

IPhO 2018

Theory problems and solutions · Lisbon, Portugal · 3 theory problems with complete solutions

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Problem 1: LIGO-GW150914

In 2015, the gravitational-wave observatory LIGO detected, for the first time, the passing of gravitational waves (GW) through Earth. This event, named GW150914, was triggered by waves produced by two black holes that were orbiting on quasi-circular orbits. This problem will make you estimate some physical parameters of the system, from the properties of the detected signal.

Part A: Newtonian (conservative) orbits (3.0 points)

A.1 1.0pt

Consider a system of two stars with masses M_1, M_2 , at locations $\vec{r}_1, \vec{r}_2$, respectively, with respect to the center-of-mass of the system, that is,

$$M_1 \vec{r}_1 + M_2 \vec{r}_2 = 0. \quad (1)$$

The stars are isolated from the rest of the Universe and moving at non-relativistic velocities. Using Newton's laws, the acceleration vector of mass M_1 can be expressed as

$$\frac{d^2 \vec{r}_1}{dt^2} = -\alpha \frac{\vec{r}_1}{r_1^n}, \quad (2)$$

where $r_1 = |\vec{r}_1|, r_2 = |\vec{r}_2|$. Find $n \in \mathbb{N}$ and $\alpha = \alpha(G, M_1, M_2)$, where G is Newton's constant [$G \simeq 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$].

A.2 1.0pt

The total energy of the 2-mass system, in circular orbits, can be expressed as:

$$E = A(\mu, \Omega, L) - G \frac{M\mu}{L}, \quad (3)$$

where

$$\mu \equiv \frac{M_1 M_2}{M_1 + M_2}, \quad M \equiv M_1 + M_2, \quad (4)$$

are the reduced mass and total mass of the system, Ω is the angular velocity of each mass and L is the total separation $L = r_1 + r_2$. Obtain the explicit form of the term $A(\mu, \Omega, L)$.

A.3 1.0pt

Equation 3 can be simplified to $E = \beta G \frac{M\mu}{L}$. Determine the number β .

Part B: Introducing relativistic dissipation (7.0 points)

The correct theory of gravity, *General Relativity*, was formulated by Einstein in 1915, and predicts that gravity travels with the speed of light. The messengers carrying information about the interaction are called GWs. GWs are emitted whenever masses are accelerated, making the system of masses lose energy.

Consider a system of two point-like particles, isolated from the rest of the Universe. Einstein proved that for small enough velocities the emitted GWs: 1) have a frequency which is twice as large as the orbital frequency; 2) can be characterized by a luminosity, i.e. emitted power $\mathcal{P}$, which is dominated by Einstein's quadrupole formula,

$$\mathcal{P} = \frac{G}{5c^5} \sum_{i=1}^3 \sum_{j=1}^3 \left(\frac{d^3 Q_{ij}}{dt^3} \right) \left(\frac{d^3 Q_{ij}}{dt^3} \right). \quad (5)$$

Here, c is the velocity of light $c \simeq 3 \times 10^8$ m/s. For a system of 2 pointlike particles orbiting on the $x - y$ plane, Q_{ij} is the following table (i, j label the row/column number)

$$Q_{11} = \sum_{A=1}^2 \frac{M_A}{3} (2x_A^2 - y_A^2),$$
$$Q_{22} = \sum_{A=1}^2 \frac{M_A}{3} (2y_A^2 - x_A^2), \quad (6)$$

$$Q_{33} = - \sum_{A=1}^2 \frac{M_A}{3} (x_A^2 + y_A^2),$$
$$Q_{12} = Q_{21} = \sum_{A=1}^2 M_A x_A y_A, \quad (7)$$

and $Q_{ij} = 0$ for all other possibilities. Here, (x_A, y_A) is the position of mass A in the center-of-mass frame.

B.1 1.0pt

For the circular orbits described in A.2 the components of Q_{ij} can be expressed as a function of time t as:

$$Q_{ii} = \frac{\mu L^2}{2} (a_i + b_i \cos kt), \quad Q_{ij} \stackrel{i \neq j}{=} \frac{\mu L^2}{2} c_{ij} \sin kt. \quad (8)$$

Determine k in terms of Ω and the numerical values of the constants a_i, b_i, c_{ij} .

B.2 1.0pt

Compute the power $\mathcal{P}$ emitted in gravitational waves for that system, and obtain:

$$\mathcal{P} = \xi \frac{G}{c^5} \mu^2 L^4 \Omega^6. \quad (9)$$

What is the number ξ ? [If you could not obtain ξ , use $\xi = 6.4$ in the following.]

B.3 1.0pt

In the absence of GW emission the two masses will orbit on a fixed circular orbit indefinitely. However, the emission of GWs causes the system to lose energy and to slowly evolve towards smaller circular orbits. Obtain that the rate of change $\frac{d\Omega}{dt}$ of the orbital angular velocity takes the form

$$\left(\frac{d\Omega}{dt}\right)^3 = (3\xi)^3 \frac{\Omega^{11}}{c^{15}} (GM_c)^5, \quad (10)$$

where M_c is called the chirp mass. Obtain M_c as a function of M and μ . This mass determines the increase in frequency during the orbital decay. [The name "chirp" is inspired by the high pitch sound (increasing frequency) produced by small birds.]

B.4 2.0pt

Using the information provided above, relate the orbital angular velocity Ω with the GW frequency f_{GW} . Knowing that, for any smooth function $F(t)$ and $a \neq 1$,

$$\frac{dF(t)}{dt} = \chi F(t)^a \Rightarrow F(t)^{1-a} = \chi(1-a)(t-t_0), \quad (11)$$

where χ is a constant and t_0 is an integration constant, show that (10) implies that the GW frequency is

$$f_{\text{GW}}^{-8/3} = 8\pi^{8/3} \xi \left(\frac{GM_c}{c^3}\right)^{(2/3)+p} (t_0 - t)^{2-p} \quad (12)$$

and determine the constant p .

On September 14, 2015 GW150914 was registered by the LIGO detectors, consisting of two L-shaped arms, each 4 km long. These arms changed by a relative length according to Fig. 1. The arms of the detector respond linearly to a passing gravitational wave, and the response pattern mimics the wave. This wave was created by two black holes on quasi-circular orbits; the loss of energy through gravitational radiation caused the orbit to shrink and the black holes to eventually collide. The collision point corresponds, roughly, to the peak of the signal after point D, in Fig. 1.

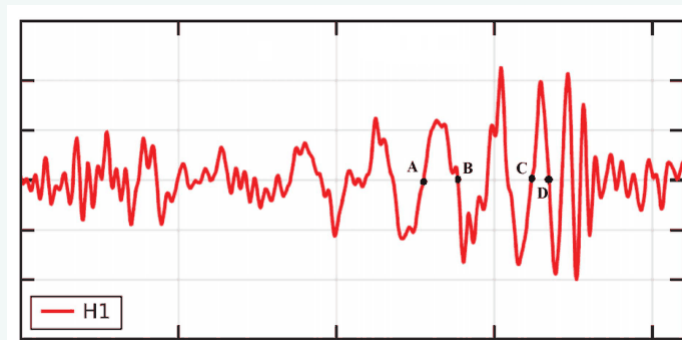


Figure 1. Strain, i.e. relative variation of the size of each arm, at the LIGO detector H1. The horizontal axis is time, and the points A, B, C, D correspond to $t = 0.000, 0.009, 0.034, 0.040$ seconds, respectively.

