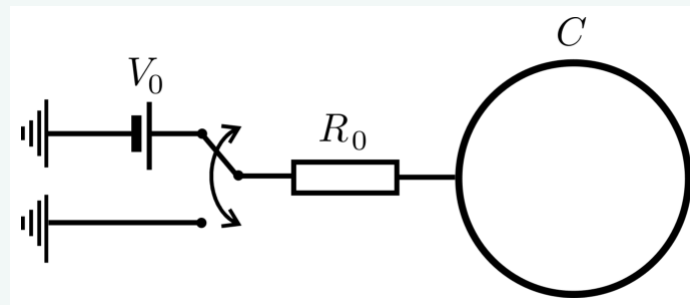


Figure 3. Charging of the electrostatic lens.

D.2	Determine the dependence of the charge on the ring as a function of time, $q(t)$, and make a schematic plot of this dependence. $t = 0$ corresponds to a time moment when electrons are in the plane of the ring. What is the charge on the ring q_0 when the absolute value $q(t)$ is maximal? The capacitance of the ring is C (i.e., you do not have to use the actual expression found in D.1). <i>Remark:</i> the drawn polarity in Figure 3 is for indicative purposes only. The sign should be chosen so that the lens is focusing.	1.0pt
-----	---	-------

Part E. Focal length of a more realistic lens: non-instantaneous charging (2 points)

In this part of the problem, we will consider the action of this more realistic lens. Here we will again neglect the width of the ring $2a$ and will assume that electrons travel parallel to the z -axis before reaching the "active region". However, the charging of the ring is no longer instantaneous.

E.1	Find the focal length f of the lens. Assume that $f/v \gg R_0 C$, but d/v and $R_0 C$ are of the same order of magnitude. Express your answer in terms of the constant β from part B and other known quantities.	1.7pt
E.2	You will see, that the result for f is similar to that obtained in part C, whereby the value q is substituted with q_{eff} . Find the expression for q_{eff} in terms of quantities given in formulation of the problem.	0.3pt

Official English text of the problem.

Answers

Use

$$k = \frac{1}{4\pi\epsilon_0}, \quad T = \frac{d}{v}, \quad \tau = R_0 C, \quad Q = CV_0.$$

Here $e > 0$, the potential vanishes at infinity, and V_0 is the signed applied voltage: focusing requires $V_0 < 0$, hence $Q < 0$. All approximate results have the accuracy stipulated in the problem.

- **A.1:** $\Phi(z) = \frac{kq}{\sqrt{R^2 + z^2}}.$
- **A.2:** $\Phi(z) \approx \frac{kq}{R} \left(1 - \frac{z^2}{2R^2}\right).$
- **A.3:** $\mathbf{F} \approx -\frac{keq}{R^3} z \hat{\mathbf{z}}.$ Axial oscillations require $q > 0$.
- **A.4:** $\omega = \sqrt{\frac{keq}{mR^3}},$ for $q > 0$.
- **B.1:** $\beta = \frac{k}{4R^3} = \frac{1}{16\pi\epsilon_0 R^3}.$
- **B.2:** $\mathbf{F}_\perp = 2e\beta q \mathbf{r}_\perp = \frac{keq}{2R^3} \mathbf{r}_\perp.$ Oscillations in the plane require $q < 0$.
- **C.1:** $f = -\frac{E}{e\beta q d} = \frac{E}{e\beta|q|d}.$
- **C.2:** $c = \frac{bE}{e\beta|q|bd - E} = \frac{bf}{b - f}.$
- **C.3:** Yes: $\frac{1}{b} + \frac{1}{c} = \frac{e\beta|q|d}{E} = \frac{1}{f}.$
- **D.1:** $C \approx \frac{4\pi^2\epsilon_0 R}{\ln(8R/a)}.$
- **D.2:**

$$q(t) = \begin{cases} 0, & t < -T/2, \\ Q \left[1 - \exp\left(-\frac{t+T/2}{\tau}\right) \right], & -T/2 \leq t \leq T/2, \\ q_0 \exp\left(-\frac{t-T/2}{\tau}\right), & t > T/2, \end{cases} \quad q_0 = Q \left(1 - e^{-T/\tau}\right) < 0.$$

The graph falls from zero to its minimum q_0 at $t = T/2$, then approaches zero exponentially from below.

- **E.1:** $f = -\frac{E}{e\beta CV_0 d} = \frac{E}{e\beta C|V_0|d}.$
- **E.2:** $q_{\text{eff}} = CV_0 = \frac{q_0}{1 - e^{-d/(vR_0 C)}}.$

Solution

For an electron, the potential energy is $U = -e\Phi$. Therefore

$$\mathbf{F} = -\nabla U = e\nabla\Phi.$$

This fixes the force signs throughout the solution.

A.1

Every charge element on the ring is at the same distance $\sqrt{R^2 + z^2}$ from the observation point on its axis. Superposition gives

$$\Phi(z) = k \int \frac{dq}{\sqrt{R^2 + z^2}} = \boxed{\frac{kq}{\sqrt{R^2 + z^2}}}.$$

A.2

For $|z| \ll R$,

$$\Phi(z) = \frac{kq}{R} \left(1 + \frac{z^2}{R^2}\right)^{-1/2}.$$

Using the supplied expansion with $\varepsilon = -1/2$,

$$\boxed{\Phi(z) \approx \frac{kq}{R} - \frac{kq}{2R^3} z^2}.$$

The first nonconstant term is quadratic because the potential is unchanged by $z \mapsto -z$.

A.3

Differentiating the axial potential,

$$F_z = e \frac{d\Phi}{dz} = -\frac{keqz}{(R^2 + z^2)^{3/2}} \approx -\frac{keq}{R^3} z.$$

Thus

$$\boxed{\mathbf{F} \approx -\frac{keq}{R^3} z \hat{\mathbf{z}}}.$$

A restoring force must point towards $z = 0$. Consequently,

$$\boxed{q > 0}$$

is required for axial oscillations.

