

IPhO 2019

Theory problems and solutions · Tel Aviv, Israel · 3 theory problems with complete solutions

eulersolve.org/ipho/2019/

Problem 1: Zero-length springs and slinky coils

A zero effective length spring (ZLS) is a spring for which the force is proportional to the spring's length, $F = kL$ for $L > L_0$ where L_0 is the minimal length of the spring as well as its unstretched length. Figure 1 shows the relation between the force F and the spring length L , for a ZLS, where the slope of the line is the spring constant k .

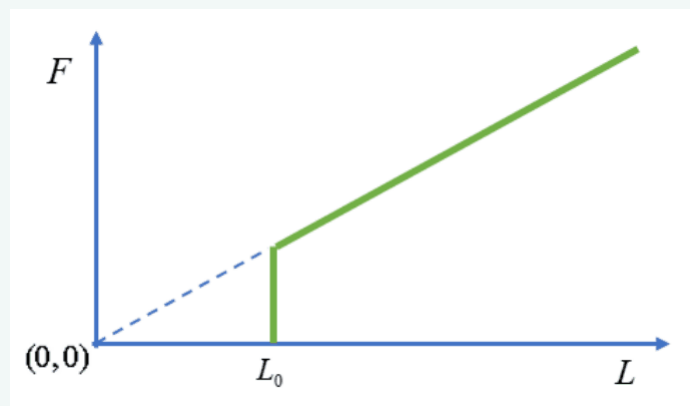


Figure 1: the relation between the force F and the spring length L

A ZLS is useful in seismography and allows very accurate measurement of changes in the gravitational acceleration g . Here, we shall consider a homogenous ZLS, whose weight Mg exceeds kL_0 . We define a corresponding dimensionless ratio, $\alpha = kL_0/Mg < 1$ to characterize the relative softness of the spring. The toy known as "slinky" may be (but not necessarily) such a ZLS.

Part A: Statics (3.0 points)

A.1	Consider a segment of length $\Delta\ell$ of the unstretched ZLS spring which is then stretched by a force F , under weightless conditions. What is the length Δy of this segment as a function of F , $\Delta\ell$ and the parameters of the spring?	0.5pt
A.2	For a segment of length $\Delta\ell$, calculate the work ΔW required to stretch it from its original length $\Delta\ell$ to a length Δy .	0.5pt

Throughout this question, we will denote a point on the spring by its distance $0 \leq \ell \leq L_0$ from the bottom of the spring when it is unstretched. In particular, for every point on the spring, ℓ remains unchanged as the spring stretches.

A.3	Suppose that we hang the spring by its top end, so that it stretches under its own weight. What is the total length H of the suspended spring in equilibrium? Express your answers in terms of L_0 and α .	2.0pt
------------	---	-------

Part B: Dynamics (5.5 points)

Experiments show that when the spring is hung at rest and then released, it gradually contracts from the top, while the lower part remains stationary (see Figure 2). As time advances, the contracting part moves as a solid chunk and accumulates additional turns of the spring, while the stationary part becomes shorter. Every point on the spring begins to move only when the moving part reaches it. The bottom end of the spring starts moving only when the spring is fully collapsed and reaches its unstretched length L_0 . After that, the contracted spring continues falling straight downwards, without tumbling, as a rigid body under the influence of gravity.

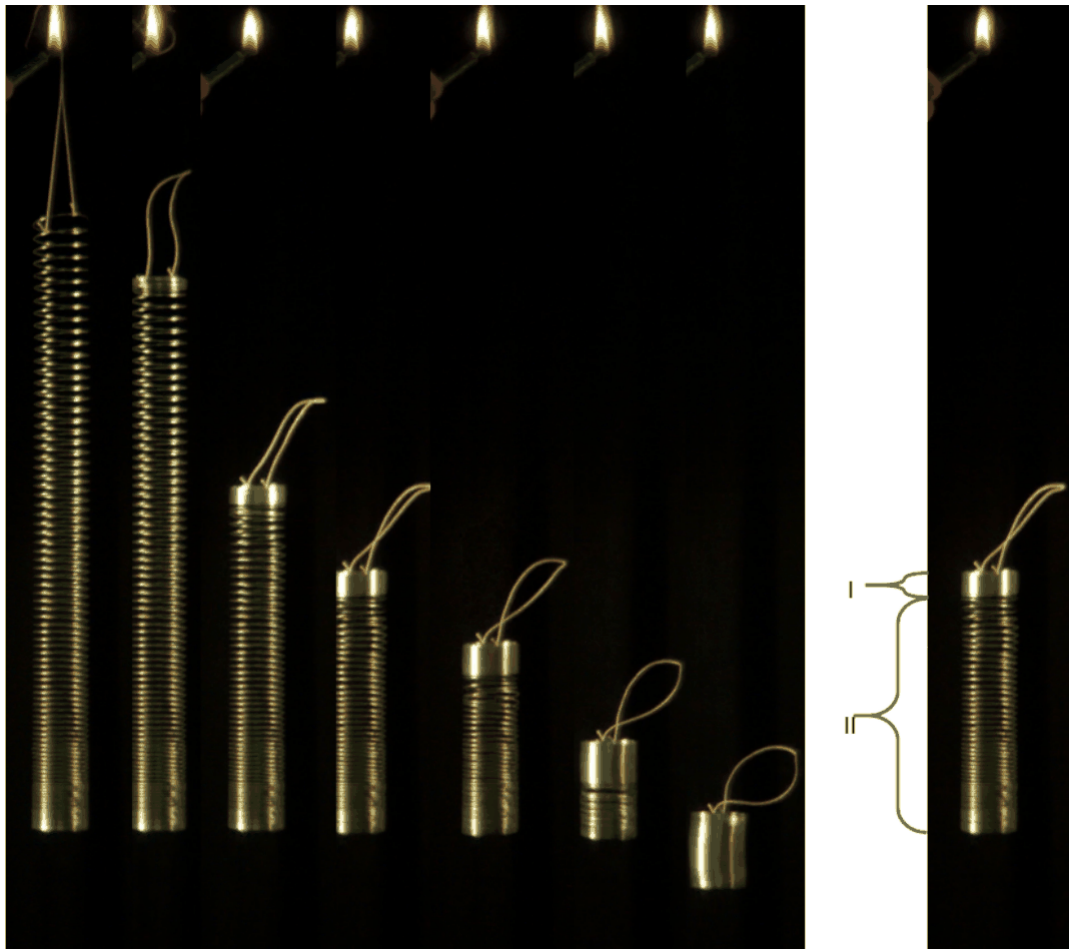


Figure 2: Left: a sequence of pictures taken during the free fall of slinky. Right: the moving part I and the stationary part II during the free fall of the spring.

In the remaining parts of the question, you are asked to base your solution on this described model. You may neglect air resistance, but you are not allowed to neglect L_0 .

B.1	Calculate the time t_c it takes from the moment the spring is released, until it fully collapses back to its minimal length L_0 . Express your answer in terms of L_0 , g and α . Compute the numerical value of t_c for a spring with $k = 1.02 \text{ N/m}$, $L_0 = 0.055 \text{ m}$ and $M = 0.201 \text{ kg}$, while taking g to be 9.80 m/s^2 .	2.5pt
B.2	In this task ℓ is used to denote the coordinate of the boundary between parts I (in figure 2, the moving part) and II (the stationary part). At a certain moment, while a stationary part still exists its mass is $m(\ell) = \frac{\ell}{L_0} M$, and the moving part moves with uniform instantaneous velocity $v_I(\ell)$. Show that at this moment (while there exists a stationary	2.5pt

	part) the velocity of the moving part is $v_I(\ell) = \sqrt{A\ell + B}$. Express the constants A and B in terms of L_0 , g and α .	
B.3	Based on B.2, find the minimum speed $v_{\min}$ of the moving part of the spring in the course of its motion, after its release and before it hits the ground. Express your answer in terms of L_0 , α , A and B .	0.5pt

Part C: Energetics (1.5 points)

C.1	Calculate the amount of mechanical energy Q that was lost by generating heat, from the moment the spring is released until just before the spring hits the ground. Express your answer in terms of L_0 , M , g and α .	1.5pt
------------	---	-------

Official English text of the problem.

Answers

- **A.1**

$$\Delta y = \Delta \ell \max\left(1, \frac{F}{kL_0}\right)$$

- **A.2**, for $\Delta y \geq \Delta \ell$:

$$\Delta W = \frac{kL_0}{2\Delta \ell} [(\Delta y)^2 - (\Delta \ell)^2]$$

- **A.3**

$$H = \frac{L_0}{2\alpha}(1 + \alpha^2)$$

- **B.1**

$$t_c = \sqrt{\frac{L_0(1 - \alpha)^3}{3\alpha g}} \quad t_c \approx 0.245 \text{ s}$$

- **B.2**, taking downward velocity as positive:

$$A = \frac{2g}{3\alpha}, \quad B = \frac{gL_0(1 - 3\alpha)}{3\alpha}$$

The collapse proceeds through $\alpha L_0 \leq \ell < L_0$.

- **B.3**

$$v_{\min} = (1 - \alpha)\sqrt{A\alpha L_0 + B}$$

This occurs immediately after the moving part collides with the initially compact bottom portion.

• C.1

$$Q = \frac{MgL_0}{6\alpha}(1 - \alpha)^2(1 + 2\alpha)$$

Solution

A.1

Under weightless conditions the tension is uniform. Since the spring is homogeneous, every segment has the same ratio of stretched length to unstretched length.

When $F > kL_0$, the whole spring has length $L = F/k$, so

$$\frac{\Delta y}{\Delta \ell} = \frac{L}{L_0} = \frac{F}{kL_0}.$$

Equivalently, the segment has spring constant

$$k_{\text{seg}} = \frac{kL_0}{\Delta \ell}.$$

For $0 \leq F \leq kL_0$, the segment remains at its minimum length. Thus

$$\Delta y = \begin{cases} \Delta \ell, & 0 \leq F \leq kL_0, \\ \frac{F\Delta \ell}{kL_0}, & F > kL_0. \end{cases}$$

A.2

Increasing the tension up to kL_0 produces no displacement and therefore requires no mechanical work. During subsequent stretching, at instantaneous segment length s ,

$$F(s) = \frac{kL_0}{\Delta \ell}s.$$

The work required for quasistatic stretching is

$$\Delta W = \int_{\Delta \ell}^{\Delta y} F(s) ds = \frac{kL_0}{\Delta \ell} \int_{\Delta \ell}^{\Delta y} s ds.$$

Hence

$$\Delta W = \frac{kL_0}{2\Delta \ell} [(\Delta y)^2 - (\Delta \ell)^2].$$

This is also the increase in elastic potential energy.