B.5 1.0pt

From the figure, estimate $f_{\text{GW}}(t)$ at

$$t_{\overline{\text{AB}}} = \frac{t_B + t_A}{2} \quad \text{and} \quad t_{\overline{\text{CD}}} = \frac{t_D + t_C}{2}. \quad (13)$$

Assuming that (12) is valid all the way until the collision (which strictly speaking is not true) and that the two objects have equal mass, estimate the chirp mass, M_c , and total mass of the system, in terms of solar masses $M_\odot \simeq 2 \times 10^{30}$ kg.

B.6 1.0pt

Estimate the minimal orbital separation between the two objects at t_{CD} . Hence estimate a maximum size for each object, R_{max} . Obtain $R_\odot/R_{\text{max}}$ to compare this size with the radius of our Sun, $R_\odot \simeq 7 \times 10^5$ km. Estimate also their orbital linear velocity at the same instant, v_{col} , comparing it with the speed of light, v_{col}/c .

Conclude that these are extremely fast moving, extremely compact objects indeed!

Official English text of the problem.

Answers

- **A.1:** $n = 3$, $\alpha = \frac{GM_2^3}{(M_1 + M_2)^2}$.

- **A.2:** $A(\mu, \Omega, L) = \frac{1}{2}\mu\Omega^2 L^2$.

- **A.3:** $\beta = -\frac{1}{2}$.

- **B.1:** With the masses on the x -axis at $t = 0$ and counterclockwise orbital motion,

$$k = 2\Omega, \quad (a_1, a_2, a_3) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right), \quad (b_1, b_2, b_3) = (1, -1, 0).$$

Also, $c_{12} = c_{21} = 1$; all other off-diagonal c_{ij} vanish.

- **B.2:** $\xi = \frac{32}{5} = 6.4$, $\mathcal{P} = \frac{32G}{5c^5}\mu^2 L^4 \Omega^6$.

- **B.3:** $M_c = \mu^{3/5} M^{2/5}$, with

$$\frac{d\Omega}{dt} = 3\xi \frac{(GM_c)^{5/3}}{c^5} \Omega^{11/3}.$$

- **B.4:** $f_{\text{GW}} = \frac{\Omega}{\pi}$, $p = 1$, and

$$f_{\text{GW}}^{-8/3} = 8\pi^{8/3} \xi \left(\frac{GM_c}{c^3}\right)^{5/3} (t_0 - t).$$

- **B.5:** $f_{\text{GW}}(0.0045 \text{ s}) \simeq 56 \text{ Hz}$, $f_{\text{GW}}(0.037 \text{ s}) \simeq 83 \text{ Hz}$; $M_c \simeq 30M_\odot$, $M \simeq 69M_\odot$.

- **B.6:** $L_{\text{min}} \simeq 5.1 \times 10^2 \text{ km}$, $R_{\text{max}} \simeq 2.6 \times 10^2 \text{ km}$, $R_\odot/R_{\text{max}} \simeq 2.7 \times 10^3$. Each object's orbital speed is $v_{\text{col}} \simeq 6.7 \times 10^7 \text{ m/s}$, so $v_{\text{col}}/c \simeq 0.22$.

Solution

A.1

Newton's law of gravitation gives the acceleration of the first star:

$$\frac{d^2\vec{r}_1}{dt^2} = -GM_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}.$$

The center-of-mass condition implies

$$\vec{r}_2 = -\frac{M_1}{M_2}\vec{r}_1, \quad \vec{r}_1 - \vec{r}_2 = \frac{M_1 + M_2}{M_2}\vec{r}_1.$$

Consequently,

$$\frac{d^2\vec{r}_1}{dt^2} = -\frac{GM_2^3}{(M_1 + M_2)^2} \frac{\vec{r}_1}{r_1^3}.$$

Comparison with (2) gives

$$\boxed{n = 3, \quad \alpha = \frac{GM_2^3}{(M_1 + M_2)^2}}.$$

A.2

The center-of-mass condition and $L = r_1 + r_2$ give

$$r_1 = \frac{M_2}{M}L, \quad r_2 = \frac{M_1}{M}L.$$

The stars have orbital speeds $v_1 = \Omega r_1$ and $v_2 = \Omega r_2$. Their total kinetic energy is therefore

$$\begin{aligned} T &= \frac{1}{2}M_1\Omega^2r_1^2 + \frac{1}{2}M_2\Omega^2r_2^2 \\ &= \frac{1}{2}\Omega^2L^2 \left(\frac{M_1M_2^2 + M_2M_1^2}{M^2} \right) \\ &= \frac{1}{2}\mu\Omega^2L^2. \end{aligned}$$

The gravitational potential energy, taking zero at infinite separation, is

$$U = -\frac{GM_1M_2}{L} = -\frac{GM\mu}{L}.$$

Thus the term $\mathcal{A}$ is the kinetic energy:

$$\boxed{\mathcal{A}(\mu, \Omega, L) = \frac{1}{2}\mu\Omega^2L^2}.$$

A.3

For the first star, gravity supplies the centripetal force:

$$M_1\Omega^2r_1 = \frac{GM_1M_2}{L^2}.$$

Using $r_1 = M_2L/M$, we obtain the circular-orbit relation

$$\boxed{\Omega^2 = \frac{GM}{L^3}}.$$

Hence

$$T = \frac{1}{2}\mu\Omega^2L^2 = \frac{GM\mu}{2L},$$

and

$$E = T + U = \frac{GM\mu}{2L} - \frac{GM\mu}{L} = -\frac{GM\mu}{2L}.$$

Therefore,

$$\boxed{\beta = -\frac{1}{2}}.$$

B.1

Choose the time origin so that the first mass lies on the positive x -axis at $t = 0$, with counterclockwise motion. Then

$$\begin{aligned}(x_1, y_1) &= (r_1 \cos \Omega t, r_1 \sin \Omega t), \\ (x_2, y_2) &= (-r_2 \cos \Omega t, -r_2 \sin \Omega t).\end{aligned}$$

From A.2,

$$M_1 r_1^2 + M_2 r_2^2 = \mu L^2.$$

Substitution into the quadrupole components gives

$$\begin{aligned}Q_{11} &= \frac{\mu L^2}{3} (2 \cos^2 \Omega t - \sin^2 \Omega t) = \frac{\mu L^2}{2} \left(\frac{1}{3} + \cos 2\Omega t \right), \\ Q_{22} &= \frac{\mu L^2}{3} (2 \sin^2 \Omega t - \cos^2 \Omega t) = \frac{\mu L^2}{2} \left(\frac{1}{3} - \cos 2\Omega t \right), \\ Q_{33} &= -\frac{\mu L^2}{3} = \frac{\mu L^2}{2} \left(-\frac{2}{3} \right), \\ Q_{12} = Q_{21} &= \mu L^2 \cos \Omega t \sin \Omega t = \frac{\mu L^2}{2} \sin 2\Omega t.\end{aligned}$$

Thus

$$\boxed{k = 2\Omega}.$$

The diagonal coefficients are

i	a_i	b_i
1	1/3	1
2	1/3	-1
3	-2/3	0

The off-diagonal coefficients are

$$\boxed{c_{12} = c_{21} = 1, \quad c_{13} = c_{31} = c_{23} = c_{32} = 0}.$$

B.2

Write $q = \mu L^2/2$ and $k = 2\Omega$. For the circular orbit, the nonzero third derivatives are

$$\begin{aligned}\ddot{Q}_{11} &= qk^3 \sin kt, \\ \ddot{Q}_{22} &= -qk^3 \sin kt, \\ \ddot{Q}_{12} = \ddot{Q}_{21} &= -qk^3 \cos kt.\end{aligned}$$

Both off-diagonal entries contribute to the double sum. Therefore,

$$\begin{aligned}
\sum_{i,j} (\ddot{Q}_{ij})^2 &= 2q^2 k^6 \sin^2 kt + 2q^2 k^6 \cos^2 kt \\
&= 2q^2 k^6 \\
&= 2 \left(\frac{\mu L^2}{2} \right)^2 (2\Omega)^6 \\
&= 32\mu^2 L^4 \Omega^6.
\end{aligned}$$

The luminosity is consequently

$$\boxed{\mathcal{P} = \frac{32G}{5c^5} \mu^2 L^4 \Omega^6}, \quad \boxed{\xi = \frac{32}{5} = 6.4}.$$

The result is independent of orbital phase.