A.4

Newton's equation becomes

$$m\ddot{z} = -\frac{keq}{R^3} z.$$

Comparison with $\ddot{z} + \omega^2 z = 0$ gives

$$\boxed{\omega = \sqrt{\frac{keq}{mR^3}} = \sqrt{\frac{eq}{4\pi\epsilon_0 mR^3}}}.$$

B.1

Place the observation point at $(r, 0, 0)$. A ring element at polar angle θ has charge

$$dq = \frac{q}{2\pi} d\theta$$

and distance

$$s = \sqrt{R^2 + r^2 - 2Rr \cos \theta}$$

from that point. Hence, with $\rho = r/R$,

$$\Phi(r) = \frac{kq}{2\pi R} \int_0^{2\pi} \frac{d\theta}{\sqrt{1 + \rho^2 - 2\rho \cos \theta}}.$$

Expanding through second order in ρ ,

$$\begin{aligned} (1 + \rho^2 - 2\rho \cos \theta)^{-1/2} &\approx 1 - \frac{1}{2}(\rho^2 - 2\rho \cos \theta) + \frac{3}{8}(4\rho^2 \cos^2 \theta) \\ &= 1 + \rho \cos \theta + \rho^2 \left(-\frac{1}{2} + \frac{3}{2} \cos^2 \theta \right). \end{aligned}$$

The angular averages are

$$\langle \cos \theta \rangle = 0, \quad \langle \cos^2 \theta \rangle = \frac{1}{2}.$$

Therefore

$$\Phi(r) \approx \frac{kq}{R} \left(1 + \frac{r^2}{4R^2} \right).$$

Comparing with $q(\alpha + \beta r^2)$,

$$\alpha = \frac{k}{R}, \quad \boxed{\beta = \frac{k}{4R^3} = \frac{1}{16\pi\epsilon_0 R^3}}.$$

B.2

Let $\mathbf{r}_\perp = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. Since $r^2 = x^2 + y^2$,

$$\boxed{\mathbf{F}_\perp = e\nabla_\perp \Phi = 2e\beta q \mathbf{r}_\perp = \frac{keq}{2R^3} \mathbf{r}_\perp}.$$

The force is restoring in the ring's plane precisely when

$$\boxed{q < 0}.$$

For this sign, each transverse coordinate satisfies a harmonic-oscillator equation. Thus axial and transverse stability require opposite signs of ring charge.

C.1

The electron spends

$$T = \frac{d}{v}$$

in the active region. Since $d \ll R$, the transverse force there can be evaluated using B.2. Also, $f \gg d$ allows us to neglect the change of transverse position during the interaction.

For an incident electron at transverse distance r , its transverse impulse is

$$m\Delta v_r = F_r T = 2e\beta q r \frac{d}{v}.$$

The incident transverse velocity is zero, so

$$v_{r,\text{out}} = \frac{2e\beta q r d}{mv} < 0$$

for the focusing choice $q < 0$.

After leaving the active region, the electron moves along a straight line. To thin-lens accuracy, the distance to the axis is

$$f = -\frac{vr}{v_{r,\text{out}}}.$$

Consequently,

$$f = -\frac{mv^2}{2e\beta qd} = -\frac{E}{e\beta qd} = \frac{E}{e\beta|q|d}.$$

The cancellation of r shows that all sufficiently near-axis parallel rays have the same focal length.

C.2

An electron emitted at angle $\gamma \ll 1$ from a source at distance b reaches the lens at

$$r \approx b\gamma.$$

Its incoming transverse velocity satisfies

$$\frac{v_{r,\text{in}}}{v} \approx \gamma \approx \frac{r}{b}.$$

The lens produces the same transverse velocity change as in C.1. Thus

$$\frac{v_{r,\text{out}}}{v} = \frac{r}{b} + \frac{2e\beta qrd}{mv^2} = r \left(\frac{1}{b} + \frac{e\beta qd}{E} \right).$$

For the outgoing trajectory to cross the axis a distance c beyond the ring,

$$\frac{v_{r,\text{out}}}{v} = -\frac{r}{c}.$$

It follows that

$$\frac{1}{c} = -\frac{1}{b} - \frac{e\beta qd}{E} = \frac{e\beta|q|d}{E} - \frac{1}{b}.$$

Therefore

$$c = \frac{bE}{e\beta|q|bd - E} = \frac{bf}{b - f}.$$

The specified condition $b > f$ ensures $c > 0$.

C.3

Using the result just obtained,

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{b} + \left(\frac{e\beta|q|d}{E} - \frac{1}{b} \right) = \frac{e\beta|q|d}{E}.$$

By C.1 this equals $1/f$, so

$$\frac{1}{b} + \frac{1}{c} = \frac{1}{f}.$$

The thin optical lens equation is therefore fulfilled within the stated approximations.

D.1

The isolated ring's capacitance is

$$C = \frac{q}{\Phi_s},$$

where Φ_s is its surface potential relative to infinity.

For a thin torus, introduce the charge per unit length

$$\lambda = \frac{q}{2\pi R}.$$

Choose an intermediate length ℓ satisfying

$$a \ll \ell \ll R.$$

We calculate the surface potential by combining a nearby, approximately straight portion of the ring with its distant curved portion.

Locally the conductor resembles a cylinder of radius a . Its exterior potential can be represented, to the required accuracy, by charge on its axis. The contribution from the segment $-\ell < s < \ell$ is therefore

$$\begin{aligned}\Phi_{\text{near}} &\approx k\lambda \int_{-\ell}^{\ell} \frac{ds}{\sqrt{s^2 + a^2}} \\ &= 2k\lambda \ln \left(\frac{\ell + \sqrt{\ell^2 + a^2}}{a} \right) \\ &\approx 2k\lambda \ln \frac{2\ell}{a}.\end{aligned}$$

For the distant portion, the thickness can be neglected. Let

$$\delta = \frac{\ell}{R} \ll 1.$$

The distance between two points on the ring separated by angle θ is $2R \sin(\theta/2)$. Hence

$$\begin{aligned}\Phi_{\text{far}} &\approx k\lambda \int_{\delta}^{2\pi-\delta} \frac{R d\theta}{2R \sin(\theta/2)} \\ &= k\lambda \int_{\delta/2}^{\pi-\delta/2} \frac{du}{\sin u} \\ &= 2k\lambda \ln \cot \frac{\delta}{4} \\ &\approx 2k\lambda \ln \frac{4R}{\ell}.\end{aligned}$$

Adding the contributions eliminates the arbitrary intermediate length:

$$\begin{aligned}\Phi_s &\approx 2k\lambda \left(\ln \frac{2\ell}{a} + \ln \frac{4R}{\ell} \right) \\ &= 2k\lambda \ln \frac{8R}{a} \\ &= \frac{q}{4\pi^2 \epsilon_0 R} \ln \frac{8R}{a}.\end{aligned}$$

Thus

$$C \approx \frac{4\pi^2 \epsilon_0 R}{\ln(8R/a)}.$$

The finite radius a supplies the short-distance cutoff that makes the surface potential finite.