A.3

The mass per unit unstretched length is

$$\mu = \frac{M}{L_0}.$$

At material coordinate ℓ , the tension supports all the spring below that point:

$$T(\ell) = \mu g \ell = \frac{Mg}{L_0} \ell.$$

Stretching occurs only where $T(\ell) > kL_0$. The boundary between the compact and stretched portions is therefore

$$\frac{Mg}{L_0}\ell_* = kL_0, \quad \boxed{\ell_* = \alpha L_0}.$$

Thus the lowest fraction α of the spring remains unstretched.

Let $y_0(\ell)$ be the equilibrium height above the bottom end. Applying A.1 locally,

$$\frac{dy_0}{d\ell} = \begin{cases} 1, & 0 \leq \ell \leq \alpha L_0, \\ \frac{\ell}{\alpha L_0}, & \alpha L_0 \leq \ell \leq L_0. \end{cases}$$

Integrating with $y_0(0) = 0$ gives

$$y_0(\ell) = \begin{cases} \ell, & 0 \leq \ell \leq \alpha L_0, \\ \frac{\ell^2}{2\alpha L_0} + \frac{\alpha L_0}{2}, & \alpha L_0 \leq \ell \leq L_0. \end{cases}$$

In particular,

$$\boxed{H = y_0(L_0) = \frac{L_0}{2\alpha}(1 + \alpha^2)}.$$

B.1

After release, gravity is the only external force. Consequently, the center of mass falls with acceleration g , even though internal collisions dissipate energy.

The initial center-of-mass height is

$$\begin{aligned} y_{\text{cm},0} &= \frac{1}{L_0} \int_0^{L_0} y_0(\ell) d\ell \\ &= \frac{1}{L_0} \left[\int_0^{\alpha L_0} \ell d\ell + \int_{\alpha L_0}^{L_0} \left(\frac{\ell^2}{2\alpha L_0} + \frac{\alpha L_0}{2} \right) d\ell \right] \\ &= \frac{L_0}{6\alpha} (1 + 3\alpha^2 - \alpha^3). \end{aligned}$$

At the instant of complete collapse, the bottom end has not yet moved. The spring occupies the interval from $y = 0$ to $y = L_0$, so

$$y_{\text{cm},c} = \frac{L_0}{2}.$$

The center of mass has therefore fallen through

$$\begin{aligned} y_{\text{cm},0} - y_{\text{cm},c} &= \frac{L_0}{6\alpha} (1 - 3\alpha + 3\alpha^2 - \alpha^3) \\ &= \frac{L_0}{6\alpha} (1 - \alpha)^3. \end{aligned}$$

Since it starts from rest,

$$\frac{1}{2}gt_c^2 = \frac{L_0}{6\alpha}(1 - \alpha)^3,$$

and hence

$$t_c = \sqrt{\frac{L_0(1-\alpha)^3}{3\alpha g}}.$$

For the supplied data,

$$\alpha = \frac{1.02(0.055)}{0.201(9.80)} = 0.0284800.$$

It is convenient to use the equivalent expression

$$t_c = \sqrt{\frac{M}{3k}}(1-\alpha)^{3/2}.$$

Numerically,

$$t_c = 0.2454228 \dots \text{ s} \approx 0.245 \text{ s}.$$

B.2

Take downward velocities as positive. Introduce

$$x = L_0 - \ell, \quad D = (1 - \alpha)L_0.$$

Here x is the unstretched length already incorporated into the moving compact chunk. Its mass and actual length are

$$m_I = \frac{M}{L_0}x, \quad L_I = x.$$

Because the remaining spring is stationary, the total downward momentum is $m_I v_I$. Integrating the external force Mg from release gives

$$\frac{M}{L_0}xv_I = Mgt,$$

or

$$xv_I = L_0gt.$$

This momentum equation includes the effect of the moving chunk gaining mass.

The lower boundary of the moving chunk is at height $y_0(\ell)$. Its top is consequently at

$$Y_{\text{top}} = y_0(\ell) + L_0 - \ell.$$

While the boundary is in the initially stretched portion,

$$\begin{aligned} v_I &= -\frac{dY_{\text{top}}}{dt} \\ &= -\left(\frac{\ell}{\alpha L_0} - 1\right)\dot{\ell} \\ &= \frac{D - x}{\alpha L_0}\dot{x}. \end{aligned}$$

Combining this with $xv_I = L_0gt$,

$$x(D - x)\dot{x} = \alpha L_0^2gt.$$

Integrating from $x = 0$, $t = 0$,

$$\frac{Dx^2}{2} - \frac{x^3}{3} = \frac{\alpha L_0^2 g t^2}{2}.$$

Therefore

$$t^2 = \frac{x^2}{\alpha L_0^2 g} \left(D - \frac{2x}{3} \right).$$

Substitution into the momentum relation gives

$$\begin{aligned} v_I^2 &= \frac{L_0^2 g^2 t^2}{x^2} \\ &= \frac{g}{\alpha} \left(D - \frac{2x}{3} \right) \\ &= \frac{g}{\alpha} \left[(1 - \alpha)L_0 - \frac{2}{3}(L_0 - \ell) \right] \\ &= \frac{2g}{3\alpha} \ell + \frac{gL_0(1 - 3\alpha)}{3\alpha}. \end{aligned}$$

Thus

$$\boxed{v_I(\ell) = \sqrt{A\ell + B}}, \quad \boxed{A = \frac{2g}{3\alpha}, \quad B = \frac{gL_0(1 - 3\alpha)}{3\alpha}}.$$

The moving boundary reaches the initially compact bottom portion at $\ell = \alpha L_0$. Hence this continuous contraction stage covers

$$\alpha L_0 \leq \ell < L_0.$$

B.3

Since $A > 0$, the moving chunk slows down as ℓ decreases. Just before it reaches the initially compact bottom portion, its speed is

$$v_- = \sqrt{A\alpha L_0 + B}.$$

At this instant, the moving mass is $(1 - \alpha)M$, while the compact bottom mass αM is still stationary. The spring has reached its minimum total length, so these portions join in a perfectly inelastic collision, as required by the stipulated rigid-body motion afterward.

The gravitational impulse during this instantaneous collision is negligible. Conservation of momentum gives

$$(1 - \alpha)Mv_- = Mv_c,$$

and therefore

$$v_c = (1 - \alpha)\sqrt{A\alpha L_0 + B}.$$

This is smaller than v_- . Subsequently the entire spring accelerates downward at g , so its speed increases. Consequently,

$$\boxed{v_{\min} = (1 - \alpha)\sqrt{A\alpha L_0 + B}}.$$

As a check,

$$A\alpha L_0 + B = \frac{gL_0(1 - \alpha)}{3\alpha},$$

so

$$v_{\min} = \sqrt{\frac{gL_0(1-\alpha)^3}{3\alpha}} = gt_c,$$

as required by the total momentum at complete collapse.

The exactly stationary state at release is excluded by "after its release." In the prescribed continuum model, the moving chunk has vanishing mass as $t \rightarrow 0^+$, allowing a finite limiting speed while total momentum tends to zero.

C.1

Choose zero elastic potential energy for the spring at its minimum length. Applying A.2 to an infinitesimal segment gives

$$dU_{\text{el}} = \frac{kL_0}{2} \left[\left(\frac{dy_0}{d\ell} \right)^2 - 1 \right] d\ell.$$

The initially compact bottom portion contributes zero. Thus

$$\begin{aligned} U_{\text{el},0} &= \frac{kL_0}{2} \int_{\alpha L_0}^{L_0} \left[\frac{\ell^2}{\alpha^2 L_0^2} - 1 \right] d\ell \\ &= \frac{kL_0^2}{6\alpha^2} (1 - 3\alpha^2 + 2\alpha^3). \end{aligned}$$

Using $kL_0 = \alpha Mg$,

$$U_{\text{el},0} = \frac{MgL_0}{6\alpha} (1 - 3\alpha^2 + 2\alpha^3).$$

Immediately after complete collapse, the elastic energy is zero and the entire spring translates with speed $v_c = gt_c$. The dissipated energy up to this moment is

$$Q = U_{\text{el},0} + Mg(y_{\text{cm},0} - y_{\text{cm},c}) - \frac{1}{2} Mv_c^2.$$

However, free fall of the center of mass gives

$$Mg(y_{\text{cm},0} - y_{\text{cm},c}) = \frac{1}{2} Mg^2 t_c^2 = \frac{1}{2} Mv_c^2.$$

These two terms cancel, leaving

$$Q = U_{\text{el},0}.$$

After collapse, the rigid spring falls without air resistance or tumbling, so no further mechanical energy is lost before impact. Factoring the result,

$$Q = \frac{MgL_0}{6\alpha} (1 - 3\alpha^2 + 2\alpha^3) = \frac{MgL_0}{6\alpha} (1 - \alpha)^2 (1 + 2\alpha).$$

The solution was checked against the official solution.

Problem 2: The Physics of a Microwave Oven

This question discusses the generation of microwave radiation in a microwave oven, and its use to heat up food. The microwave radiation is generated in a device called “magnetron”. Part A concerns the operation of the magnetron, while part B deals with the absorption of microwave radiation in food.

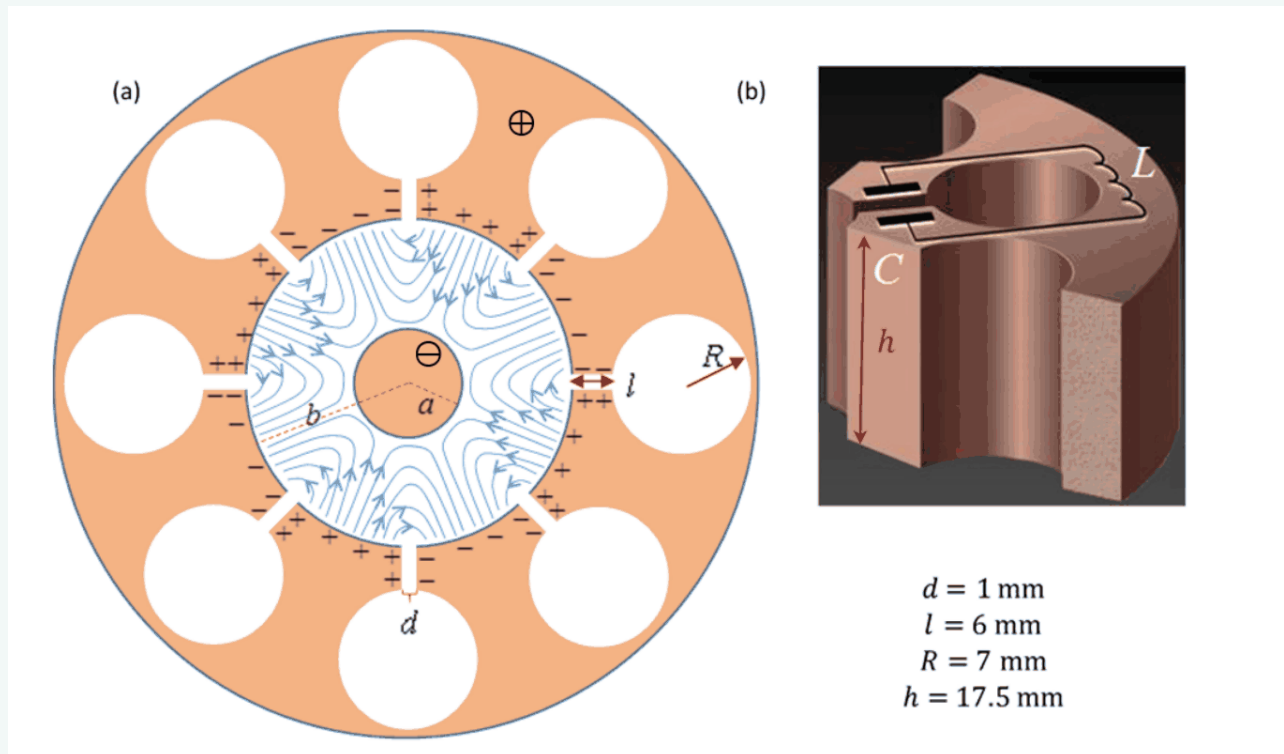


Figure 1

Part A: The structure and operation of a magnetron (6.6 points)

A magnetron is a device for the generation of microwave radiation, either in pulses (for radar applications), or continuously (e.g., in a microwave oven). The magnetron has a mode of self-amplifying oscillations. Supplying the magnetron with static (non-alternating) voltage quickly excites this mode. The microwave radiation thus created is transmitted out of the magnetron.