B.3

We approximate the inspiral as a slowly evolving sequence of circular orbits. The masses remain constant, and the energy carried away by the waves obeys

$$\frac{dE}{dt} = -\mathcal{P}.$$

Eliminating L with the circular-orbit relation,

$$L = (GM)^{1/3} \Omega^{-2/3},$$

gives

$$E = -\frac{1}{2} \mu (GM)^{2/3} \Omega^{2/3}.$$

Its time derivative is

$$\frac{dE}{dt} = -\frac{1}{3} \mu (GM)^{2/3} \Omega^{-1/3} \frac{d\Omega}{dt}.$$

The luminosity, expressed in terms of Ω , is

$$\mathcal{P} = \xi \frac{G}{c^5} \mu^2 (GM)^{4/3} \Omega^{10/3}.$$

Equating the energy-loss rate to $-\mathcal{P}$ yields

$$\frac{d\Omega}{dt} = 3\xi \frac{G^{5/3} \mu M^{2/3}}{c^5} \Omega^{11/3}.$$

Cubing,

$$\left(\frac{d\Omega}{dt} \right)^3 = (3\xi)^3 \frac{\Omega^{11}}{c^{15}} G^5 \mu^3 M^2.$$

This has the required form if

$$M_c^5 = \mu^3 M^2.$$

Thus

$$\boxed{M_c = \mu^{3/5} M^{2/5} = \frac{(M_1 M_2)^{3/5}}{(M_1 + M_2)^{1/5}}}.$$

In particular, $d\Omega/dt > 0$: as the system loses energy, its orbit shrinks and its orbital frequency increases.

B.4

The orbital frequency is $f_{\text{orb}} = \Omega/(2\pi)$. The given factor of two between GW and orbital frequencies therefore implies

$$f_{\text{GW}} = 2f_{\text{orb}} = \frac{\Omega}{\pi}.$$

Taking the positive cube root of (10),

$$\frac{d\Omega}{dt} = 3\xi \frac{(GM_c)^{5/3}}{c^5} \Omega^{11/3}.$$

With $\Omega = \pi f_{\text{GW}}$, this becomes

$$\frac{df_{\text{GW}}}{dt} = 3\xi \pi^{8/3} \left(\frac{GM_c}{c^3} \right)^{5/3} f_{\text{GW}}^{11/3}.$$

Using (11) with $a = 11/3$, or directly differentiating $f_{\text{GW}}^{-8/3}$, gives

$$\frac{d}{dt} \left(f_{\text{GW}}^{-8/3} \right) = -8\xi \pi^{8/3} \left(\frac{GM_c}{c^3} \right)^{5/3}.$$

Integration yields

$$f_{\text{GW}}^{-8/3} = 8\pi^{8/3} \xi \left(\frac{GM_c}{c^3} \right)^{5/3} (t_0 - t).$$

Comparing the exponents with (12),

$$\frac{2}{3} + p = \frac{5}{3}, \quad 2 - p = 1,$$

so

$$p = 1.$$

Here t_0 is the formal time at which this approximate inspiral model predicts a divergent frequency.

B.5

A and B mark consecutive zero crossings on either side of one positive crest; C and D do likewise for a later crest. Each interval therefore spans approximately half a GW cycle.

Approximating the frequency over each short interval by its midpoint value,

$$f_{\text{GW}}(t_{\overline{\text{AB}}}) \simeq \frac{1}{2(t_{\text{B}} - t_{\text{A}})} = \frac{1}{2(0.009 \text{ s})} \simeq 55.6 \text{ Hz},$$

$$f_{\text{GW}}(t_{\overline{\text{CD}}}) \simeq \frac{1}{2(t_{\text{D}} - t_{\text{C}})} = \frac{1}{2(0.006 \text{ s})} \simeq 83.3 \text{ Hz}.$$

The corresponding times are

$$t_{\overline{\text{AB}}} = 0.0045 \text{ s}, \quad t_{\overline{\text{CD}}} = 0.037 \text{ s},$$

so their difference is 0.0325 s.

Let f_{AB} and f_{CD} denote these frequency estimates. Subtracting the B.4 relation evaluated at the two times eliminates t_0 :

$$f_{AB}^{-8/3} - f_{CD}^{-8/3} = 8\pi^{8/3}\xi \left(\frac{GM_c}{c^3} \right)^{5/3} (t_{CD} - t_{AB}).$$

Solving for the chirp mass,

$$M_c = \frac{c^3}{G} \left[\frac{f_{AB}^{-8/3} - f_{CD}^{-8/3}}{8\pi^{8/3}\xi (t_{CD} - t_{AB})} \right]^{3/5}.$$

Using the supplied constants and $\xi = 6.4$,

$$M_c \simeq 6.02 \times 10^{31} \text{ kg},$$

or, to appropriate precision,

$$\boxed{M_c \simeq 30M_\odot}.$$

For equal component masses, $M_1 = M_2 = M/2$, so

$$\mu = \frac{M}{4}, \quad M_c = \left(\frac{M}{4} \right)^{3/5} M^{2/5} = \frac{M}{4^{3/5}}.$$

Thus

$$M = 4^{3/5} M_c \simeq 1.38 \times 10^{32} \text{ kg},$$

and

$$\boxed{M \simeq 69M_\odot}.$$

Each component therefore has a mass of approximately $35M_\odot$.

B.6

At the later measured time,

$$\Omega_{CD} = \pi f_{CD} \simeq 2.62 \times 10^2 \text{ rad/s}.$$

The circular-orbit relation gives

$$L = \left(\frac{GM}{\Omega_{CD}^2} \right)^{1/3}.$$

The equal-mass choice also gives the minimum total mass, and hence the minimum separation, compatible with the inferred chirp mass and frequency. Indeed,

$$\frac{\mu}{M} = \frac{M_1 M_2}{(M_1 + M_2)^2} \leq \frac{1}{4},$$

so

$$M_c = M \left(\frac{\mu}{M} \right)^{3/5} \leq \frac{M}{4^{3/5}}.$$

Equality holds for equal masses. Using the mass found in B.5,

$$L_{\min} \simeq \left[\frac{(6.67 \times 10^{-11})(1.383 \times 10^{32})}{(2.618 \times 10^2)^2} \right]^{1/3} \text{ m},$$

which gives

$$L_{\min} \simeq 5.1 \times 10^5 \text{ m} = 5.1 \times 10^2 \text{ km}.$$

Using a common radius for the equal-mass objects, their remaining distinct requires approximately $2R \lesssim L$. Hence

$$R_{\max} \simeq \frac{L_{\min}}{2} \simeq 2.6 \times 10^2 \text{ km}.$$

The comparison with the Sun is

$$\frac{R_{\odot}}{R_{\max}} \simeq \frac{7 \times 10^5 \text{ km}}{2.56 \times 10^2 \text{ km}},$$

so

$$\frac{R_{\odot}}{R_{\max}} \simeq 2.7 \times 10^3.$$

Each object orbits the center of mass at radius $L/2$. Its orbital speed at this instant is therefore

$$v_{\text{col}} = \Omega_{\text{CD}} \frac{L}{2} \simeq (2.618 \times 10^2)(2.562 \times 10^5) \text{ m/s},$$

giving

$$v_{\text{col}} \simeq 6.7 \times 10^7 \text{ m/s}, \quad \frac{v_{\text{col}}}{c} \simeq 0.22.$$

Thus each object contains roughly 35 solar masses within a radius no greater than about 260 km, while moving at about one fifth of the speed of light. These are extremely compact and fast-moving objects. The numerical estimates use the stipulated Newtonian circular-orbit and quadrupole approximations, whose accuracy deteriorates near merger.