D.2

The electrons enter and leave the active region at

$$t_- = -\frac{T}{2}, \quad t_+ = \frac{T}{2}, \quad T = \frac{d}{v}.$$

The ring is initially neutral. Define

$$\tau = R_0 C, \quad Q = CV_0 < 0.$$

While connected to the source, the circuit equation is

$$R_0 \frac{dq}{dt} + \frac{q}{C} = V_0,$$

or

$$\tau \dot{q} + q = Q.$$

With $q(t_-) = 0$,

$$q(t) = Q \left[1 - e^{-(t+T/2)/\tau} \right], \quad -T/2 \leq t \leq T/2.$$

When the ring is grounded,

$$\tau \dot{q} + q = 0.$$

Charge is continuous at switching, so

$$q(t) = q_0 e^{-(t-T/2)/\tau}, \quad t > T/2,$$

where

$$q_0 = Q(1 - e^{-T/\tau}) = CV_0 \left[1 - \exp \left(-\frac{d}{vR_0 C} \right) \right].$$

Combining the intervals,

$$q(t) = \begin{cases} 0, & t < -T/2, \\ Q \left[1 - e^{-(t+T/2)/\tau} \right], & -T/2 \leq t \leq T/2, \\ Q(1 - e^{-T/\tau}) e^{-(t-T/2)/\tau}, & t > T/2. \end{cases}$$

For the requested sketch, draw $q(t)$ below the time axis after charging begins. It is initially zero, falls continuously and with decreasing steepness during charging, reaches its most negative value at $t = T/2$, and then rises exponentially towards zero. Its slope changes discontinuously at both switching times, while $q(t)$ remains continuous.

Useful points are:

Time	Charge
$-T/2$	0
0	$Q(1 - e^{-T/(2\tau)})$
$T/2$	$q_0 = Q(1 - e^{-T/\tau})$

Time	Charge
$T/2 + \tau$	$q_0 \exp(-1)$
$t \rightarrow +\infty$	0, approached from below

Thus $|q(t)|$ is maximal when the electrons leave the active region.

E.1

Here τ and $T = d/v$ are comparable. Therefore the charging and appreciable discharge occur while the electrons remain within a longitudinal distance of order d from the ring. Since $d \ll R$, the transverse force from B.2 remains applicable:

$$F_r(t) \approx 2e\beta q(t)r.$$

Also, $f/v \gg \tau$ means that focusing occurs long after the charge has substantially decayed. During the force pulse, the transverse displacement is small compared with r , so r can be held constant when calculating the impulse:

$$m\Delta v_r = 2e\beta r \int_{-T/2}^{\infty} q(t) dt.$$

The charging contribution is

$$\begin{aligned} I_{\text{ch}} &= Q \int_{-T/2}^{T/2} [1 - e^{-(t+T/2)/\tau}] dt \\ &= Q [T - \tau(1 - e^{-T/\tau})]. \end{aligned}$$

The discharge contribution is

$$\begin{aligned} I_{\text{dis}} &= q_0 \int_{T/2}^{\infty} e^{-(t-T/2)/\tau} dt \\ &= q_0 \tau = QT(1 - e^{-T/\tau}). \end{aligned}$$

Their sum is

$$\boxed{\int_{-T/2}^{\infty} q(t) dt = QT = CV_0 \frac{d}{v}}.$$

Consequently,

$$\Delta v_r = \frac{2e\beta r Q d}{mv}.$$

Using the same thin-lens geometry as in C.1,

$$\boxed{f = -\frac{vr}{\Delta v_r} = -\frac{mv^2}{2e\beta CV_0 d} = -\frac{E}{e\beta CV_0 d} = \frac{E}{e\beta C|V_0|d}}.$$

The resistance changes the shape and duration of the charge pulse, but its contributions to the total impulse cancel between charging and discharging. The finite extent of the interaction produces corrections beyond the assumed thin-lens accuracy.

E.2

Comparing

$$f = -\frac{E}{e\beta qd} \quad \text{and} \quad f = -\frac{E}{e\beta CV_0 d},$$

we obtain

$$q_{\text{eff}} = CV_0.$$

Equivalently, using D.2,

$$q_{\text{eff}} = \frac{q_0}{1 - \exp[-d/(vR_0C)]}.$$

This effective charge equals the charge the ring would reach if held at the applied voltage indefinitely. Its magnitude exceeds the actual peak charge because the discharge tail also contributes to focusing.

The solution was checked against the official solution.

Problem 3: Particles and Waves

Wave–particle duality, which states that each particle can be described as a wave and vice versa, is one of the central concepts of quantum mechanics. In this problem, we will rely on this notion and just a few other basic assumptions to explore a selection of quantum phenomena covering the two distinct types of particles of the microworld—fermions and bosons.

Part A. Quantum particle in a box (1.4 points)

Consider a particle of mass m moving in a one-dimensional potential well, where its potential energy $V(x)$ is given by

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L; \\ \infty, & x < 0 \text{ or } x > L. \end{cases} \quad (1)$$

While classical particle can move in such a potential having any kinetic energy, for quantum particle only some specific positive discrete energy levels are allowed. In any such allowed state, the particle can be described as a standing de Broglie wave with nodes at the walls.

A.1 — 0.4pt

Determine the minimal possible energy $E_{\min}$ of the quantum particle in the well. Express your answer in terms of m , L , and the Planck's constant $\hbar$.

The particle's state with minimal possible energy is called the ground state, and all the rest allowed states are called excited states. Let us sort all the possible energy values in the increasing order and denote them as E_n , starting from E_1 for the ground state.

A.2 — 0.6pt

Find the general expression for the energy E_n (here $n = 1, 2, 3, \dots$).

A.3 — 0.4pt

Particle can undergo instantaneous transition from one state to another only by emitting or absorbing a photon of the corresponding energy difference. Find the wavelength λ_{21} of the photon emitted during the transition of the particle from the first excited state (E_2) to the ground state (E_1).

Part B. Optical properties of molecules (2.1 points)

In this part, we will study several optical properties of the cyanine Cy5 molecule—a widely used dye molecule, schematically shown in Fig. 1a. Its optical properties are determined mainly by the carbon backbone, composed of the alternating single and double bonds between carbon atoms, shown in Fig. 1b, while the influence of the rings at the molecule's ends as well as radicals R is much smaller. Three of the four valence electrons of each C atom (and of N atoms) in the backbone form the chemical bonds, while the remaining valence electrons are “shared” and can move along the whole backbone. The net potential energy of each such electron is shown with oscillating thin line in Fig. 1c, with minima corresponding to the positions of the C and N atoms.