A typical microwave oven magnetron consists of a solid copper cylindrical cathode (with radius a) and a surrounding anode (with radius b). The latter has the shape of a thick cylindrical shell into which cylindrical cavities are drilled. These cavities are known as “resonators”. One of the resonators is coupled to an antenna which will transmit the microwave energy out; we will ignore the antenna in the following. All internal spaces are in vacuum. We will consider a typical magnetron with eight resonators, as depicted in Figure 1(a). The three-dimensional structure of a single resonator is shown in Figure 1(b). As indicated there, each of the eight cavities behaves as an inductor-capacitor (LC) resonator, with operating frequency $f = 2.45 \text{ GHz}$.

A static uniform magnetic field is applied along the magnetron's longitudinal axis, pointing out of the page in Figure 1(a). In addition, a constant voltage is applied between the anode (positive potential) and the cathode (negative potential). Electrons emitted from the cathode reach the

anode and charge it, such that they excite an oscillation mode in which the sign of the charge is opposite between every two adjacent resonators. The oscillation of the cavities amplify these oscillations.

The process described above creates an alternating electric field with the aforementioned frequency $f = 2.45 \text{ GHz}$ (blue lines in Figure 1(a); the static field is not plotted) in the space between the cathode and the anode, in addition to the static field caused by the applied constant voltage. In the steady state, the typical amplitude of the alternating electric field between the anode and the cathode is approximately $\frac{1}{3}$ of the static electric field there. The electron motion in the space between the cathode and the anode is affected by both the static and the alternating parts of the field. This causes electrons that reach the anode to transfer about 80% of the energy they acquire from the static field into the alternating field. A minority of the ejected electrons returns to the cathode and releases additional electrons, further amplifying the alternating field.

Each resonator can be thought of as a capacitor and an inductor, see Figure 1(b). The capacitance mainly arises from the planar parts of the resonator surface, while the inductance stems from the cylindrical part. Assume that the current in the resonator flows uniformly very close to the surface of its cylindrical cavity, and that the strength of the magnetic field generated by this current is 0.6 times that of an ideal infinite solenoid. The various lengths defining the resonator geometry are given in Figure 1(b). The vacuum permittivity and permeability are $\epsilon_0 = 8.85 \cdot 10^{-12} \text{ F/m}$ and $\mu_0 = 4\pi \cdot 10^{-7} \text{ H/m}$, respectively.

A.1 — 0.4pt

Use the above data to estimate the frequency f_{est} of a single resonator. (Your result may differ from the actual value, $f = 2.45 \text{ GHz}$. Use the **actual** value in the remainder of the question.)

Task A.2 below does not deal with the magnetron itself, but helps to introduce some of the relevant physics. Consider an electron moving in free space under the influence of a uniform electric field directed along the negative y axis, $\vec{E} = -E_0\hat{y}$, and a uniform magnetic field directed along the positive z axis, $\vec{B} = B_0\hat{z}$ (E_0 and B_0 are positive; $\hat{x}, \hat{y}, \hat{z}$ are unit vectors oriented in the conventional manner). Let us denote the electron velocity at time t by $\vec{u}(t)$. The drift velocity $\vec{u}_D$ of the electron is defined as its average velocity. We denote by m and $-e$ the mass and charge of the electron, respectively.

A.2 — 1.5pt

In each of the following two cases, find $\vec{u}_D$. In addition, draw in the Answer Sheet the electron's trajectory (in the lab frame) during the time interval $0 < t < \frac{4\pi m}{eB_0}$ if:

1. at $t = 0$ the electron velocity is $\vec{u}(0) = (3E_0/B_0)\hat{x}$,
2. at $t = 0$ the electron velocity is $\vec{u}(0) = -(3E_0/B_0)\hat{x}$.

We now resume our discussion of the magnetron. The distance between the cathode and the anode is 15 mm. Assume that, due to the aforementioned energy loss to the alternating fields, the maximal kinetic energy of each electron does not exceed $K_{\text{max}} = 800 \text{ eV}$. The static magnetic field strength is $B_0 = 0.3 \text{ T}$. The electron mass and charge are $m = 9.1 \cdot 10^{-31} \text{ kg}$ and $-e = -1.6 \cdot 10^{-19} \text{ C}$, respectively.

A.3 — 0.4pt

Numerically estimate the maximal radius r of the electron motion trajectory in the reference frame in which this motion is approximately circular, considering this reference frame as approximately inertial.

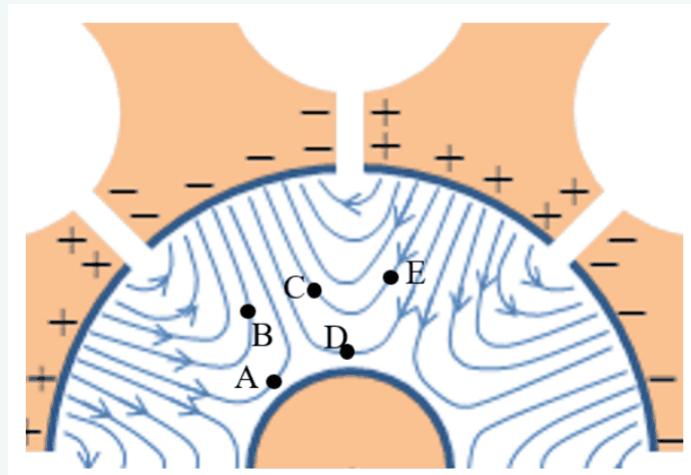


Figure 2

A.4 — 1.2pt

Figure 2 depicts the alternating electric field lines between the anode and the cathode at a given moment in time (the static field is not plotted). Indicate in the Answer Sheet which of the electrons positioned at A,B,C,D and E will drift towards the anode, which will drift towards the cathode and which will drift at a direction perpendicular to the radius at that moment.

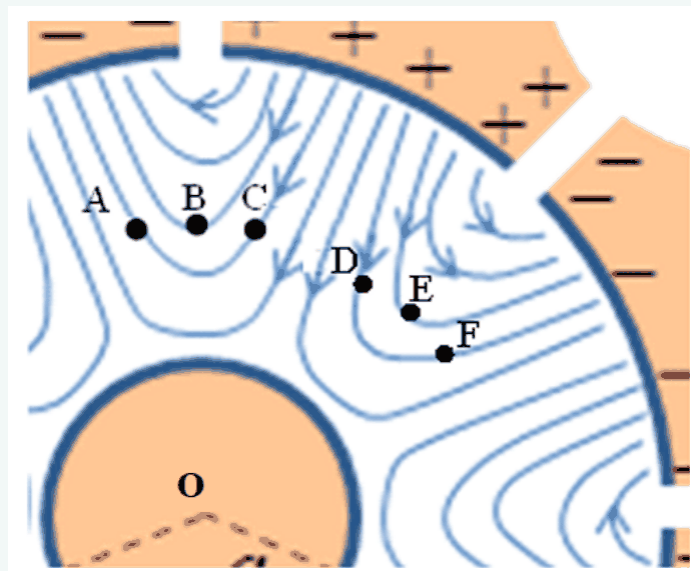


Figure 3

Figure 3 depicts the alternating electric field lines between the anode and the cathode (the static field is not plotted) at a given moment in time. The positions of six electrons at that moment are denoted by A, B, C, D, E and F. All electrons are at the same distance from the cathode.

A.5 — 1.2pt

Consider the situation shown in Figure 3. For each of the six electron pairs AB, AC, BC, DE, DF, EF, indicate in the Answer Sheet whether their drift will cause the angle between their position vectors (measured from the cathode's center O) to increase or decrease at that moment.

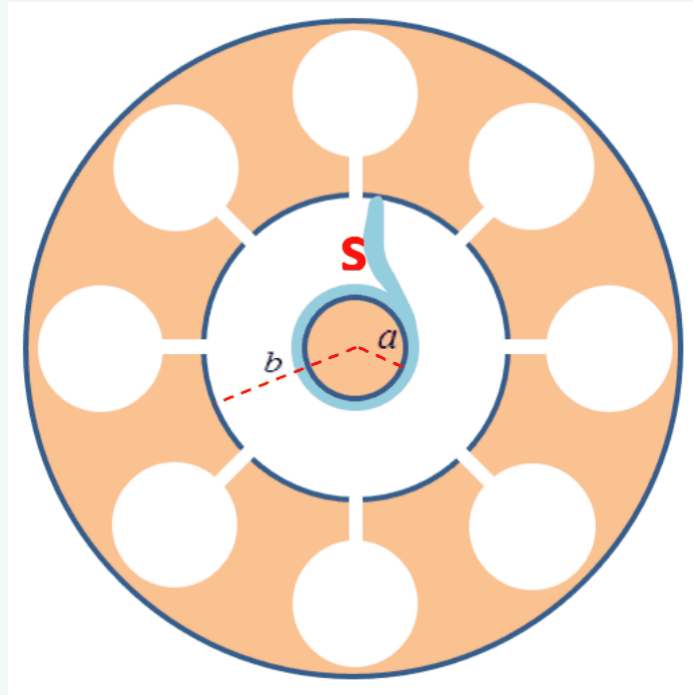


Figure 4

The pattern you have discovered in Task A.5 acts as a focusing mechanism, concentrating the electrons in the space between the cathode and anode into spokes. Figure 4 depicts one such spoke, denoted by S.