The solution was checked against the official solution.

Problem 2: Where is the neutrino?

When two protons collide with a very high energy at the Large Hadron Collider (LHC), several particles may be produced as a result of that collision, such as electrons, muons, neutrinos, quarks, and their respective anti-particles. Most of these particles can be detected by the particle detector surrounding the collision point. For example, quarks undergo a process called *hadronisation* in which they become a shower of subatomic particles, called "jet". In addition, the high magnetic field present in the detectors allows even very energetic charged particles to curve enough for their momentum to be determined. The ATLAS detector uses a superconducting solenoid system that produces a constant and uniform 2.00 Tesla magnetic field in the inner part of the detector, surrounding the collision point. Charged particles with momenta below a certain value will be curved so strongly that they will loop repeatedly in the field and most likely not be measured. Due to its nature, the neutrino is not detected at all, as it escapes through the detector without interacting.

Data: Electron rest mass, $m = 9.11 \times 10^{-31} \text{ kg}$; Elementary charge, $e = 1.60 \times 10^{-19} \text{ C}$;

Speed of light, $c = 3.00 \times 10^8 \text{ m s}^{-1}$; Vacuum permittivity, $\epsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1}$

Part A. ATLAS Detector physics (4.0 points)

A.1 — 0.5pt

Derive an expression for the cyclotron radius, r , of the circular trajectory of an electron acted upon by a magnetic force perpendicular to its velocity, and express that radius as a function of its kinetic energy, K ; charge modulus, e ; mass, m ; and magnetic field, B . Assume that the electron is a non-relativistic classical particle.

Electrons produced inside the ATLAS detector must be treated relativistically. However, the formula for the cyclotron radius also holds for relativistic motion when the relativistic momentum is considered.

A.2 — 0.5pt

Calculate the minimum value of the momentum of an electron that allows it to escape the inner part of the detector in the radial direction. The inner part of the detector has a cylindrical shape with a radius of 1.1 meters, and the electron is produced in the collision point exactly in the center of the cylinder. Express your answer in MeV/c .

When accelerated perpendicularly to the velocity, relativistic particles of charge e and rest mass m emit electromagnetic radiation, called synchrotron radiation. The emitted power is given by

$$P = \frac{e^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3}$$

where a is the acceleration and $\gamma = [1 - (v/c)^2]^{-1/2}$.

A.3 — 1.0pt

A particle is called ultrarelativistic when its speed is very close to the speed of light. For an ultrarelativistic particle the emitted power can be expressed as:

$$P = \xi \frac{e^4}{\epsilon_0 m^k c^n} E^2 B^2,$$

where ξ is a real number, n, k are integers, E is the energy of the charged particle and B is the magnetic field. Find ξ, n and k .

A.4 — 1.0pt

In the ultrarelativistic limit, the energy of the electron as a function of time is:

$$E(t) = \frac{E_0}{1 + \alpha E_0 t},$$

where E_0 is the initial energy of the electron. Find α as a function of e, c, B, ϵ_0 and m .

A.5 — 0.5pt

Consider an electron produced at the collision point along the radial direction with an energy of 100 GeV. Estimate the amount of energy that is lost due to synchrotron radiation until the electron escapes the inner part of the detector? Express your answer in MeV.

A.6 — 0.5pt

Find an expression for the cyclotron frequency of the electron as a function of time in the ultrarelativistic limit.

Part B. Finding the neutrino (6.0 points)

The collision between the two protons shown in Figure 1 leads to the production of a top quark (t) and an anti-top quark ($\bar{t}$), the heaviest elementary particles ever detected. The top quark decays into a W^+ boson and a bottom quark (b), while the anti-top quark decays into a W^- boson and an anti-bottom quark ($\bar{b}$). In the case depicted in Figure 1, the W^+ boson decays into a anti-muon (μ^+) and a neutrino (ν), and the W^- boson decays into a quark and an anti-quark. The task of this problem is to reconstruct the full momentum of the neutrino using the momenta of some detected particles. **For simplicity, all particles and jets in this problem will be considered massless, except for the top quark and $W^\pm$ bosons.**

The momenta of the top quark decay products can be determined from the experiment (see Table), except for the neutrino momentum component along the (z -axis). The total linear momentum of the final state particles caught by the detector is only zero on the transverse plane (xy plane), and not along the collision line (z -axis). As such, one can find the transverse momentum of the neutrino from the missing momentum in the transverse plane.

On June 4, 2015, the ATLAS experiment at the LHC recorded a proton-proton collision at 00:21:24 GMT+1 like the one represented in Figure 1.

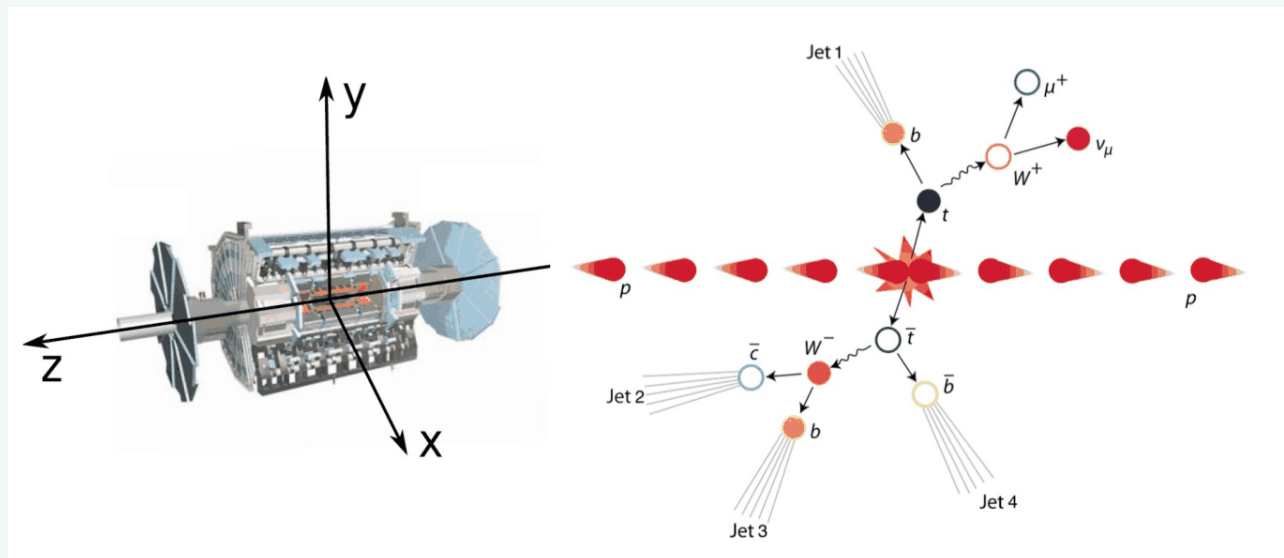


Figure 1. Schematic representation of the ATLAS detector coordinate system (left) and proton-proton collision (right).