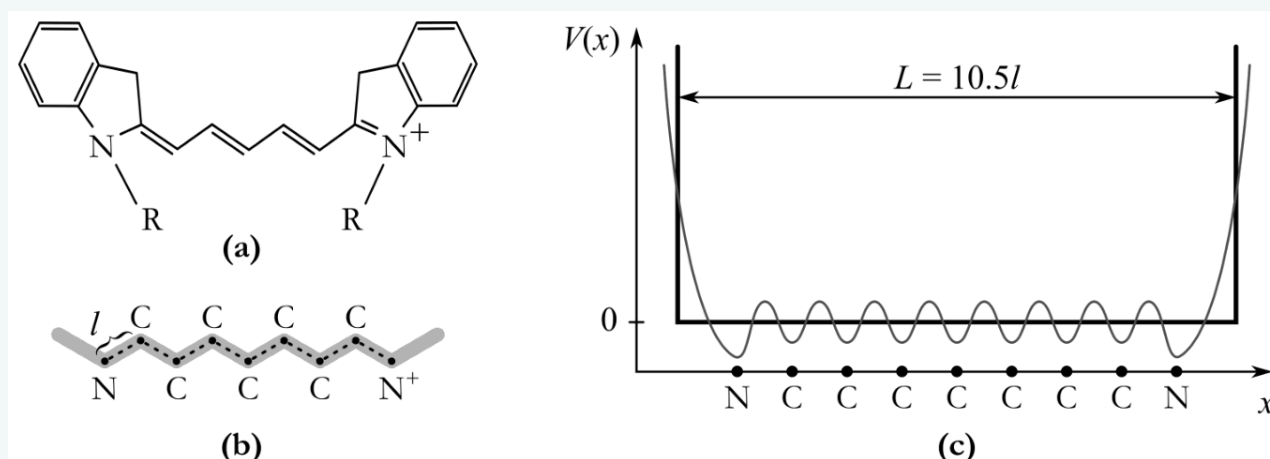


Figure 1. (a) Chemical structure of the cyanine Cy5 molecule (for simplicity, hydrogen atoms are not shown, and R denote some radicals). (b) The backbone of the Cy5 molecule, with mean inter-atomic distance l . (c) Potential energy of the electron along the backbone (thin line) and its approximation by the step function given by Eq. 1 (thick line).

For simplicity, we will approximate this potential energy profile by a simple function given in Eq. 1 with the width $L = 10.5l$ (see thick line in Fig. 1c), here $l = 140 \text{ pm}$ is the mean inter-atomic distance (see also Fig. 1b). As a result, we obtain the “electronic gas” composed of 10 electrons (7 from C atoms, 2 from the N atom, and 1 from the N^+ ion), moving in a one-dimensional potential well discussed in part A. In our evaluation, we can neglect the mutual interaction of these electrons; however, we should account for the fact that electrons are fermions and thus obey the Pauli exclusion principle. We also neglect the influence of other electrons as well as motion of the nuclei.

B.1 — 0.8pt

Evaluate the largest wavelength λ of the photon that can be absorbed by the Cy5 molecule assuming that the electron system is initially in its ground state. Express your answer in terms of l , physical constants and some numerical prefactor, and calculate the numerical value.

B.2 — 0.4pt

Another dye molecule Cy3 has similar structure, but its backbone is shorter by 2 carbon atoms. Is its absorption spectrum shifted to the bluer or to the redder spectral region compared to the Cy5 molecule? Evaluate numerically the magnitude $\Delta\lambda$ of this spectral shift. You can assume that removing two carbon atoms doesn't change the molecule shape and only makes the backbone length shorter by two interatomic distances.

Being in the excited state, molecule can undergo a spontaneous transition to the ground state while emitting photon. The mean rate K of such events (i.e. the relative decrease of the molecules being in the excited state, dN/N , over time dt , $K = \frac{1}{N} \frac{dN}{dt}$) is determined by the wavelength λ of the emitted photon, the transition electrical dipole moment d (which is of the order of $d \simeq el$, here e is elementary charge) as well as vacuum permittivity ε_0 and Planck's constant h .

B.3 — 0.7pt

Using dimension analysis, determine the expression for the rate of spontaneous emission in terms of ε_0 , h , λ , and d . The numerical prefactor for your expression is $= \frac{16}{3}\pi^3$.

B.4 — 0.2pt

For Cy5 molecule, $d \approx 2.4 el$. Evaluate the mean fluorescence lifetime of the lowest excited state of Cy5 molecule, τ_{Cy5} , which is reciprocal to the rate of its emissive transition to the ground state.

Part C. Bose-Einstein condensation (1.5 points)

This part is not directly related to Parts A and B. Here, we will study the collective behaviour of bosonic particles. Bosons do not respect the Pauli exclusion principle, and—at low temperatures or high densities—experience a dramatic phenomenon known as the Bose–Einstein condensation (BEC). This is a phase transition to an intriguing collective quantum state: a large number of identical particles ‘condense’ into a single quantum state and start behaving as a single wave. The transition is typically reached by cooling a fixed number of particles below the critical temperature. In principle, it can also be induced by keeping the temperature fixed and driving the particle density past its critical value.

We begin by exploring the relation between the temperature and the particle density at the transition. As it turns out, estimates of their critical values can be deduced from a simple observation: *Bose–Einstein condensation takes place when the de Broglie wavelength corresponding to the mean square speed of the particles is equal to the characteristic distance between the particles in a gas.*

C.1 — 0.4pt

Given a non-interacting gas of ^{87}Rb atoms in thermal equilibrium, write the expressions for their typical linear momentum p and the typical de Broglie wavelength λ_{dB} as a function of atom's mass m , temperature T and physical constants.

C.2 — 0.5pt

Calculate the typical distance between the particles in a gas, ℓ , as a function of particle density n . Hence deduce the critical temperature T_c as a function of atom's mass, their density and physical constants.

To realize BEC in the lab, the experimentalists have to cool gases to temperatures as low as $T_c = 100 \text{ nK}$.

C.3 — 0.6pt

What is the particle density of the Rb gas n_c if the transition takes place at such a temperature? For the sake of comparison, calculate also the ‘ordinary’ particle density n_0 of an ideal gas at the standard temperature and pressure (STP), i.e. $T_0 = 300 \text{ K}$ and $p_0 = 10^5 \text{ Pa}$. How many times is

the 'ordinary' gas denser? You may assume that the mass of the atoms is equal to 87 atomic mass units (m_{amu}).