A.6 — 0.8pt

Depict in the Answer Sheet the other spokes at that moment. Indicate by arrows their direction of rotation, and calculate their average angular velocity ω_s .

Make the approximation that the total electric field half-way between the cathode and the anode is equal to its average static value along a radial line from the cathode to the anode, and that the spokes are approximately radial in that region. The cathode and anode radii (a and b , respectively) are defined in Figure 4.

A.7 — 1.1pt

Find an approximate expression for the static voltage V_0 required for operating the magnetron in the manner described. (The expression you will find gives an approximation for the minimal value required for the magnetron operation; the optimal voltage is somewhat higher.)

Part B: The interaction of microwave radiation with water molecules (3.4 points)

This part deals with the usage of microwave radiation (radiated by the magnetron antenna into the food chamber) for cooking, that is, heating up a lossy dielectric material such as water, either pure or salty (which is our model for, say, soup).

An electric dipole is a configuration of two equal and opposite electric charges q and $-q$ a small distance d apart. The electric dipole vector points from the negative to the positive charge, and its magnitude is $p = qd$.

A time-dependent electric field $\vec{E}(t) = E(t)\hat{x}$ is applied on a single dipole of moment $\vec{p}(t)$ with constant magnitude $p_0 = |\vec{p}(t)|$. The angle between the dipole and the electric field is $\theta(t)$.

B.1 — 0.5pt

Write expressions for both the magnitude of the torque $\tau(t)$ applied by the electric field on the dipole and the power $H_i(t)$ delivered by the field to the dipole, in terms of p_0 , $E(t)$, $\theta(t)$ and their derivatives.

Water molecules are polar, hence can be treated as electric dipoles. Due to the strong hydrogen bonds between water molecules in liquid water, one cannot treat them as independent dipoles. Rather, one should refer to the polarization vector $\vec{P}(t)$, which is the dipole moment density (average dipole moment per unit volume of an ensemble of water molecules). The polarization $\vec{P}(t)$ is parallel to the local applied alternating electric field (of the microwave radiation), $\vec{E}(t)$, and oscillates in time with an amplitude that is proportional to the amplitude of the local alternating electric field, but with a phase lag δ .

The local alternating electric field at a given location inside the water is $\vec{E}(t) = E_0 \sin(\omega t)\hat{x}$, where $\omega = 2\pi f$, giving rise to polarization $\vec{P}(t) = \beta\epsilon_0 E_0 \sin(\omega t - \delta)\hat{x}$, where the dimensionless constant β is a property of water.

B.2 — 0.5pt

Find an expression for the time-averaged power $\langle H(t) \rangle$ per unit volume absorbed by the water.

The time-average for a time dependent periodic variable $f(t)$ over its period T is defined as:

$$\langle f(t) \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt. \quad (1)$$

Let us now consider the propagation of the radiation through the water. The relative dielectric constant of water (at the electromagnetic field frequency) is ϵ_r , and the corresponding index of refraction of water is $n = \sqrt{\epsilon_r}$. The momentary energy density of the electric field is given by $\frac{1}{2}\epsilon_r\epsilon_0 E^2$. The time-averaged energy density of the electric and magnetic fields are equal.

B.3 — 1.1pt

Let us denote the time-averaged radiation energy flux density by $I(z)$ (average radiation power flow per unit area). Here z is the depth of penetration into the water, and the radiation propagates in the z direction. Find an expression for the dependence of the flux density $I(z)$ on z . The flux density at the water surface, $I(0)$, may appear in your result.

The phase lag δ is the result of the interaction between the water molecules. It depends on the dimensionless dielectric loss coefficient ϵ_ℓ and the relative dielectric constant ϵ_r (both of which depend on the radiation angular frequency ω and the temperature) via the relation $\tan \delta = \epsilon_\ell / \epsilon_r$. When δ is small enough, the electric field at penetration depth z into the water is given by:

$$\vec{E}(z, t) = \vec{E}_0 e^{-\frac{1}{2}nk_0 z \tan \delta} \sin(nk_0 z - \omega t) \quad (2)$$

where $k_0 = \omega/c$ and $c = 3.0 \cdot 10^8$ m/s is the speed of light in vacuum.

B.4 — 0.6pt

Employ the approximation $\tan \delta \approx \sin \delta$ and find an expression for the coefficient β defined in Task B.2 in terms of the other parameters.

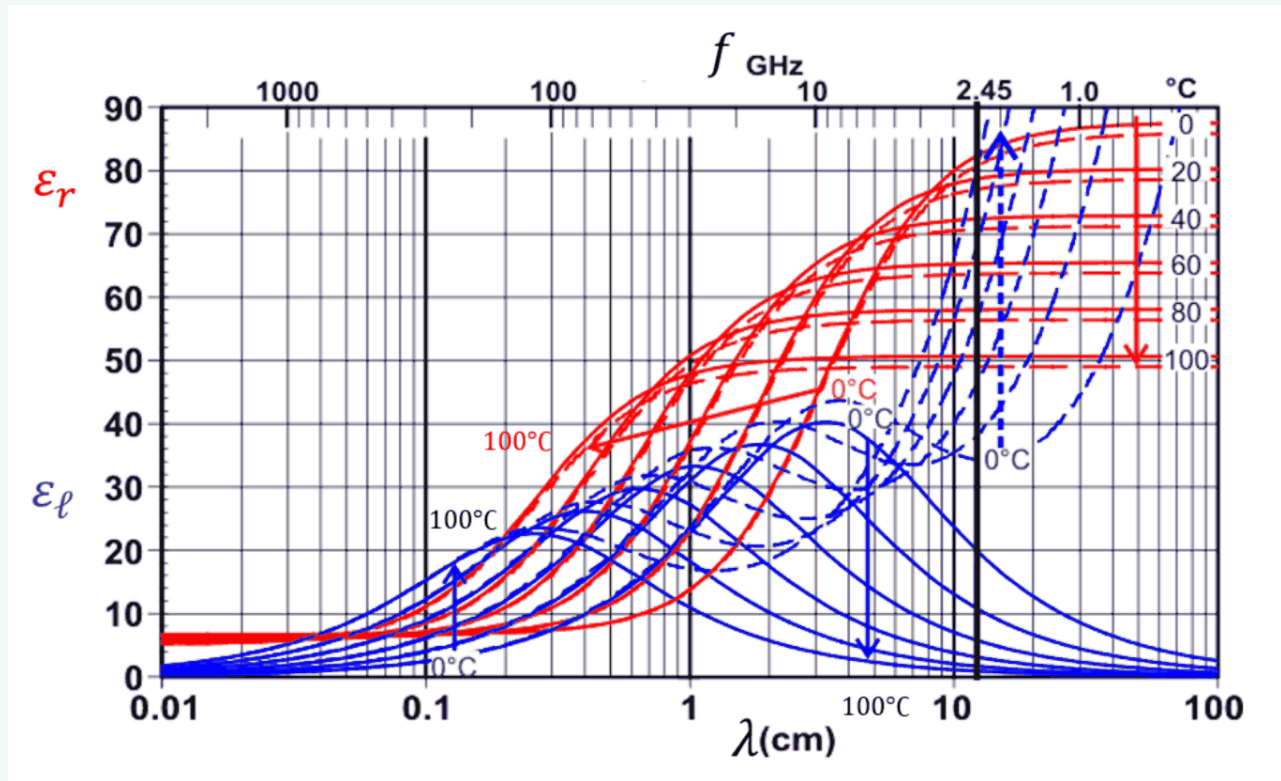


Figure 5. The arrows indicate the variation with temperature across the curves from 0°C to 100°C .

Figure 5 depicts ϵ_l (blue) and ϵ_r (red) for both pure water (solid lines) and a dilute solution of salt in water (dashed lines) as functions of wavelength or frequency, at several different temperatures. The angular frequency $\omega = 2\pi \cdot 2.45 \cdot 10^9 \text{ s}^{-1}$ is indicated by a bold vertical line. Below we will consider microwave radiation at this frequency only.

B.5 — 0.7pt

Use Figure 5 to address the following questions:

1. For water at 20°C , find the penetration depth $z_{1/2}$ at which the power per unit volume is reduced to half of its value at $z = 0$.
2. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into water increases, decreases or remains the same with temperature.
3. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into soup (dilute salt solution) increases, decreases or remains the same with temperature.

Official English text of the problem.

Answers

- **A.1:** $C \simeq 0.93 \text{ pF}$, $L \simeq 6.6 \text{ nH}$, giving

$$f_{\text{est}} \simeq 2.03 \text{ GHz.}$$

- **A.2:** In both cases,

$$\vec{u}_D = -\frac{E_0}{B_0} \hat{x}.$$

Put $\omega_c = eB_0/m$, $\ell = mE_0/(eB_0^2)$, and $s = \omega_c t$. With the initial position at the origin, the requested trajectories are

$$\frac{(x, y)_1}{\ell} = (-s + 4 \sin s, 4[1 - \cos s]),$$

$$\frac{(x, y)_2}{\ell} = (-s - 2 \sin s, -2[1 - \cos s]), \quad 0 < s < 4\pi.$$

They are looping trochoids drifting left, with counterclockwise gyration.

- **A.3:**

$$r_{\text{max}} \simeq 3.2 \times 10^{-4} \text{ m} = 0.32 \text{ mm.}$$

- **A.4:** My reading of the field-line tangents is **A and B toward the cathode; C, D and E toward the anode; none exactly perpendicular to the radius.** The diagram-reading uncertainty is noted below.
- **A.5:** **AB, AC and BC decrease; DE, DF and EF increase.**
- **A.6:** There are **four spokes**, separated by 90° . Add copies of S rotated by 90° , 180° , and 270° . They rotate **counterclockwise**, with

$$\omega_s = \frac{2\pi f}{4} = 3.85 \times 10^9 \text{ rad s}^{-1}.$$

- **A.7:**

$$V_0 \simeq \frac{B_0 \omega_s}{2} (b^2 - a^2) = \frac{\pi f B_0}{4} (b^2 - a^2).$$

- **B.1:** For field magnitude $|E|$ and the angle between the dipole and the instantaneous field,

$$\tau = p_0 |E| \sin \theta, \quad H_i = -p_0 |E| \sin \theta \dot{\theta}.$$

The convention-independent power formula is $H_i = \vec{E} \cdot \dot{\vec{p}}$.

- **B.2:**

$$\langle H \rangle = \frac{1}{2} \beta \epsilon_0 \omega E_0^2 \sin \delta.$$

• B.3:

$$I(z) = I(0) \exp \left[-\frac{\beta \omega \sin \delta}{nc} z \right].$$

• B.4:

$$\beta \simeq n^2 = \epsilon_r.$$

• B.5:

1.

$$z_{1/2} = \frac{c\sqrt{\epsilon_r} \ln 2}{\omega \epsilon_\ell} \simeq 1.2 \text{ cm}$$

from approximate graph readings $\epsilon_r \simeq 78$, $\epsilon_\ell \simeq 10$.