The linear momenta of the three final-state particles coming from the top quark decay, including the neutrino, is presented below for each component.

Particle	p_x (GeV/c)	p_y (GeV/c)	p_z (GeV/c)
anti-muon (μ^+)	-24.7	-24.9	-12.4
jet 1 (j_1)	-14.2	+50.1	+94.1
neutrino (ν)	-104.1	+5.3	---

B.1 — 1.5pt

Find an equation which relates the square of the W^+ boson mass, m_W^2 , with the neutrino and anti-muon momentum components presented in the table above. Express your answer in terms of the neutrino and anti-muon transverse momentum,

$$\vec{p}_T^{(\nu)} = p_x^{(\nu)}\hat{i} + p_y^{(\nu)}\hat{j} \quad \text{and} \quad \vec{p}_T^{(\mu)} = p_x^{(\mu)}\hat{i} + p_y^{(\mu)}\hat{j},$$

and their z -axis momentum components, $p_z^{(\mu)}$ and $p_z^{(\nu)}$.

B.2 — 1.5pt

Assuming a W^+ boson mass of $m_W = 80.4 \text{ GeV}/c^2$ calculate the two possible solutions for the neutrino momentum along the z -axis, $p_z^{(\nu)}$. Express your answer in GeV/c .

B.3 — 1.0pt

Calculate the top quark mass for each one of the two previous solutions. Express your answer in GeV/c^2 .

[If you did not obtain the two solutions in B.2, use $p_z^{(\nu)} = 70 \text{ GeV}/c$ and $p_z^{(\nu)} = -180 \text{ GeV}/c$.]

The normalised number of collision events for the measurement of the top quark mass (as determined from the experiment), has two components: the so-called "signal" (corresponding to the decay of top quarks) and "background" (corresponding to events from other processes that do not include top quarks). Experimental data include both processes, see Fig. 2.

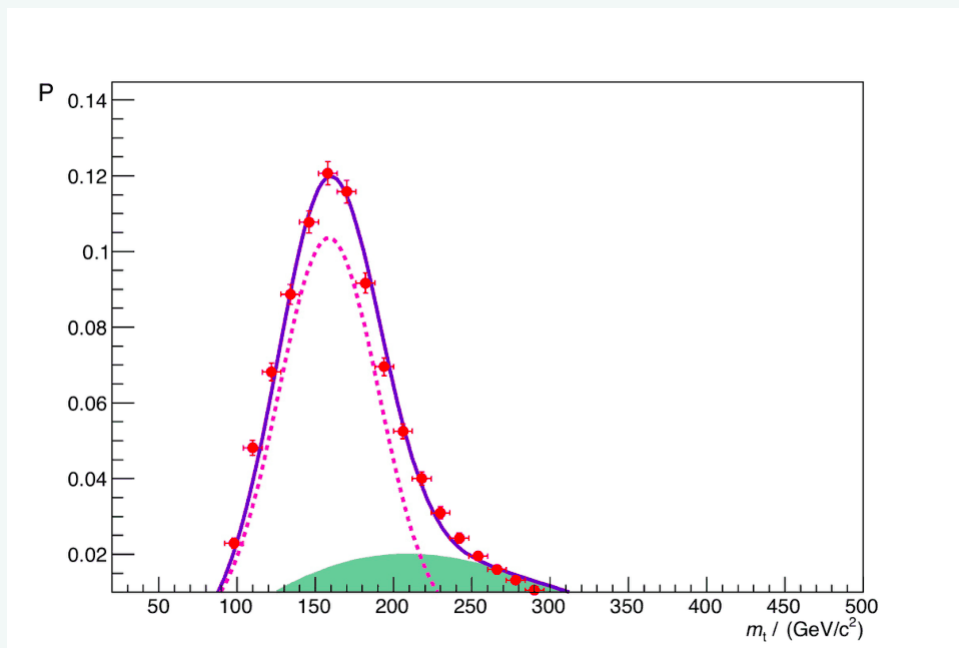


Figure 2. Top quark mass distribution as determined from the experiment, i.e. the normalised number of events plotted against the top quark mass. The dots correspond to the data. The dashed line corresponds to the "signal" and the shade to the "background".

B.4 — 1.0pt

According to the top quark mass distribution, which one of the two previous solutions is more likely to be the right one? Estimate the probability for the most likely solution.

B.5 — 1.0pt

Calculate the distance traveled by the top quark before decaying, using the most likely solution. Assume the top quark has a mean lifetime at rest of 5×10^{-25} s.

Official English text of the problem.

Answers

Here $p_T^{(a)} = |\vec{p}_T^{(a)}|$.

- **A.1:** $r = \frac{\sqrt{2mK}}{eB}$.
- **A.2:** $p_{\min} = \frac{eBR}{2} = 3.3 \times 10^2 \text{ MeV}/c$.
- **A.3:** $\xi = \frac{1}{6\pi}$, $n = 5$, $k = 4$.
- **A.4:** $\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}$.
- **A.5:** $\Delta E \simeq 55 \text{ MeV}$.
- **A.6:** The instantaneous frequency is

$$f(t) = \frac{eBc^2}{2\pi E_0} (1 + \alpha E_0 t).$$

The angular frequency is $\omega(t) = 2\pi f(t)$.

- **B.1:**

$$\frac{m_W^2 c^2}{2} = \sqrt{\left[(p_T^{(\mu)})^2 + (p_z^{(\mu)})^2\right] \left[(p_T^{(\nu)})^2 + (p_z^{(\nu)})^2\right]} - \vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} - p_z^{(\mu)} p_z^{(\nu)}.$$

- **B.2:** $p_z^{(\nu)} \simeq +74.0 \text{ GeV}/c$ or $-188 \text{ GeV}/c$.
- **B.3:** Respectively, $m_t \simeq 169 \text{ GeV}/c^2$ or $311 \text{ GeV}/c^2$.
- **B.4:** The $p_z^{(\nu)} \simeq +74.0 \text{ GeV}/c$ solution is more likely. The plotted probability at its reconstructed mass is approximately $P \simeq 0.12$, or 12%.
- **B.5:** The mean distance is $\bar{L} \simeq 1.9 \times 10^{-16} \text{ m}$.

Solution

A.1

For motion perpendicular to the magnetic field, the magnetic Lorentz force provides the centripetal force:

$$evB = \frac{mv^2}{r}.$$

Thus $r = mv/(eB)$. Using the non-relativistic kinetic energy $K = \frac{1}{2}mv^2$,

$$r = \frac{\sqrt{2mK}}{eB}.$$

Equivalently, $r = p/(eB)$, which also applies relativistically when p is the relativistic momentum.

A.2

Let the detector radius be $R = 1.1$ m. The electron's circular orbit passes through the collision point, so its orbit centre is a distance r from that point. Its greatest radial distance from the collision point is therefore $2r$.

The limiting orbit touches the cylindrical boundary when

$$2r = R.$$

The minimum total momentum occurs for motion perpendicular to the solenoid field. Hence the escape threshold is

$$p_{\min} = eB \frac{R}{2} = (1.60 \times 10^{-19})(2.00) \frac{1.1}{2} = 1.76 \times 10^{-19} \text{ kg m s}^{-1}.$$

Since

$$1 \text{ MeV}/c = \frac{10^6(1.60 \times 10^{-19})}{3.00 \times 10^8} = 5.33 \times 10^{-22} \text{ kg m s}^{-1},$$

we obtain

$$p_{\min} = 3.3 \times 10^2 \text{ MeV}/c.$$

Just above this threshold, the orbit crosses the cylindrical boundary.