Part D. Three-beam optical lattices (5 points)

The first Bose-Einstein condensates were produced back in 1995, and since then the experimental work has branched out in diverse directions. In this part, you will investigate one particularly fruitful idea to load the condensate into spatially periodic potentials created by interfering a number of coherent laser beams. Due to the periodic nature of the resulting interference patterns, they are referred to as *optical lattices*. The potential energy $V(\vec{r})$ of an atom moving in an optical lattice is proportional to the local intensity of the light, and in your calculations you may assume that

$$V(\vec{r}) = -\alpha \left\langle \left| \vec{E}(\vec{r}, t) \right|^2 \right\rangle. \quad (2)$$

Here, α is a positive constant, and the angle brackets indicate time-averaging which eliminates the rapidly oscillating terms. The electric field produced by the i -th laser is described by

$$\vec{E}_i = E_{0,i} \vec{e}_i \cos(\vec{k}_i \cdot \vec{r} - \omega t), \quad (3)$$

with the amplitude $E_{0,i}$, the wave vector $\vec{k}_i$, and the unit-length polarization vector $\vec{e}_i$.

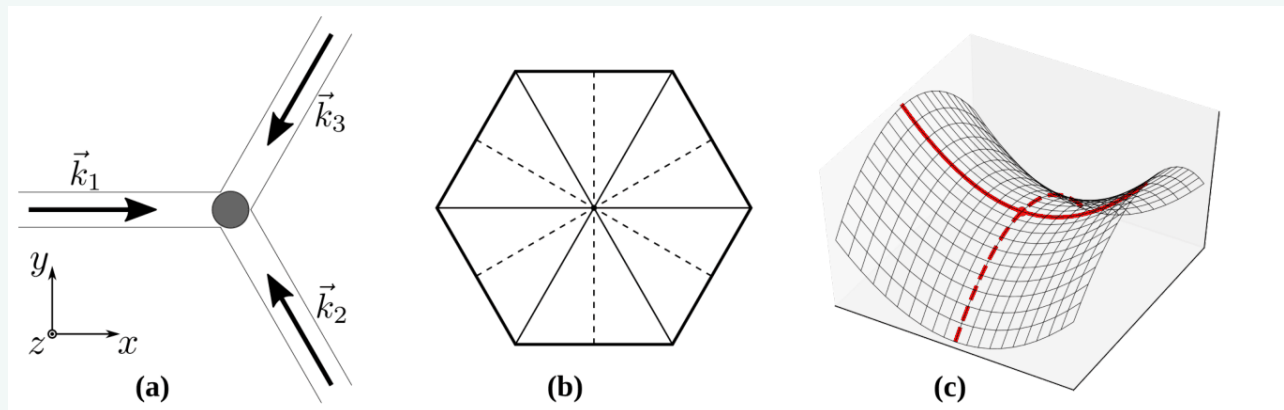


Figure 2. (a) Three-beam optical lattice: three plane waves with wave vectors $\vec{k}_{1,2,3}$ intersect and interfere in the area indicated by the grey circle. (b) Symmetries of a regular hexagon: solid and dashed lines show two sets of symmetry axes. (c) Saddle point: a point on a surface where the slopes in orthogonal directions are all zero, but which is not a local extremum of the plotted function. Travelling along the trajectory marked by the full line one encounters an apparent minimum. Additional analysis of the perpendicular direction (dashed line) is needed to distinguish a true minimum from a saddle point (shown).

Your task is to study *triangular optical lattices* that are produced by interfering three coherent laser beams of equal intensity. A typical setup is shown in Fig. 2a. Here, all three beams are polarized in the z direction, propagate in the xy plane and intersect at equal angles of 120° . Choose the direction of the x axis parallel to the wave vector $\vec{k}_1$.

D.1 – 1.4pt

Using Eqs. 2 and 3 obtain the expression for the potential energy $V(\vec{r})$ as a function of $\vec{r} = (x, y)$ in the plane of the beams.

Hint: the result can be neatly expressed as a constant term plus a sum of three cosine functions of arguments $\vec{b}_i \cdot \vec{r}$. Please write your result in this form and identify the vectors $\vec{b}_i$.

D.2 — 0.5pt

The resulting potential energy has a sixfold rotational symmetry axis, i.e., the potential distribution is invariant with respect to a rotation by a multiple of 60° around the origin. Provide a simple argument to prove that this is indeed the case.

The above observation of symmetry simplifies the analysis of the two-dimensional potential distribution $V(\vec{r})$. As shown in Fig. 2b, a regular hexagon has symmetry lines that, respectively, connect opposite vertices (solid lines) and midpoints of opposite edges (dashed lines). Therefore, in our situation one does not need to produce and study two-dimensional potential plots as many insights can be deduced by focusing on the coordinate axes x and y that run along the symmetry lines.

D.3 — 1.2pt

Derive the behavior of the potential $V(\vec{r})$ along the coordinate axes, i.e., determine the functions $V_X(x) \equiv V(x, 0)$ and $V_Y(y) \equiv V(0, y)$. Identify the locations of the extrema of $V_X(x)$ and $V_Y(y)$ as functions of a single argument. As these functions are periodic, please include in your lists only one representative from each family of periodically repeated minima and maxima.

We are interested in determining the locations of so-called *lattice sites*, i.e., the minima of the full two-dimensional potential $V(\vec{r})$. The obtained minima of single-argument functions V_X and V_Y identify their suspected positions but still have to be checked to eliminate the saddle points. As shown in Fig. 2c, when studied along a single line, saddle points may disguise as minima but they are not.

D.4 — 0.8pt

Review your results in the previous question to determine actual minima of the optical lattice: Identify all equivalent minima nearest to (but not coinciding with) the origin. What is the distance a between the nearest minima, in other words—the lattice constant of our optical lattice? Express the answer in terms of the laser wavelength λ_{las} .

Charge neutrality of ultracold atoms suggests that their interactions become relevant only when two or more atoms occupy the same site of an optical lattice. However, experimentalists are also able to explore setups that sustain long-range atomic interactions. A possible approach relies on creation of the so-called *Rydberg atoms* that are physically large and feature other exaggerated properties. Rydberg atoms are excited atoms with one electron promoted to a state with a very high principal quantum number n . The size of a Rydberg atom can be estimated by calculating the radius of the classical circular orbit of that electron with the orbital angular momentum $n\hbar$, here $\hbar$ is the reduced Planck constant.

D.5 — 1.1pt

Calculate the value of n that corresponds to the radius of the Rb Rydberg atom comparable to the wavelength of laser light $\lambda_{\text{las}} = 380 \text{ nm}$. Give your answer in terms of λ_{las} and physical constants and find its numerical value.