2. The penetration depth in pure water **increases** with temperature.

3. The penetration depth in soup **decreases** with temperature.

Solution

A.1

The approximately parallel conducting faces forming the slot have area hl and separation d . Neglecting fringing, their capacitance is

$$C \simeq \frac{\epsilon_0 hl}{d}.$$

Using the dimensions in Figure 1,

$$C = \frac{(8.85 \times 10^{-12})(17.5 \times 10^{-3})(6 \times 10^{-3})}{1 \times 10^{-3}} = 9.29 \times 10^{-13} \text{ F}.$$

Let I be the total current circulating around the cylindrical cavity. Its uniform distribution over height h gives an effective surface current density I/h . An ideal infinite solenoid with this surface current would produce $\mu_0 I/h$. Applying the specified correction,

$$B \simeq 0.6 \frac{\mu_0 I}{h}.$$

Approximating the flux through the cavity by $B\pi R^2$,

$$L = \frac{\Phi}{I} \simeq 0.6 \frac{\mu_0 \pi R^2}{h} = 6.63 \times 10^{-9} \text{ H}.$$

Thus

$$f_{\text{est}} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi} \sqrt{\frac{d}{0.6\mu_0\epsilon_0\pi R^2 l}} \simeq 2.03 \times 10^9 \text{ Hz}.$$

The height cancels because $C \propto h$ and $L \propto h^{-1}$. The estimate is approximate because the resonator is represented by idealized lumped capacitance and inductance.

A.2

The electron obeys the Lorentz-force equation,

$$m \frac{d\vec{u}}{dt} = -e(\vec{E} + \vec{u} \times \vec{B}).$$

For the stated fields, its components are

$$\dot{u}_x = -\omega_c u_y, \quad \dot{u}_y = \omega_c \left(u_x + \frac{E_0}{B_0} \right), \quad \omega_c = \frac{eB_0}{m}.$$

Define $v_D = E_0/B_0$. A constant velocity $-v_D \hat{x}$ makes the electric and magnetic forces cancel. More generally,

$$\vec{u}_D = \frac{\vec{E} \times \vec{B}}{B_0^2} = -v_D \hat{x}.$$

In the frame translating with this velocity, the remaining velocity rotates uniformly. For $u_y(0) = 0$,

$$u_x = -v_D + A \cos(\omega_c t), \quad u_y = A \sin(\omega_c t),$$

where $A = u_x(0) + v_D$. The oscillatory terms average to zero, so both initial conditions give the same drift.

Taking $x(0) = y(0) = 0$, integration gives

$$x = -v_D t + \frac{A}{\omega_c} \sin(\omega_c t), \quad y = \frac{A}{\omega_c} [1 - \cos(\omega_c t)].$$

For case 1, $A = 4v_D$; for case 2, $A = -2v_D$. This gives the parametric trajectories stated in the answers.

For a precise sketch, use $\ell = v_D/\omega_c$. The following points describe the first cyclotron period:

$s = \omega_c t$	Case 1: $(x/\ell, y/\ell)$	Case 2: $(x/\ell, y/\ell)$
0	(0, 0)	(0, 0)
$\pi/2$	$(4 - \pi/2, 4)$	$(-2 - \pi/2, -2)$
π	$(-\pi, 8)$	$(-\pi, -4)$
$3\pi/2$	$(-4 - 3\pi/2, 4)$	$(2 - 3\pi/2, -2)$
2π	$(-2\pi, 0)$	$(-2\pi, 0)$

Repeat this curve translated by $-2\pi\ell$ in x to obtain the second period.

In case 1, the electron initially moves right and bends upward; the entire curve lies in $0 \leq y \leq 8\ell$. In case 2, it initially moves left and bends downward; the curve lies in $-4\ell \leq y \leq 0$. Both end at $(-4\pi\ell, 0)$ after the specified interval. Their gyration is counterclockwise, and neither trajectory has cusps because its velocity never vanishes.

A.3

In the approximately inertial local drift frame, the electric force is approximately cancelled by the magnetic force associated with the frame velocity. The remaining motion is approximately circular:

$$\frac{mv_\perp^2}{r} = ev_\perp B_0, \quad r = \frac{mv_\perp}{eB_0}.$$

Using the full kinetic-energy bound as an upper estimate for the gyration energy,

$$v_\perp \lesssim \sqrt{\frac{2K_{\max}}{m}}.$$

Here

$$K_{\max} = 800(1.6 \times 10^{-19}) = 1.28 \times 10^{-16} \text{ J},$$

so

$$v_{\max} \simeq 1.68 \times 10^7 \text{ m s}^{-1},$$

and

$$r_{\max} \simeq \frac{\sqrt{2mK_{\max}}}{eB_0} = \frac{\sqrt{2(9.1 \times 10^{-31})(1.28 \times 10^{-16})}}{(1.6 \times 10^{-19})(0.3)} = 3.18 \times 10^{-4} \text{ m}.$$

Thus $r_{\max} \simeq 0.32 \text{ mm}$, much smaller than the 15 mm gap. This supports treating the gyration as a small motion superposed on the drift of its centre.

A.4

Let $\hat{r}$ point radially outward and $\hat{\phi}$ counterclockwise. With the magnetic field pointing out of the page,

$$\vec{E} = E_r \hat{r} + E_\phi \hat{\phi}$$

gives

$$\vec{u}_D = \frac{\vec{E} \times \vec{B}}{B_0^2} = \frac{E_\phi}{B_0} \hat{r} - \frac{E_r}{B_0} \hat{\phi}.$$

Therefore:

- $E_\phi > 0$ produces drift toward the anode.
- $E_\phi < 0$ produces drift toward the cathode.
- $E_\phi = 0$ produces drift perpendicular to the radius.

The static electric field is radial, so it contributes only azimuthal drift. The radial classification is determined entirely by the tangential component of the plotted alternating field.

My reading of the local tangents gives:

Electron	Tangential alternating-field component	Radial drift
A	Clockwise	Toward cathode
B	Clockwise	Toward cathode
C	Counterclockwise	Toward anode
D	Counterclockwise	Toward anode
E	Counterclockwise	Toward anode

These classifications concern the radial component; the electrons also have azimuthal drift. The tangents are read from a schematic figure, so these classifications carry some reading uncertainty.

A.5

At a common distance ρ from the cathode centre,

$$\dot{\phi} = \frac{u_\phi}{\rho} = -\frac{E_r}{B_0 \rho}.$$

The static field is radially inward and gives all six electrons the same counterclockwise angular velocity at that radius. Thus only differences in the radial component of the alternating field affect their instantaneous angular separations.

For A, B and C, Figure 3 shows the radial alternating-field component changing from outward at A, through approximately zero at B, to inward at C. Consequently,

$$\dot{\phi}_A < \dot{\phi}_B < \dot{\phi}_C.$$

Their positions are ordered $\phi_A > \phi_B > \phi_C$. The electron behind therefore moves faster than the one ahead, so all three separations decrease:

$$\frac{d}{dt}(\phi_A - \phi_B) < 0, \quad \frac{d}{dt}(\phi_A - \phi_C) < 0, \quad \frac{d}{dt}(\phi_B - \phi_C) < 0.$$

For D, E and F, the radial alternating-field component changes in the opposite way: inward at D, approximately zero at E, and outward at F. Hence

$$\dot{\phi}_D > \dot{\phi}_E > \dot{\phi}_F.$$

Since $\phi_D > \phi_E > \phi_F$, all three separations increase.

Pair	Change of angular separation
AB	Decreases
AC	Decreases
BC	Decreases
DE	Increases
DF	Increases
EF	Increases

A.6

Adjacent resonators have opposite oscillation phases. Thus, at a given instant, alternate angular regions focus electrons while the intervening regions spread them apart. Eight resonators therefore produce four spokes.

The drawing should contain three additional copies of S, obtained by rotations of 90° , 180° , and 270° about the cathode centre. Their shapes should have the same sense of curvature.

The static field points inward. Its drift is

$$(-\hat{r}) \times \hat{z} = +\hat{\phi},$$

so the spokes rotate counterclockwise.

After half an oscillation period, the resonator polarities reverse. A spoke must advance to the next resonator position, an angle $2\pi/8 = \pi/4$. Therefore

$$\omega_s = \frac{\pi/4}{T/2} = \frac{\pi}{2T} = \frac{2\pi f}{4}.$$

Using $f = 2.45 \times 10^9$ Hz,

$$\boxed{\omega_s = 3.85 \times 10^9 \text{ rad s}^{-1}.}$$

Equivalently, the spokes make 6.125×10^8 revolutions per second.

A.7

Under the prescribed approximation, the field magnitude at the midpoint of the gap is

$$E_{\text{mid}} \simeq \frac{V_0}{b-a}.$$

The midpoint radius is

$$\rho_{\text{mid}} = \frac{a+b}{2}.$$

The electron drift there must have the tangential speed of the spoke:

$$\frac{E_{\text{mid}}}{B_0} \simeq \omega_s \rho_{\text{mid}}.$$

Consequently,

$$\frac{V_0}{B_0(b-a)} \simeq \omega_s \frac{a+b}{2},$$

which yields

$$V_0 \simeq \frac{B_0 \omega_s}{2} (b^2 - a^2) = \frac{\pi f B_0}{4} (b^2 - a^2).$$

With the given B_0 and f , this can also be written

$$V_0 \simeq (5.77 \times 10^8 \text{ V m}^{-2})(b^2 - a^2),$$

with a and b measured in metres.

B.1

The forces on the two charges form a couple. The torque is

$$\vec{\tau} = \vec{p} \times \vec{E}.$$

Thus, for θ defined as the angle between the dipole and the instantaneous field,

$$\tau = p_0 |E| \sin \theta.$$

The torque tends to reduce this angle. Its signed component conjugate to θ is therefore $-p_0 |E| \sin \theta$, giving the mechanical power

$$H_i = -p_0 |E| \sin \theta \dot{\theta}.$$

In particular, an aligning dipole has $\dot{\theta} < 0$, so the field delivers positive power.

The result can be derived without an angular convention. If $\vec{r}_+$ and $\vec{r}_-$ are the charge positions,

$$H_i = q \vec{E} \cdot \dot{\vec{r}}_+ - q \vec{E} \cdot \dot{\vec{r}}_- = \vec{E} \cdot \frac{d}{dt} [q(\vec{r}_+ - \vec{r}_-)].$$

Hence

$$H_i = \vec{E} \cdot \dot{\vec{p}}.$$

If instead θ is measured from the fixed positive x axis and $E(t)$ is a signed component, this becomes

$$H_i = E(t) \frac{d}{dt} (p_0 \cos \theta) = -p_0 E(t) \sin \theta \dot{\theta}.$$

There is no $\dot{\vec{E}}$ term in the mechanical power delivered to the dipole.