A.3

For acceleration perpendicular to the velocity, the relativistic transverse equation of motion is

$$evB = \gamma ma,$$

so the curvature acceleration is

$$a = \frac{evB}{\gamma m}.$$

Substitution into the given radiation formula gives

$$P = \frac{e^2}{6\pi\epsilon_0 c^3} \frac{e^2 v^2 B^2}{\gamma^2 m^2} \gamma^4 = \frac{e^4 B^2 v^2 \gamma^2}{6\pi\epsilon_0 m^2 c^3}.$$

In the ultrarelativistic limit, $v \simeq c$, while the total energy is $E = \gamma mc^2$. Therefore,

$$P \simeq \frac{e^4 B^2}{6\pi\epsilon_0 m^2 c} \left(\frac{E}{mc^2} \right)^2 = \frac{e^4 E^2 B^2}{6\pi\epsilon_0 m^4 c^5}.$$

Comparison with the requested form yields

$$\xi = \frac{1}{6\pi}, \quad n = 5, \quad k = 4.$$

A.4

The magnetic force does no work, so the electron's energy decreases by the energy radiated:

$$\frac{dE}{dt} = -P = -\alpha E^2,$$

where, from A.3,

$$\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}.$$

Integrating with $E(0) = E_0$,

$$\int_{E_0}^{E(t)} \frac{dE}{E^2} = -\alpha \int_0^t dt,$$

which gives

$$\frac{1}{E(t)} - \frac{1}{E_0} = \alpha t.$$

Consequently,

$$E(t) = \frac{E_0}{1 + \alpha E_0 t},$$

as stated in the question.

A.5

First check the trajectory curvature. For an ultrarelativistic electron, $p \simeq E/c$, so its initial cyclotron radius is

$$r_0 \simeq \frac{E_0}{eBc}.$$

Here

$$E_0 = 100 \text{ GeV} = 1.60 \times 10^{-8} \text{ J},$$

and therefore

$$r_0 = \frac{1.60 \times 10^{-8}}{(1.60 \times 10^{-19})(2.00)(3.00 \times 10^8)} = 1.67 \times 10^2 \text{ m}.$$

Since $r_0 \gg R = 1.1 \text{ m}$, the path through the detector is almost straight. Thus

$$t_{\text{exit}} \simeq \frac{R}{c} = \frac{1.1}{3.00 \times 10^8} = 3.67 \times 10^{-9} \text{ s}.$$

Using the supplied constants,

$$\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5} = 9.39 \times 10^{12} \text{ J}^{-1} \text{ s}^{-1},$$

and

$$\alpha E_0 t_{\text{exit}} = 5.51 \times 10^{-4}.$$

The energy lost is

$$\Delta E = E_0 - \frac{E_0}{1 + \alpha E_0 t_{\text{exit}}} = \frac{\alpha E_0^2 t_{\text{exit}}}{1 + \alpha E_0 t_{\text{exit}}}.$$

Numerically,

$$\Delta E = 8.81 \times 10^{-12} \text{ J} = 55.1 \text{ MeV}.$$

Thus, to sensible precision,

$$\boxed{\Delta E \simeq 55 \text{ MeV}}.$$

This is only about 5.5×10^{-4} of the initial energy, confirming that the radius remains very large throughout the flight.

A.6

The instantaneous angular rate of rotation of the momentum is

$$\omega = \frac{v}{r} = \frac{eB}{\gamma m}.$$

Using $E = \gamma mc^2$,

$$\omega(t) = \frac{eBc^2}{E(t)}.$$

Substituting the result of A.4,

$$\boxed{\omega(t) = \frac{eBc^2}{E_0}(1 + \alpha E_0 t)}.$$

The ordinary frequency, measured in cycles per second, is therefore

$$\boxed{f(t) = \frac{eBc^2}{2\pi E_0}(1 + \alpha E_0 t)}.$$

Both increase linearly with time within the ultrarelativistic approximation.

B.1

Four-momentum conservation in $W^+ \rightarrow \mu^+ + \nu$ gives

$$E_W = E_\mu + E_\nu, \quad \vec{p}_W = \vec{p}^{(\mu)} + \vec{p}^{(\nu)}.$$

The invariant mass of the two-particle system satisfies

$$m_W^2 c^4 = (E_\mu + E_\nu)^2 - c^2 |\vec{p}^{(\mu)} + \vec{p}^{(\nu)}|^2.$$

Both daughter particles are treated as massless, so

$$E_\mu = c|\vec{p}^{(\mu)}|, \quad E_\nu = c|\vec{p}^{(\nu)}|.$$

The individual energy–momentum terms cancel on expanding the invariant:

$$m_W^2 c^4 = 2 \left(E_\mu E_\nu - c^2 \vec{p}^{(\mu)} \cdot \vec{p}^{(\nu)} \right).$$

Now

$$|\vec{p}^{(a)}| = \sqrt{(p_T^{(a)})^2 + (p_z^{(a)})^2},$$

and

$$\vec{p}^{(\mu)} \cdot \vec{p}^{(\nu)} = \vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} + p_z^{(\mu)} p_z^{(\nu)}.$$

Hence the requested relation is

$$\frac{m_W^2 c^2}{2} = \sqrt{\left[(p_T^{(\mu)})^2 + (p_z^{(\mu)})^2\right] \left[(p_T^{(\nu)})^2 + (p_z^{(\nu)})^2\right] - \vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} - p_z^{(\mu)} p_z^{(\nu)}}.$$

B.2

Define

$$q = p_z^{(\nu)}, \quad \ell = p_z^{(\mu)}, \quad a = (p_T^{(\mu)})^2, \quad b = (p_T^{(\nu)})^2,$$

and let

$$p_\mu = |\vec{p}^{(\mu)}|, \quad D = \frac{m_W^2 c^2}{2} + \vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)}.$$

The equation from B.1 becomes

$$p_\mu \sqrt{b + q^2} = D + \ell q.$$

Squaring and using $p_\mu^2 - \ell^2 = a$,

$$a q^2 - 2 D \ell q + p_\mu^2 b - D^2 = 0.$$

Its roots are

$$q = \frac{D \ell \pm p_\mu \sqrt{D^2 - a b}}{a}.$$

The measured components give

$$\begin{aligned} a &= 24.7^2 + 24.9^2 = 1230.1 \text{ (GeV/c)}^2, \\ b &= 104.1^2 + 5.3^2 = 10864.9 \text{ (GeV/c)}^2, \\ p_\mu &= \sqrt{24.7^2 + 24.9^2 + 12.4^2} = 37.2003 \text{ GeV/c}, \\ \vec{p}_T^{(\mu)} \cdot \vec{p}_T^{(\nu)} &= (-24.7)(-104.1) + (-24.9)(5.3) \\ &= 2439.3 \text{ (GeV/c)}^2, \\ D &= \frac{80.4^2}{2} + 2439.3 = 5671.38 \text{ (GeV/c)}^2. \end{aligned}$$

Thus the two numerical roots are

$$q_+ = 73.9532 \text{ GeV/c}, \quad q_- = -188.2937 \text{ GeV/c}.$$

Both satisfy the original, unsquared equation. Therefore,

$$p_z^{(\nu)} \simeq +74.0 \text{ GeV/c} \quad \text{or} \quad -188 \text{ GeV/c}.$$