Answers

Here m_e denotes the electron mass. For Part D, define the common field amplitude E_0 , $U = \alpha E_0^2 > 0$, and

$$k = \frac{2\pi}{\lambda_{\text{las}}}, \quad P_X = \frac{4\pi}{3k}, \quad P_Y = \frac{4\pi}{\sqrt{3}k}.$$

- **A.1:** $E_{\min} = \frac{h^2}{8mL^2}$.
- **A.2:** $E_n = \frac{n^2 h^2}{8mL^2}, n = 1, 2, 3, \dots$
- **A.3:** $\lambda_{21} = \frac{8mcL^2}{3h}$.
- **B.1:** $\lambda_{\text{Cy}5} = \frac{882}{11} \frac{m_e c l^2}{h} \simeq 648 \text{ nm}$.
- **B.2:** A **blue shift**: $\lambda_{\text{Cy}3} \simeq 519 \text{ nm}$, and $\Delta\lambda = \frac{1580}{99} \frac{m_e c l^2}{h} \simeq 129 \text{ nm}$.
- **B.3:** $K = \frac{16\pi^3 d^2}{3\varepsilon_0 h \lambda^3}$.
- **B.4:** $\tau_{\text{Cy}5} = \frac{3\varepsilon_0 h \lambda_{\text{Cy}5}^3}{16\pi^3 (2.4 \text{ eV})^2} \simeq 3.3 \text{ ns}$.
- **C.1:** $p = \sqrt{3mk_{\text{B}}T}, \lambda_{\text{dB}} = \frac{h}{\sqrt{3mk_{\text{B}}T}}$.
- **C.2:** $\ell = n^{-1/3}, T_c = \frac{h^2 n^{2/3}}{3mk_{\text{B}}}$, within the stated transition estimate.
- **C.3:** $n_c \simeq 1.59 \times 10^{18} \text{ m}^{-3}, n_0 \simeq 2.41 \times 10^{25} \text{ m}^{-3}$; the ordinary gas is 1.52×10^7 times denser.
- **D.1:** $V(\vec{r}) = -U \left[\frac{3}{2} + \sum_{i=1}^3 \cos(\vec{b}_i \cdot \vec{r}) \right]$, with $\vec{b}_1 = k \left(\frac{3}{2}, -\frac{\sqrt{3}}{2} \right), \vec{b}_2 = k(0, \sqrt{3})$, and $\vec{b}_3 = k \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2} \right)$.
- **D.2:** The six vectors $\{\pm\vec{b}_1, \pm\vec{b}_2, \pm\vec{b}_3\}$ form a regular hexagon. A 60° rotation permutes them, and cosine is even, so V is unchanged.
- **D.3:** $V_X(x) = -U \left[\frac{5}{2} + 2 \cos \left(\frac{3kx}{2} \right) \right]$, and $V_Y(y) = -\frac{U}{2} \left[1 + 2 \cos \left(\frac{\sqrt{3}ky}{2} \right) \right]^2$.
Modulo P_X , the x -section has a minimum at 0 and a maximum at $P_X/2$. Modulo P_Y , the y -section has minima at 0, $P_Y/2$ and maxima at $P_Y/3, 2P_Y/3$.
- **D.4:** $a = \frac{2}{3} \lambda_{\text{las}}$. The six nearest minima are $(\pm a, 0)$ and $(\pm a/2, \pm\sqrt{3}a/2)$, with independent signs in the latter pair.

- **D.5:** $n = \sqrt{\frac{m_e e^2 \lambda_{\text{las}}}{4\pi\epsilon_0 \hbar^2}} \simeq 84.7$, corresponding to $n \approx 85$.

Solution

The numerical evaluations use

$$\begin{aligned} h &= 6.62607015 \times 10^{-34} \text{ J s}, & c &= 2.99792458 \times 10^8 \text{ m s}^{-1}, \\ m_e &\simeq 9.10938 \times 10^{-31} \text{ kg}, & e &= 1.602176634 \times 10^{-19} \text{ C}, \\ \epsilon_0 &\simeq 8.85419 \times 10^{-12} \text{ F m}^{-1}, & k_B &= 1.380649 \times 10^{-23} \text{ J K}^{-1}, \\ m_{\text{amu}} &\simeq 1.66054 \times 10^{-27} \text{ kg}. \end{aligned}$$

A.1

The wave must have a node at each wall. The longest nonzero standing wave fitting into the well has one half-wavelength across its width:

$$L = \frac{\lambda_{\text{dB},1}}{2}.$$

The de Broglie relation therefore gives

$$p_1 = \frac{h}{\lambda_{\text{dB},1}} = \frac{h}{2L}.$$

Using nonrelativistic kinetic energy, with zero potential energy inside the well,

$$E_{\text{min}} = \frac{p_1^2}{2m} = \frac{h^2}{8mL^2}.$$

A zero-wave-number state cannot satisfy both boundary conditions unless the wave is identically zero, so it is not an allowed particle state.

A.2

In the n -th standing-wave mode, n half-wavelengths fit between the walls:

$$L = n \frac{\lambda_{\text{dB},n}}{2}, \quad n = 1, 2, 3, \dots$$

Consequently,

$$\lambda_{\text{dB},n} = \frac{2L}{n}, \quad p_n = \frac{nh}{2L},$$

and

$$E_n = \frac{n^2 h^2}{8mL^2}.$$

A.3

Energy conservation equates the emitted photon energy to the decrease in the particle's energy:

$$\frac{hc}{\lambda_{21}} = E_2 - E_1 = \frac{(4-1)h^2}{8mL^2}.$$

Hence

$$\lambda_{21} = \frac{8mcL^2}{3h}.$$

B.1

Each spatial energy level can hold two electrons with opposite spins. Thus the ground state of the ten-electron system fills levels $n = 1, \dots, 5$, two electrons per level.

The smallest possible excitation promotes an electron from $n = 5$ to the first unoccupied level, $n = 6$. Its energy is

$$\Delta E_{\text{Cy}5} = E_6 - E_5 = \frac{(6^2 - 5^2)h^2}{8m_e L^2} = \frac{11h^2}{8m_e L^2}.$$

The smallest absorbed photon energy corresponds to the largest wavelength:

$$\lambda_{\text{Cy}5} = \frac{hc}{\Delta E_{\text{Cy}5}} = \frac{8m_e c L^2}{11h}.$$

With $L = 10.5l$,

$$\lambda_{\text{Cy}5} = \frac{8(10.5)^2}{11} \frac{m_e c l^2}{h} = \frac{882}{11} \frac{m_e c l^2}{h}.$$

Substituting $l = 140 \times 10^{-12} \text{ m}$,

$$\lambda_{\text{Cy}5} \simeq 6.48 \times 10^{-7} \text{ m} = 648 \text{ nm}.$$

B.2

Removing two carbon atoms removes two shared electrons and shortens the well by $2l$. Therefore Cy3 has eight electrons and

$$L_{\text{Cy}3} = 8.5l.$$

Its ground state fills levels $n = 1, \dots, 4$. The lowest excitation is $4 \rightarrow 5$, so

$$\Delta E_{\text{Cy}3} = \frac{(5^2 - 4^2)h^2}{8m_e (8.5l)^2} = \frac{9h^2}{8m_e (8.5l)^2}.$$