B.2

Summing the single-dipole power over unit volume gives

$$H(t) = \vec{E}(t) \cdot \frac{d\vec{P}}{dt}.$$

The given polarization has derivative

$$\frac{d\vec{P}}{dt} = \beta\epsilon_0 E_0 \omega \cos(\omega t - \delta) \hat{x}.$$

Therefore

$$H(t) = \beta\epsilon_0 \omega E_0^2 \sin(\omega t) \cos(\omega t - \delta).$$

Using

$$\cos(\omega t - \delta) = \cos(\omega t) \cos \delta + \sin(\omega t) \sin \delta,$$

and the period averages

$$\langle \sin(\omega t) \cos(\omega t) \rangle = 0, \quad \langle \sin^2(\omega t) \rangle = \frac{1}{2},$$

we obtain

$$\boxed{\langle H \rangle = \frac{1}{2} \beta \epsilon_0 \omega E_0^2 \sin \delta.}$$

For a positive phase lag, the mean absorbed power is positive.

B.3

Let $\mathcal{E}(z)$ denote the local electric-field amplitude. The average electric energy density is

$$\langle u_E \rangle = \frac{1}{2} \epsilon_r \epsilon_0 \langle E^2 \rangle = \frac{1}{4} \epsilon_r \epsilon_0 \mathcal{E}^2.$$

The average magnetic energy density is equal to this, so

$$\langle u \rangle = \frac{1}{2} \epsilon_r \epsilon_0 \mathcal{E}^2.$$

Using the propagation speed c/n in the problem's dielectric-wave model,

$$I(z) = \frac{c}{n} \langle u \rangle = \frac{1}{2} n c \epsilon_0 \mathcal{E}^2(z),$$

where $n^2 = \epsilon_r$.

Conservation of energy for a thin layer of thickness dz gives

$$-\frac{dI}{dz} = \langle H \rangle.$$

Substituting the result of B.2 and eliminating $\mathcal{E}^2$,

$$-\frac{dI}{dz} = \frac{1}{2} \beta \epsilon_0 \omega \mathcal{E}^2 \sin \delta = \frac{\beta \omega \sin \delta}{nc} I.$$

For a homogeneous medium at fixed temperature, the coefficient is constant. Integration gives

$$I(z) = I(0) \exp \left[-\frac{\beta \omega \sin \delta}{nc} z \right].$$

The attenuation coefficient for power is therefore

$$\alpha = \frac{\beta \omega \sin \delta}{nc}.$$

B.4

Equation (2) gives an electric-field amplitude proportional to

$$e^{-\frac{1}{2}nk_0z \tan \delta}.$$

Since intensity is proportional to the square of this amplitude,

$$I(z) = I(0)e^{-nk_0z \tan \delta}.$$

Comparison with B.3, using $k_0 = \omega/c$, gives

$$\frac{\beta \omega \sin \delta}{nc} = \frac{n\omega}{c} \tan \delta.$$

Thus

$$\beta \sin \delta = n^2 \tan \delta.$$

Within the requested approximation $\tan \delta \simeq \sin \delta$,

$$\boxed{\beta \simeq n^2 = \varepsilon_r.}$$

This result follows within the dielectric and small-loss approximations specified in the problem.

B.5

Using the supplied relation $\tan \delta = \varepsilon_\ell / \varepsilon_r$, the power attenuation coefficient is

$$\alpha = nk_0 \tan \delta = \frac{\omega}{c} \frac{\varepsilon_\ell}{\sqrt{\varepsilon_r}}.$$

At uniform temperature, the local absorbed power per unit volume is proportional to the local intensity:

$$\langle H(z) \rangle = \alpha I(z).$$

It therefore has the same exponential dependence,

$$\frac{\langle H(z) \rangle}{\langle H(0) \rangle} = e^{-\alpha z}.$$

1. Half-power depth at 20°C.

The condition $e^{-\alpha z_{1/2}} = 1/2$ gives

$$z_{1/2} = \frac{\ln 2}{\alpha} = \frac{c\sqrt{\varepsilon_r} \ln 2}{\omega \varepsilon_\ell}.$$

At the marked frequency, the solid curves for 20°C give approximately

$$\varepsilon_r \simeq 78, \quad \varepsilon_\ell \simeq 10.$$

Consequently,

$$z_{1/2} \simeq \frac{(3.0 \times 10^8) \sqrt{78} \ln 2}{(2\pi)(2.45 \times 10^9)(10)} = 1.19 \times 10^{-2} \text{ m}.$$

Thus the graph supports a half-power depth of approximately

$$z_{1/2} \simeq 1.2 \text{ cm.}$$

This is the half-power depth; the field-amplitude decay has an exponent smaller by a factor of two.

2. Pure water as temperature increases.

At fixed frequency,

$$z_{1/2} \propto \frac{\sqrt{\epsilon_r}}{\epsilon_\ell}.$$

The solid blue curves show a substantial decrease of ϵ_ℓ with temperature. The decrease of $\sqrt{\epsilon_r}$ is much smaller in relative terms. Therefore the ratio increases:

The penetration depth in pure water increases with temperature.

3. Soup as temperature increases.

At the same frequency, the dashed blue curves show increasing dielectric loss with increasing temperature, while the dashed red curves show decreasing ϵ_r . Both changes reduce $\sqrt{\epsilon_r}/\epsilon_\ell$. Therefore

The penetration depth in soup decreases with temperature.

The solution was checked against the official solution.

Problem 3: Thermoacoustic Engine

A thermoacoustic engine is a device that converts heat into acoustic power, or sound waves - a form of mechanical work. Like many other heat machines, it can be operated in reverse to become a refrigerator, using sound to pump heat from a cold to a hot reservoir. The high operating frequencies reduce heat conduction and eliminate the need for any working chamber confinement. Unlike many other engine types, the thermoacoustic engine has no moving parts except the working fluid itself.

The efficiencies of thermoacoustic machines are typically lower than other engine types, but they have advantages in set up and maintenance costs. This creates opportunities for renewable energy applications, such as solar-thermal power plants and utilization of waste heat. Our analysis will focus on the creation of acoustic energy within the system, ignoring the extraction or conversion for powering external devices.

Part A: Sound wave in a closed tube (3.7 points)

Consider a thermally insulating tube of length L and cross-sectional area S , whose axis lies along the x direction. The two ends of the tube are located at $x = 0$ and $x = L$. The tube is filled with an ideal gas and is sealed on both ends. At equilibrium, the gas has temperature T_0 , pressure p_0 and mass density ρ_0 . Assume that viscosity can be ignored and that the gas motion is only in the x direction. The gas properties are uniform in the perpendicular y and z directions.

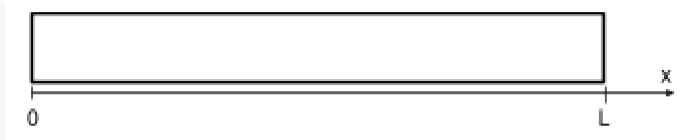


Figure 1

A.1 (0.3pt)

When a standing sound wave forms, the gas elements oscillate in the x direction with angular frequency ω . The amplitude of the oscillations depends on each element's equilibrium position x along the tube. The longitudinal displacement of each gas element from its equilibrium position x is given by

$$u(x, t) = a \sin(kx) \cos(\omega t) = u_1(x) \cos(\omega t) \quad (1)$$

(please note the u here describes the displacement of a gas element)

where $a \ll L$ is a positive constant, $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength. What is the maximum possible wavelength $\lambda_{\max}$ in this system?

We will assume throughout the question an oscillation mode of $\lambda = \lambda_{\max}$.

Now, consider a narrow parcel of gas, located at rest between x and $x + \Delta x$ ($\Delta x \ll L$). As a result of the displacement wave of Task A.1, the parcel oscillates along the x axis and undergoes a change in volume and other thermodynamic properties.

Throughout the following tasks assume all these changes to the thermodynamic properties to be small compared to the unperturbed values.

A.2 (0.5pt)

The parcel volume $V(x, t)$ oscillates around the equilibrium value of $V_0 = S\Delta x$ and has the form

$$V(x, t) = V_0 + V_1(x) \cos(\omega t). \quad (2)$$

Obtain an expression for $V_1(x)$ in terms of V_0 , a , k and x .

A.3 (0.7pt)

Assume that the total pressure of the gas, as a result of the sound wave, takes the approximate form

$$p(x, t) = p_0 - p_1(x) \cos(\omega t). \quad (3)$$

Considering the forces acting on the parcel of gas, compute the amplitude $p_1(x)$ of the pressure oscillation to leading order, in terms of the position x , the equilibrium density ρ_0 , the displacement amplitude a and the wave parameters k and ω .

At acoustic frequencies, the thermal conductivity of the gas can be neglected. We will treat the expansion and contraction of gas parcels as purely adiabatic, satisfying the relation $pV^\gamma = \text{const}$, where γ is the adiabatic constant.

A.4 (0.3pt)

Use the relation above and the results of the previous tasks to obtain an expression for the speed of sound waves $c = \omega/k$ in the tube, to first order. Express your answer in terms of p_0 , ρ_0 and the adiabatic constant γ .

A.5 (0.7pt)

The change in the gas temperature due to the adiabatic expansion and contraction, as a result of the sound wave, takes the form:

$$T(x, t) = T_0 - T_1(x) \cos(\omega t). \quad (4)$$

Compute the amplitude $T_1(x)$ of the temperature oscillations in terms of T_0 , γ , a , k and x .

A.6 (1.2pt)

For the purpose of this task only, we assume a weak thermal interaction between the tube and the gas. As a result, the standing sound wave remains almost unchanged, but the gas can exchange a small amount of heat with the tube. The heating due to viscosity can be neglected.

For each of the points in Figure 2 (A, C at the edges of the tube, B at the center) state whether the temperature of the tube at that point will increase, decrease or remain the same over a long time.

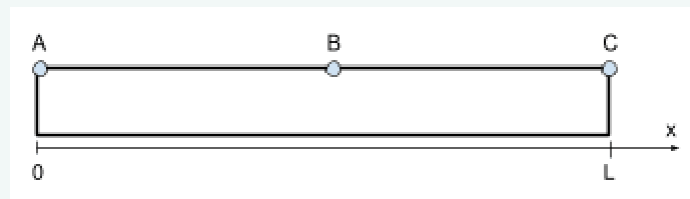


Figure 2

Part B: Sound wave amplification induced by external thermal contact (6.3 points)

A stack of thin well-spaced solid plates is placed inside the tube. The plates of the stack are aligned in parallel to the tube axis, so as not to obstruct the flow of gas along the tube. The center of the stack is positioned at $x_0 = L/4$, and spans a width of $\ell \ll L$ along the tube axis, filling its entire cross section. The right and left edges of the stack are held at temperature difference τ . The left edge of the stack, at $x_H = x_0 - \ell/2$, is held by an external thermal reservoir at temperature $T_H = T_0 + \tau/2$, and at the same time, its right edge, at $x_C = x_0 + \ell/2$, is held at a temperature $T_C = T_0 - \tau/2$.