B.3

The top quark decays into the anti-muon, neutrino and bottom-quark jet. Conservation of energy and momentum gives

$$E_t = E_\mu + E_\nu + E_{j_1}, \quad \vec{p}_t = \vec{p}^{(\mu)} + \vec{p}^{(\nu)} + \vec{p}^{(j_1)}.$$

Because all three decay products are treated as massless,

$$E_t = c \left(|\vec{p}^{(\mu)}| + |\vec{p}^{(\nu)}| + |\vec{p}^{(j_1)}| \right).$$

The jet momentum magnitude is

$$|\vec{p}^{(j_1)}| = \sqrt{(-14.2)^2 + 50.1^2 + 94.1^2} = 107.5475 \text{ GeV}/c.$$

Writing q as its numerical value in GeV/c ,

$$E_t = \left[37.2003 + 107.5475 + \sqrt{10864.9 + q^2} \right] \text{ GeV},$$

and

$$\vec{p}_t = (-143.0, 30.5, 81.7 + q) \text{ GeV}/c.$$

The reconstructed top mass follows from

$$m_t^2 c^4 = E_t^2 - c^2 |\vec{p}_t|^2.$$

Keeping extra digits during the calculation gives:

Neutrino solution	E_ν (GeV)	E_t (GeV)	p_{tz} (GeV/c)	m_t (GeV/c ²)
$q_+ = 73.9532 \text{ GeV}/c$	127.804	272.552	155.653	169.344
$q_- = -188.2937 \text{ GeV}/c$	215.219	359.967	-106.594	311.183

Therefore,

$$m_t \simeq 169 \text{ GeV}/c^2 \quad \text{for } p_z^{(\nu)} > 0,$$

$$m_t \simeq 311 \text{ GeV}/c^2 \quad \text{for } p_z^{(\nu)} < 0.$$

B.4

The mass $169 \text{ GeV}/c^2$ lies close to the maximum of the distribution in Figure 2, whereas $311 \text{ GeV}/c^2$ lies in its small upper tail. The positive- $p_z^{(\nu)}$ solution is therefore more likely.

Reading the ordinate of the plotted total-distribution curve near $169 \text{ GeV}/c^2$ gives

$$P \simeq 0.12 \quad \text{or approximately } 12\%.$$

The corresponding reconstructed neutrino momentum is

$$\vec{p}^{(\nu)} \simeq (-104.1, +5.3, +74.0) \text{ GeV}/c.$$

B.5

For the more likely solution,

$$\vec{p}_t = (-143.0, 30.5, 155.653) \text{ GeV}/c,$$

so

$$|\vec{p}_t| = 213.558 \text{ GeV}/c.$$

The top quark's Lorentz factor and speed are

$$\gamma_t = \frac{E_t}{m_t c^2} = \frac{272.552}{169.344} = 1.609,$$

$$\beta_t = \frac{|\vec{p}_t| c}{E_t} = \frac{213.558}{272.552} = 0.7836.$$

The mean lifetime in the detector frame is $\gamma_t \tau_0$. Thus the mean distance travelled is

$$\bar{L} = v_t \gamma_t \tau_0 = \beta_t \gamma_t c \tau_0 = \frac{|\vec{p}_t| c}{m_t c^2} c \tau_0.$$

Substitution gives

$$\bar{L} = \frac{213.558}{169.344} (3.00 \times 10^8) (5 \times 10^{-25}) = 1.89 \times 10^{-16} \text{ m}.$$

Therefore,

$$\boxed{\bar{L} \simeq 1.9 \times 10^{-16} \text{ m}}.$$

This is a mean distance because the supplied rest-frame lifetime is a mean lifetime.

The solution was checked against the official solution.

Problem 3: Physics of Live Systems

Data: Normal atmospheric pressure, $P_0 = 1.013 \times 10^5 \text{ Pa} = 760 \text{ mmHg}$

Part A. The physics of blood flow (4.5 points)

In this part you will analyse two simplified models of blood flow in vessels.

Blood vessels are approximately cylindrical in shape, and it is known that for a steady, non turbulent flow of an incompressible fluid in a rigid cylinder, the difference in pressure of the fluid at the two ends of the cylinder is given by

$$\Delta P = \frac{8\ell\eta}{\pi r^4} Q, \quad (1)$$

where ℓ and r are the length and radius of the cylinder, η is the fluid viscosity and Q is the volumetric flow rate, i.e. the fluid volume that passes the cylinder cross section per unit time. This expression is often able to provide the correct order of magnitude for the pressure difference in a vessel, even without taking into account the pulsatile flow, the vessel's compressibility and irregular shape, and the fact that blood is not a simple fluid but a mixture of cells and plasma. Moreover, this expression has the same form as Ohm's law, with the volumetric flow rate being interpreted as a current, the difference in pressure as a voltage, and the factor $R = \frac{8\ell\eta}{\pi r^4}$ as a resistance.

Consider for example the symmetrical network of arterioles (small arteries) depicted in Figure 1 that delivers blood to the capillary bed of a tissue. In this network, at each bifurcation a vessel is divided in two identical vessels. However, the vessels of higher levels are thinner and shorter: consider that the radii and lengths of vessels in two consecutive levels, i and $i + 1$, are related by $r_{i+1} = r_i/2^{1/3}$ and by $\ell_{i+1} = \ell_i/2^{1/3}$.

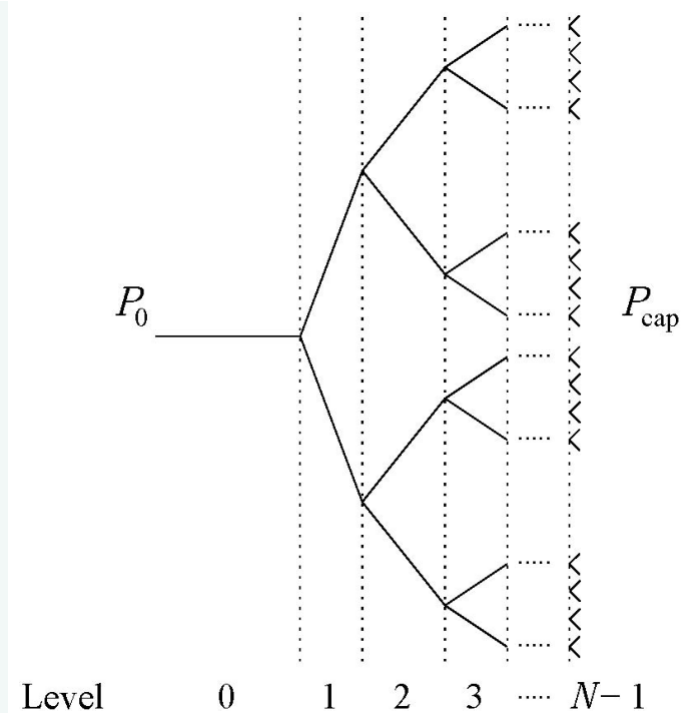


Figure 1. Network of arterioles.

A.1 (1.3pt)

Obtain an expression for the volumetric flow rate, Q_i , in a vessel at any level i , as a function of the total number of levels N , of the viscosity η , of the radius r_0 and length ℓ_0 of the first vessel, and of the difference $\Delta P = P_0 - P_{\text{cap}}$ between the pressure at the arteriole at level 0, P_0 , and the pressure at the capillary bed, P_{cap} .