Thus

$$\lambda_{\text{Cy}3} = \frac{8(8.5)^2}{9} \frac{m_e c l^2}{h} = \frac{578}{9} \frac{m_e c l^2}{h} \simeq 519 \text{ nm}.$$

This is shorter than the Cy5 wavelength, so the shift is toward the **blue**. Its magnitude is

$$\begin{aligned} \Delta\lambda &= \lambda_{\text{Cy}5} - \lambda_{\text{Cy}3} \\ &= \left(\frac{882}{11} - \frac{578}{9} \right) \frac{m_e c l^2}{h} \\ &= \frac{1580}{99} \frac{m_e c l^2}{h}. \end{aligned}$$

Therefore

$$\Delta\lambda \simeq 1.29 \times 10^{-7} \text{ m} = 129 \text{ nm}.$$

B.3

The dipole moment has dimensions of charge times length:

$$[d] = \text{C m}.$$

Because

$$[\varepsilon_0] = \frac{\text{C}^2}{\text{J m}},$$

we have

$$\left[\frac{d^2}{\varepsilon_0} \right] = \text{J m}^3.$$

Dividing by h , whose dimensions are J s, and by λ^3 produces the required inverse-time dimension:

$$\left[\frac{d^2}{\varepsilon_0 h \lambda^3} \right] = \text{s}^{-1}.$$

Using the supplied dimensionless prefactor,

$$K = \frac{16\pi^3 d^2}{3\varepsilon_0 h \lambda^3}.$$

Here K is the positive decay rate. For the number $N(t)$ of molecules remaining excited, the decay equation is $dN/dt = -KN$; the statement's "relative decrease" is interpreted as the positive loss.

B.4

Under the given model, the lowest excited state emits a photon of wavelength λ_{Cy5} when it returns to the ground state. Its dipole moment is

$$d = 2.4 \text{ e}l \simeq 5.38 \times 10^{-29} \text{ C m}.$$

The spontaneous-emission rate is

$$K = \frac{16\pi^3 (2.4 \text{ e}l)^2}{3\varepsilon_0 h \lambda_{\text{Cy5}}^3} \simeq 3.01 \times 10^8 \text{ s}^{-1}.$$

For exponential decay, the mean lifetime is $1/K$:

$$\tau_{\text{Cy5}} = \frac{3\varepsilon_0 h \lambda_{\text{Cy5}}^3}{16\pi^3 (2.4 \text{ e}l)^2} \simeq 3.3 \text{ ns}.$$

C.1

For the temperature estimate specified in the problem, equipartition gives the mean translational kinetic energy in three dimensions:

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T.$$

The typical momentum associated with the root-mean-square speed is therefore

$$p = m \sqrt{\langle v^2 \rangle} = \sqrt{3mk_B T}.$$

Its de Broglie wavelength is

$$\lambda_{\text{dB}} = \frac{h}{p} = \frac{h}{\sqrt{3mk_B T}}.$$

C.2

At number density n , the mean volume per particle is $1/n$. The characteristic linear spacing is consequently

$$\ell = n^{-1/3}.$$

Applying the problem's transition criterion,

$$\lambda_{\text{dB}}(T_c) = \ell,$$

gives

$$\frac{h}{\sqrt{3mk_{\text{B}}T_c}} = n^{-1/3}.$$

Squaring and rearranging,

$$T_c = \frac{h^2 n^{2/3}}{3mk_{\text{B}}}.$$

This is the critical-temperature estimate obtained from the stated wavelength–spacing criterion.

C.3

For rubidium,

$$m = 87m_{\text{amu}} \simeq 1.44467 \times 10^{-25} \text{ kg}, \quad T_c = 100 \text{ nK} = 1.00 \times 10^{-7} \text{ K}.$$

Inverting the result of C.2,

$$n_c = \left(\frac{3mk_{\text{B}}T_c}{h^2} \right)^{3/2}.$$

Numerically,

$$n_c \simeq 1.59 \times 10^{18} \text{ m}^{-3}.$$

The corresponding spacing is approximately $8.57 \times 10^{-7} \text{ m}$.

For the ordinary ideal gas, the ideal-gas law gives

$$p_0 = n_0 k_{\text{B}} T_0.$$

Using the temperature and pressure specified in the question,

$$n_0 = \frac{10^5}{k_{\text{B}}(300)} \simeq 2.41 \times 10^{25} \text{ m}^{-3}.$$

Thus

$$\frac{n_0}{n_c} \simeq 1.52 \times 10^7.$$

The ordinary gas is about fifteen million times denser.

D.1

Let all three field amplitudes be E_0 . Choosing the wave vectors consistently with Fig. 2a,

$$\vec{k}_1 = k(1, 0), \quad \vec{k}_2 = k\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \vec{k}_3 = k\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

where $k = 2\pi/\lambda_{\text{las}}$.

All polarizations are along z , so the fields add as scalars in that direction:

$$\vec{E}(\vec{r}, t) = E_0 \hat{z} \sum_{i=1}^3 \cos(\vec{k}_i \cdot \vec{r} - \omega t).$$

The required time averages are

$$\langle \cos^2(\phi - \omega t) \rangle = \frac{1}{2},$$

and

$$\langle \cos(\phi_i - \omega t) \cos(\phi_j - \omega t) \rangle = \frac{1}{2} \cos(\phi_i - \phi_j).$$

Therefore

$$\langle |\vec{E}|^2 \rangle = E_0^2 \left[\frac{3}{2} + \sum_{i < j} \cos((\vec{k}_i - \vec{k}_j) \cdot \vec{r}) \right].$$

Define

$$\begin{aligned} \vec{b}_1 &= \vec{k}_1 - \vec{k}_2 = k \left(\frac{3}{2}, -\frac{\sqrt{3}}{2} \right), \\ \vec{b}_2 &= \vec{k}_2 - \vec{k}_3 = k(0, \sqrt{3}), \\ \vec{b}_3 &= \vec{k}_3 - \vec{k}_1 = k \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2} \right). \end{aligned}$$

Since cosine is even, the sign chosen for each difference is immaterial. With $U = \alpha E_0^2$,

$$V(\vec{r}) = -U \left[\frac{3}{2} + \cos(\vec{b}_1 \cdot \vec{r}) + \cos(\vec{b}_2 \cdot \vec{r}) + \cos(\vec{b}_3 \cdot \vec{r}) \right].$$

Combining the first and third cosines gives the useful coordinate form

$$V(x, y) = -U \left[\frac{3}{2} + 2 \cos\left(\frac{3kx}{2}\right) \cos\left(\frac{\sqrt{3}ky}{2}\right) + \cos(\sqrt{3}ky) \right].$$

D.2

The three $\vec{b}_i$ have equal magnitude $\sqrt{3}k$ and are separated by 120° . Including their negatives gives six vectors at successive angular intervals of 60° .