The plate stack allows a slight longitudinal heat flow to maintain a constant temperature gradient between its edges, such that $T_{\text{plate}}(x) = T_0 - \frac{x-x_0}{\ell} \tau$.

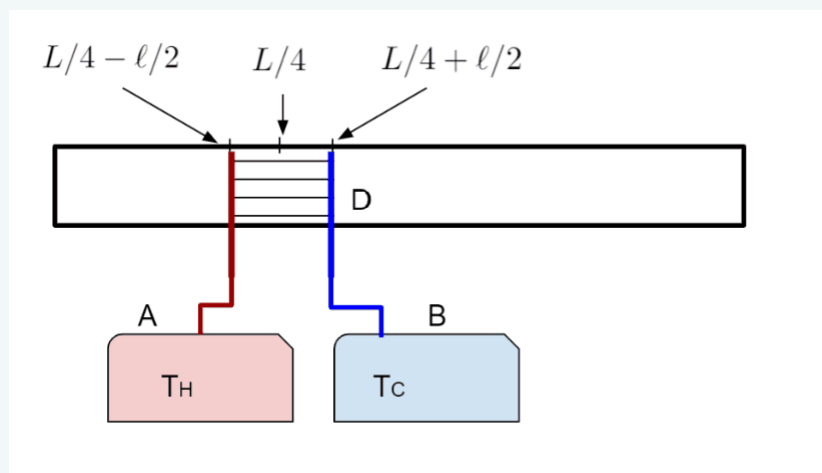


Figure 3. A sketch of the system. (A) and (B) denote the hot and cold heat reservoirs respectively. (D) denotes the stack.

To analyze the effect of the thermal contact between the plate stack and the gas on the sound waves in the tube, make the following assumptions:

- As in the previous part, all changes to the thermodynamic properties are small compared to the unperturbed values.
- The system operates in the fundamental standing-wave mode of the longest possible wavelength. It is only slightly modified by the presence of the plate stack.
- The stack is much shorter than the wavelength $\ell \ll \lambda_{\max}$, and can be positioned far enough from both displacement and pressure nodes, so that the displacement $u(x, t) \approx u(x_0, t)$ and the pressure $p(x, t) \approx p(x_0, t)$ may be considered uniform over the entire length of the stack.
- We may neglect any edge effects, caused by the parcels moving in and out of the stack.
- The temperature difference between the ends of the plate stack, i.e. between the hot and the cold reservoirs, is small compared to the absolute temperature: $\tau \ll T_0$.
- Heat conduction through the stack, through the gas, and along the tube are all negligible. The only significant sources of heat transfer are convection due to the motion of the gas and conduction between the gas and the stack.

B.1 (0.4pt)

Consider a specific parcel of gas in the region of the stack, originally at $x_0 = L/4$. As the parcel moves within the stack, the local temperature of the nearby part of the stack changes as follows:

$$T_{\text{env}}(t) = T_0 - T_{\text{st}} \cos(\omega t). \quad (5)$$

Express T_{st} in terms of a , τ and ℓ .

B.2 (1.0pt)

Above which critical temperature difference τ_{cr} will the gas be conveying heat from the hot reservoir to the cold one? Express τ_{cr} in terms of T_0 , γ , k and ℓ .

B.3 (0.8pt)

Obtain the general approximate expression for the heat flow $\frac{dQ}{dt}$ into a small parcel of gas as a linear function of its volume and pressure change rates. Express your answer in terms of the rate of volume change $\frac{dV}{dt}$, the rate of pressure change $\frac{dp}{dt}$, the unperturbed equilibrium values of parcel pressure and volume p_0 , V_0 and the adiabatic index γ . (You may use the expression for the molar heat capacity at constant volume $c_v = \frac{R}{\gamma-1}$, where R is the gas constant.)

The limited heat flow rate between the parcel and the stack causes a phase difference between the pressure and volume oscillations of the parcel. We will see how this generates work.

Let the heat flux into the parcel from the stack be proportional to the temperature difference between the parcel and the neighboring element of the stack, given approximately by $\frac{dQ}{dt} = -\beta V_0 (T_{\text{st}} - T_1) \cos(\omega t)$. Here T_1 and T_{st} are the temperature oscillation amplitudes of the gas parcel and the neighbouring stack from Tasks A.5 and B.1, respectively, and $\beta > 0$ is a constant. Assume that at the machine's operating frequencies, the change in gas temperature as a result of this heat flow is insignificant compared to both T_1 and T_{st} .

B.4 (1.9pt)

In order to calculate work, we will consider a change to the volume of the moving parcel as a result of the thermal contact with the stack. Let us write the pressure and the volume of the parcel under the stack's influence in the form:

$$\begin{aligned} p &= p_0 + p_a \sin(\omega t) - p_b \cos(\omega t), \\ V &= V_0 + V_a \sin(\omega t) + V_b \cos(\omega t), \end{aligned} \quad (6)$$

Given p_a and p_b , find the coefficients V_a and V_b . Express your answer in terms of $p_a, p_b, p_0, V_0, \gamma, \tau, \tau_{cr}, \beta, \omega, a$ and ℓ .

B.5 (0.8pt)

Obtain an approximate expression for the acoustic work per unit volume w produced by the gas parcel over one cycle. Integrate over the volume of the stack to obtain the total work W_{tot} generated by the gas over one cycle. Express W_{tot} in terms of $\gamma, \tau, \tau_{cr}, \beta, \omega, a, k$ and S .

B.6 (0.8pt)

Obtain an approximate expression for the heat Q_{tot} transported from the left side of the plane $x = x_0$ to the right, over a cycle. Express your answer in terms of $\tau, \tau_{cr}, \beta, \omega, a, S, \ell$.

(Hint: you may use the formula $j = Q \frac{du}{dt}$ for the heat current due convection.)

B.7 (0.6pt)

Find the efficiency η of the thermoacoustic engine. The efficiency is defined as the ratio of the generated acoustic work to the heat drawn from the hot reservoir. Express your answer in terms of the temperature difference τ between the hot and the cold reservoir, the critical temperature difference τ_{cr} and the Carnot efficiency $\eta_c = 1 - T_C/T_H$.

Official English text of the problem.

Answers

The coefficients V_1, p_1, T_1 are signed coefficients in the specified oscillatory forms. Throughout, $k = \pi/L$.

- A.1

$$\lambda_{\max} = 2L.$$

- A.2

$$V_1(x) = V_0 a k \cos(kx).$$

- A.3

$$p_1(x) = \frac{\rho_0 a \omega^2}{k} \cos(kx).$$

- A.4

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}}.$$

- A.5

$$T_1(x) = (\gamma - 1)T_0ak \cos(kx).$$

- A.6 A: **increases**; B: **decreases**; C: **increases**.

- B.1

$$T_{\text{st}} = \frac{a\tau}{\sqrt{2}\ell}.$$

- B.2

$$\tau_{\text{cr}} = (\gamma - 1)T_0k\ell.$$

Heat is transported from hot to cold when $\tau > \tau_{\text{cr}}$.

- B.3

$$\frac{dQ}{dt} \simeq \frac{\gamma p_0 dV/dt + V_0 dp/dt}{\gamma - 1}.$$

- B.4

$$V_a = -\frac{V_0}{\gamma p_0} \left[p_a + \frac{(\gamma - 1)\beta a(\tau - \tau_{\text{cr}})}{\sqrt{2}\omega\ell} \right], \quad V_b = \frac{V_0 p_b}{\gamma p_0}.$$

- B.5

$$w \simeq \frac{\pi(\gamma - 1)\beta a^2 k}{2\omega\ell} (\tau - \tau_{\text{cr}}),$$

$$W_{\text{tot}} \simeq \frac{\pi(\gamma - 1)\beta a^2 k S}{2\omega} (\tau - \tau_{\text{cr}}).$$

- B.6

$$Q_{\text{tot}} \simeq \frac{\pi\beta a^2 S}{2\omega\ell} (\tau - \tau_{\text{cr}}).$$

- B.7

$$\eta \simeq \frac{\tau_{\text{cr}}}{\tau} \eta_c, \quad \tau > \tau_{\text{cr}}.$$

Solution

We retain first-order perturbations when finding the oscillating thermodynamic quantities. Work and transported heat have their first nonzero cycle averages at second order in the wave amplitude. Thermal coupling is weak, so unperturbed wave amplitudes can be used in expressions already proportional to that coupling.

A.1

The sealed, rigid ends require zero longitudinal displacement:

$$u(0, t) = u(L, t) = 0.$$

The first condition is already satisfied. The second gives

$$\sin(kL) = 0 \implies kL = n\pi, \quad n = 1, 2, \dots$$

The choice $n = 0$ gives no wave. Thus

$$\lambda_n = \frac{2\pi}{k} = \frac{2L}{n}.$$

The longest wavelength corresponds to $n = 1$:

$$\lambda_{\max} = 2L, \quad k = \frac{\pi}{L}.$$

A.2

The parcel's two boundaries move to

$$x + u(x, t), \quad x + \Delta x + u(x + \Delta x, t).$$

Consequently,

$$V(x, t) = S [\Delta x + u(x + \Delta x, t) - u(x, t)].$$

For $k\Delta x \ll 1$,

$$V(x, t) \simeq S\Delta x \left(1 + \frac{\partial u}{\partial x} \right) = V_0 [1 + ak \cos(kx) \cos(\omega t)].$$

Therefore,

$$V_1(x) = V_0 ak \cos(kx).$$

A.3

The pressure forces on the parcel give, to first order,

$$F_x = S[p(x, t) - p(x + \Delta x, t)] \simeq -S\Delta x \frac{\partial p}{\partial x}.$$

The parcel's mass is $\rho_0 S\Delta x$. Newton's second law therefore becomes

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{\partial p}{\partial x}.$$

Substituting the given oscillations,

$$-\rho_0 a \omega^2 \sin(kx) \cos(\omega t) = p'_1(x) \cos(\omega t),$$

so

$$p'_1(x) = -\rho_0 a \omega^2 \sin(kx).$$

Integration yields

$$p_1(x) = \frac{\rho_0 a \omega^2}{k} \cos(kx) + C.$$

There is no spatially uniform acoustic compression: the total volume change is proportional to

$$\int_0^L \frac{\partial u}{\partial x} dx = u(L, t) - u(0, t) = 0.$$

The linear adiabatic pressure–compression relation, made explicit in A.4, then requires the spatially averaged pressure perturbation to vanish. Since $\int_0^L \cos(kx) dx = 0$, this fixes $C = 0$. Hence

$$p_1(x) = \frac{\rho_0 a \omega^2}{k} \cos(kx).$$

A.4

Linearizing $pV^\gamma = p_0 V_0^\gamma$,

$$\frac{\delta p}{p_0} + \gamma \frac{\delta V}{V_0} = 0.$$

Using $\delta p = -p_1 \cos(\omega t)$ and $\delta V = V_1 \cos(\omega t)$,

$$p_1 = \gamma p_0 \frac{V_1}{V_0} = \gamma p_0 a k \cos(kx).$$

Comparison with A.3 gives

$$\frac{\rho_0 a \omega^2}{k} = \gamma p_0 a k,$$

and therefore

$$c = \frac{\omega}{k} = \sqrt{\frac{\gamma p_0}{\rho_0}}.$$

A.5

For a fixed amount of ideal gas,

$$pV = nRT.$$

Its linearized form is

$$\frac{\delta T}{T_0} = \frac{\delta p}{p_0} + \frac{\delta V}{V_0}.$$

Using the adiabatic relation from A.4,

$$\frac{\delta T}{T_0} = -(\gamma - 1) \frac{\delta V}{V_0} = -(\gamma - 1) a k \cos(kx) \cos(\omega t).$$

Comparing with the specified temperature oscillation,

$$T_1(x) = (\gamma - 1) T_0 a k \cos(kx).$$

A.6

Consider a parcel in the left half of the tube, where $\cos(kx) > 0$.