Rotating the position vector by 60° is equivalent, in each dot product, to rotating the corresponding $\vec{b}_i$ by -60° . This sends every $\vec{b}_i$ to another member of the set $\{\pm \vec{b}_j\}$. Since

$$\cos(-\vec{b}_j \cdot \vec{r}) = \cos(\vec{b}_j \cdot \vec{r}),$$

the sum is unchanged. Thus the potential has the stated sixfold rotational symmetry.

D.3

Setting $y = 0$,

$$V_X(x) = -U \left[\frac{5}{2} + 2 \cos \left(\frac{3kx}{2} \right) \right].$$

Its fundamental period is

$$P_X = \frac{4\pi}{3k} = \frac{2\lambda_{\text{las}}}{3}.$$

The derivative is

$$V'_X(x) = 3Uk \sin \left(\frac{3kx}{2} \right).$$

Within $0 \leq x < P_X$, the minimum is at $x = 0$, and the maximum is at $x = P_X/2$.

For the y -axis, introduce

$$q = \frac{\sqrt{3}ky}{2}.$$

Then

$$V_Y(y) = -U \left(\frac{3}{2} + 2 \cos q + \cos 2q \right).$$

Using $\cos 2q = 2 \cos^2 q - 1$,

$$V_Y(y) = -\frac{U}{2}(1 + 2 \cos q)^2.$$

Its fundamental period is

$$P_Y = \frac{4\pi}{\sqrt{3}k} = \frac{2\lambda_{\text{las}}}{\sqrt{3}}.$$

Differentiation gives

$$V'_Y(y) = \sqrt{3}Uk(1 + 2 \cos q) \sin q.$$

Hence the stationary points satisfy either $\sin q = 0$ or $\cos q = -1/2$.

One representative of every family of extrema is:

Section	Coordinate in one period	Character along that axis	Potential
V_X	$x = 0$	Minimum	$-9U/2$
V_X	$x = P_X/2$	Maximum	$-U/2$
V_Y	$y = 0$	Minimum	$-9U/2$
V_Y	$y = P_Y/3$	Maximum	0
V_Y	$y = P_Y/2$	Minimum	$-U/2$
V_Y	$y = 2P_Y/3$	Maximum	0

Thus V_X is a sinusoid about $-5U/2$. Over one period, V_Y rises from a deep minimum to zero, falls to a shallower minimum, rises to zero again, and returns to the deep minimum at the next period boundary.

D.4

First check the apparently shallow minimum at

$$(x, y) = \left(0, \frac{P_Y}{2}\right).$$

Here the second derivatives of the full potential are

$$V_{xx} = -\frac{9}{2}Uk^2, \quad V_{yy} = \frac{3}{2}Uk^2, \quad V_{xy} = 0.$$

The curvatures have opposite signs, so this point is a **saddle**, not a lattice site.

At the origin,

$$V_{xx} = V_{yy} = \frac{9}{2}Uk^2, \quad V_{xy} = 0,$$

so the origin is a true minimum.

To locate all true minima, write

$$t = \frac{3kx}{2}, \quad q = \frac{\sqrt{3}ky}{2}.$$

The stationarity conditions are

$$\sin t \cos q = 0, \quad \sin q(\cos t + 2 \cos q) = 0.$$

If $\cos q \neq 0$, stationarity in x requires $\sin t = 0$. Positive curvature in x then requires

$$\cos t \cos q > 0.$$

Since $\cos t$ and $\cos q$ have the same sign, $\cos t + 2 \cos q$ cannot vanish. Thus $\sin q = 0$. The minima therefore have

$$t = \pi r, \quad q = \pi s,$$

where integers r, s have the same parity. The remaining stationary possibility, $\cos q = 0$, also requires $\cos t = 0$; its Hessian has negative determinant, so it gives only saddles.

Writing $r = 2j + s$, the lattice sites are

$$\vec{r}_{j,s} = a \left(j + \frac{s}{2}, \frac{\sqrt{3}s}{2} \right), \quad j, s \in \mathbb{Z},$$

with

$$a = \frac{4\pi}{3k}.$$

At these points all three interference cosines equal 1, and

$$V_{\min} = -\frac{9}{2}U.$$

The squared distance from the origin is

$$|\vec{r}_{j,s}|^2 = a^2(j^2 + js + s^2).$$

For nonzero integer pairs, the smallest value of the positive integer $j^2 + js + s^2$ is 1. It occurs at six sites:

$$\boxed{(a, 0), (-a, 0), \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right), \left(-\frac{a}{2}, \frac{\sqrt{3}a}{2}\right), \left(\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right), \left(-\frac{a}{2}, -\frac{\sqrt{3}a}{2}\right)}.$$

They form a regular hexagon around the origin. Therefore

$$a = \frac{4\pi}{3k} = \frac{2}{3}\lambda_{\text{las}}.$$

D.5

For a highly excited outer electron, the nucleus and inner electrons act approximately as a core of net charge $+e$. We use the stated classical circular-orbit model, neglecting core-penetration effects and the motion of the much heavier core.

Coulomb attraction supplies the centripetal force:

$$\frac{m_e v^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2}.$$

The prescribed angular momentum is

$$m_e v r = n\hbar.$$

Eliminating v ,

$$\frac{n^2 \hbar^2}{m_e r^3} = \frac{e^2}{4\pi\epsilon_0 r^2},$$

so

$$r = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} n^2 = a_0 n^2,$$

where

$$a_0 \simeq 5.292 \times 10^{-11} \text{ m}.$$

Taking the estimated radius to equal λ_{las} ,

$$n = \sqrt{\frac{\lambda_{\text{las}}}{a_0}} = \sqrt{\frac{m_e e^2 \lambda_{\text{las}}}{4\pi\epsilon_0 \hbar^2}}.$$

For $\lambda_{\text{las}} = 380 \text{ nm}$,

$$n = \sqrt{\frac{380 \times 10^{-9}}{5.292 \times 10^{-11}}} \simeq 84.7.$$

Thus the corresponding integer principal quantum number is

$$n \approx 85.$$

The solution was checked against the official solution.