When it is displaced to the right, it expands and cools. It consequently absorbs heat from the tube at a relatively rightward position. When displaced to the left, it is compressed and heated, so it releases heat to the tube at a relatively leftward position. Repetition transfers heat toward the left end.

In the right half, $\cos(kx) < 0$, so the correspondence reverses: a parcel absorbs heat when displaced toward the center and releases heat when displaced toward the right end.

Thus the acoustic motion pumps heat away from the central region toward both ends:

$$A : \text{temperature increases}, \quad B : \text{temperature decreases}, \quad C : \text{temperature increases}.$$

Although the parcel whose equilibrium position is exactly at the center has zero first-order temperature oscillation, neighboring parcels move through the central region and remove heat. The mean thermal effect is second order in amplitude; it is not determined by the first-order temperature at a single equilibrium position.

B.1

At the stack center,

$$kx_0 = \frac{\pi}{L} \frac{L}{4} = \frac{\pi}{4},$$

so the parcel displacement is

$$u(x_0, t) = \frac{a}{\sqrt{2}} \cos(\omega t).$$

Its instantaneous position is $x_0 + u(x_0, t)$. Substitution into the plate temperature profile gives

$$T_{\text{env}}(t) = T_0 - \frac{\tau}{\ell} u(x_0, t) = T_0 - \frac{a\tau}{\sqrt{2}\ell} \cos(\omega t).$$

Therefore,

$$T_{\text{st}} = \frac{a\tau}{\sqrt{2}\ell}.$$

B.2

The adiabatic gas-temperature amplitude at the stack center is

$$T_1(x_0) = \frac{(\gamma - 1)T_0 a k}{\sqrt{2}}.$$

At the leftward extreme of the parcel's motion, the gas temperature is $T_0 + T_1$, while the neighboring stack temperature is $T_0 + T_{\text{st}}$. To absorb heat there, the parcel must satisfy

$$T_{\text{st}} > T_1.$$

At the rightward extreme this same condition makes the stack colder than the gas, so the parcel releases heat there. Thus it conveys heat from hot to cold precisely when

$$\frac{a\tau}{\sqrt{2}\ell} > \frac{(\gamma - 1)T_0 a k}{\sqrt{2}}.$$

The critical temperature difference is therefore

$$\tau_{\text{cr}} = (\gamma - 1)T_0 k \ell.$$

At equality, the gas and neighboring stack have matching temperature oscillations, so the leading thermal exchange vanishes. Below this threshold, the heat transport reverses.

B.3

Take $dQ > 0$ for heat entering the parcel and $p dV > 0$ for work done by the gas. The first law is

$$dQ = dU + p dV.$$

For the ideal gas,

$$U = nc_v T = \frac{nRT}{\gamma - 1} = \frac{pV}{\gamma - 1}.$$

Hence

$$dQ = \frac{V dp + p dV}{\gamma - 1} + p dV = \frac{V dp + \gamma p dV}{\gamma - 1}.$$

To first order, the coefficients multiplying the small rates can be replaced by equilibrium values:

$$\boxed{\frac{dQ}{dt} \simeq \frac{V_0 dp/dt + \gamma p_0 dV/dt}{\gamma - 1}}.$$

B.4

From B.1 and B.2,

$$T_{\text{st}} - T_1 = \frac{a}{\sqrt{2}\ell}(\tau - \tau_{\text{cr}}).$$

For brevity, define

$$\Delta T \equiv \frac{a(\tau - \tau_{\text{cr}})}{\sqrt{2}\ell}.$$

The prescribed heat input is then

$$\dot{Q} = -\beta V_0 \Delta T \cos(\omega t).$$

Differentiating the pressure and volume expressions,

$$\begin{aligned}\dot{p} &= \omega p_a \cos(\omega t) + \omega p_b \sin(\omega t), \\ \dot{V} &= \omega V_a \cos(\omega t) - \omega V_b \sin(\omega t).\end{aligned}$$

Substitution into B.3 gives

$$\dot{Q} = \frac{\omega}{\gamma - 1} [(V_0 p_a + \gamma p_0 V_a) \cos(\omega t) + (V_0 p_b - \gamma p_0 V_b) \sin(\omega t)].$$

Matching the sine coefficients,

$$V_0 p_b - \gamma p_0 V_b = 0,$$

so

$$\boxed{V_b = \frac{V_0 p_b}{\gamma p_0}}.$$

Matching the cosine coefficients,

$$\frac{\omega}{\gamma - 1} (V_0 p_a + \gamma p_0 V_a) = -\beta V_0 \Delta T.$$

Thus

$$\boxed{V_a = -\frac{V_0}{\gamma p_0} \left[p_a + \frac{(\gamma - 1)\beta a(\tau - \tau_{\text{cr}})}{\sqrt{2}\omega\ell} \right]}.$$

The term proportional to β supplies the volume component needed for nonzero cycle work.

B.5

The work done by a parcel over one cycle is

$$W_{\text{parcel}} = \oint p dV.$$

Writing $\theta = \omega t$, the constant-pressure contribution integrates to zero, leaving

$$W_{\text{parcel}} = \int_0^{2\pi} (p_a \sin \theta - p_b \cos \theta)(V_a \cos \theta - V_b \sin \theta) d\theta.$$

Using

$$\int_0^{2\pi} \sin \theta \cos \theta d\theta = 0, \quad \int_0^{2\pi} \sin^2 \theta d\theta = \int_0^{2\pi} \cos^2 \theta d\theta = \pi,$$

we obtain

$$W_{\text{parcel}} = -\pi(p_a V_b + p_b V_a).$$

The terms containing $p_a p_b$ cancel upon substituting B.4:

$$W_{\text{parcel}} = \frac{\pi(\gamma - 1)\beta V_0 p_b}{\gamma p_0 \omega} \Delta T.$$

Therefore the work per unit equilibrium volume is

$$w = \frac{W_{\text{parcel}}}{V_0} = \frac{\pi(\gamma - 1)\beta p_b}{\gamma p_0 \omega} \frac{a(\tau - \tau_{\text{cr}})}{\sqrt{2} \ell}.$$

Since this is already first order in thermal coupling, use the unperturbed pressure coefficient:

$$p_b \simeq p_1(x_0) = \gamma p_0 a k \cos\left(\frac{\pi}{4}\right) = \frac{\gamma p_0 a k}{\sqrt{2}}.$$

Hence

$$w \simeq \frac{\pi(\gamma - 1)\beta a^2 k}{2\omega \ell} (\tau - \tau_{\text{cr}}).$$

The stack is short, so this work density is approximately uniform over its gas volume $S\ell$. Thus

$$W_{\text{tot}} \simeq w S \ell,$$

giving

$$W_{\text{tot}} \simeq \frac{\pi(\gamma - 1)\beta a^2 k S}{2\omega} (\tau - \tau_{\text{cr}}).$$

The work is positive in the engine regime $\tau > \tau_{\text{cr}}$.

B.6

Let $q(t)$ denote the heat acquired from the stack per unit equilibrium gas volume, measured relative to an arbitrary constant. To the retained order,

$$\dot{q} = \frac{\dot{Q}}{V_0} = -\beta \Delta T \cos(\omega t).$$

Integrating,

$$q(t) = -\frac{\beta \Delta T}{\omega} \sin(\omega t) + q_*.$$

The parcel velocity at the stack center is

$$v(t) = \dot{u}(x_0, t) = -\frac{a\omega}{\sqrt{2}} \sin(\omega t).$$

With q expressed per unit volume, the convective heat flux per unit area is

$$j(t) = q(t)v(t).$$

The arbitrary constant contributes no net transport because the velocity averages to zero. The purely adiabatic contribution also gives zero cycle average: it is proportional to $\cos(\omega t)$, whereas velocity is proportional to $\sin(\omega t)$. The net heat transported through the cross section is therefore

$$\begin{aligned} Q_{\text{tot}} &= S \int_0^{2\pi/\omega} q(t)v(t) dt \\ &= \frac{S\beta a \Delta T}{\sqrt{2}} \int_0^{2\pi/\omega} \sin^2(\omega t) dt \\ &= \frac{\pi S\beta a \Delta T}{\sqrt{2} \omega}. \end{aligned}$$

Substituting ΔT ,

$$Q_{\text{tot}} \simeq \frac{\pi\beta a^2 S}{2\omega\ell} (\tau - \tau_{\text{cr}}).$$

The sign is positive for left-to-right transport when $\tau > \tau_{\text{cr}}$.

B.7

To the accuracy retained for the short stack and small temperature difference, the heat drawn from the hot reservoir is approximately the heat transported across the stack center:

$$Q_H \simeq Q_{\text{tot}}.$$

Their difference is of order the acoustic work. Since the efficiency itself is small, accounting for that difference would change the efficiency only at higher order.

Using B.5 and B.6,

$$\eta = \frac{W_{\text{tot}}}{Q_H} \simeq \frac{W_{\text{tot}}}{Q_{\text{tot}}} = (\gamma - 1)k\ell = \frac{\tau_{\text{cr}}}{T_0}.$$

Meanwhile,

$$\eta_c = 1 - \frac{T_0 - \tau/2}{T_0 + \tau/2} = \frac{\tau}{T_0 + \tau/2} \simeq \frac{\tau}{T_0},$$

because $\tau \ll T_0$. Therefore,

$$\eta \simeq \frac{\tau_{\text{cr}}}{\tau} \eta_c.$$

For engine operation, $\tau > \tau_{\text{cr}}$, so the efficiency is below the Carnot efficiency. As the threshold is approached from above, the efficiency approaches the Carnot value at this order, while both work output and heat transport vanish. At the threshold itself their ratio does not define a finite-output engine efficiency.

The solution was checked against the official solution.