A.2 (0.5pt)

Calculate the numerical value of the volumetric flow rate Q_0 of the arteriole at level 0, if its radius is $6.0 \times 10^{-5} \text{ m}$ and its length is $2.0 \times 10^{-3} \text{ m}$. Consider that the pressure at the arteriole inlet is 55 mmHg and the vessel network has $N = 6$ levels linking this arteriole to the capillary bed at the pressure 30 mmHg. Consider that the blood viscosity is $\eta = 3.5 \times 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}$. Express your result in ml/h.

A blood vessel as an LCR circuit

The approximation of rigid cylindrical vessels falls short for several reasons. It is particularly important to include the time dependent flow and to take into account the change in vessel diameter that occurs when the pressure varies during a blood pumping cycle done by the heart. Moreover, it is observed that in the larger vessels the blood pressure varies significantly during a cycle, while in the smaller vessels the amplitude of the oscillations in pressure is much smaller, and the flow is almost time independent.

When the pressure increases in a single elastic vessel, there will be an increase in its diameter, thus permitting to store more fluid in the vessel, and to deliver it when the pressure drops. For this reason, the elastic behaviour of the vessel can be simulated by adding a capacitor to our initial description. Moreover, when taking into account the time dependent blood flow rate, one has to

consider the inertia of the fluid, proportional to its density $\rho = 1.05 \times 10^3 \text{ kg m}^{-3}$. This inertia can be described by an inductance in our model. In Figure 2 we represent the equivalent circuit for a single vessel in this model. The equivalent capacitance and inductance are given by

$$C = \frac{3\ell\pi r^3}{2Eh} \quad \text{and} \quad L = \frac{9\ell\rho}{4\pi r^2}, \quad (2)$$

respectively, where h is the width of the vessel wall and E is the artery Young's modulus, a coefficient that describes the alteration in size of the vessel tissue when a force is applied. The Young's modulus has units of pressure and is on the order of $E = 0.06 \text{ MPa}$ for arterioles.

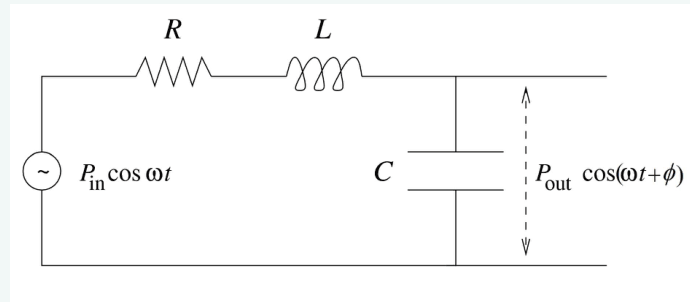


Figure 2. Equivalent electric circuit for a single vessel.

A.3 (2.0pt)

Obtain, in the stationary regime, the pressure amplitude at the vessel outlet, P_{out} , as a function of the pressure amplitude at the inlet, P_{in} , the equivalent resistance, R , inductance, L and capacitance, C , for a flow with angular frequency ω . Establish the condition between η , ρ , E , h , r and ℓ so that, for low frequencies, the pressure oscillation amplitude at the outlet is smaller than that of P_{in} .

A.4 (0.7pt)

For the vessel network in **A.2** estimate the maximum arteriole wall thickness h so that the condition established in **A.3** is satisfied (consider that h is level independent).

Part B. Tumour growth (5.5 points)

Tumour growth is a very complex process where biological mechanisms such as cell proliferation and natural selection are intertwined with physics. In this problem we will consider a simplified model of tumour growth that addresses the increase in pressure commonly observed in solid tumors.

Consider a group of normal cells forming a tissue surrounded by an inextensible basement membrane, which forces the tissue to maintain always the same form: a sphere of radius R (Figure 3).

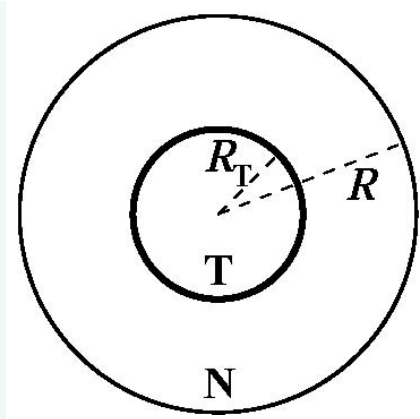


Figure 3. Simplified tumour.

Initially the tissue does not have residual stresses, i.e. the pressure at every point is equal to the atmospheric pressure.

At time $t = 0$, a tumour starts growing at the centre of this sphere and, as it grows, the pressure inside the tissue increases. Consider that both tissues (normal, N, and tumour, T) are compressible such that their densities, ρ_N and ρ_T , increase linearly with pressure:

$$\rho_N = \rho_0 \left(1 + \frac{p}{K_N} \right), \quad \rho_T = \rho_0 \left(1 + \frac{p}{K_T} \right), \quad (3)$$

where ρ_0 is the rest tissue density, p is the pressure difference to the atmospheric pressure and K_N , K_T are the compressibility moduli (bulk moduli) of the normal and tumour tissues, respectively. In general, tumours are stiffer and so they have a higher bulk modulus.

B.1 (1.0pt)

The mass of normal cells is not altered while the tumour is growing. Obtain the ratio between the tumour volume and the total tissue volume, $v = V_T/V$, as a function of the ratio between the tumour mass (M_T) and the normal tissue mass (M_N), $\mu = M_T/M_N$ and the ratio of the bulk moduli, $\kappa = K_N/K_T$.

Hyperthermia is sometimes used together with chemotherapy and radiotherapy in the treatment of cancer. In hyperthermia the cancer cells are selectively heated from the normal human body temperature, 37°C , to temperatures above 43°C , inducing their death. Researchers are currently developing carbon nanotubes covered with special proteins capable of binding to tumour cells. When the tissue is irradiated with near-infrared radiation, the nanotubes absorb it in a much greater extent than the surrounding tissues and therefore can be selectively heated as well as the tumour cells to which they are attached.

Consider that the tumour, the normal cells and the surrounding tissue have a constant thermal conductivity k , i.e. in the geometry of this problem, the energy that crosses a spherical surface of radius r per unit time and per unit area is equal to k times the derivative of the temperature with respect to r . The nanotubes are uniformly distributed in the tumour volume and are able to deliver a power $\mathcal{P}$ of thermal energy per unit volume. Assume that the temperature is equal to the normal human body temperature very far away from the tumour.

B.2 (1.7pt)

Obtain, for the stationary state, the temperature at the centre of the tumour as a function of $\mathcal{P}$, k , the human body temperature and the tumour radius, R_T .

B.3 (0.5pt)

Obtain the minimum power per unit volume, $\mathcal{P}_{\min}$, needed to heat up all tumour cells in a tumour with 5.0 cm radius to a temperature larger than 43.0°C . Take the thermal conductivity of the tissue to be equal to $k = 0.60 \text{ W K}^{-1} \text{ m}^{-1}$.

Consider that the tumour is irrigated by a vessel network with a branched structure like in question **A.1**. As the tumour grows, when its pressure p becomes larger than the pressure P_{cap} at the thinnest vessels, the radii of these vessels will decrease by a small amount δr . If this pressure reaches a critical value p_c (which would correspond to a radius decrease of δr_c), the thinnest vessels would collapse, compromising seriously the irrigation to the tumour. The pressure and the radius change can be related by the following phenomenological relation:

$$\frac{p}{P_{\text{cap}}} - 1 = \left(\frac{p_c}{P_{\text{cap}}} - 1 \right) \left(2 - \frac{\delta r}{\delta r_c} \right) \frac{\delta r}{\delta r_c}. \quad (4)$$

Consider that just the smallest vessels (of level $N - 1$) have their radius altered when the tumour increases its pressure.

B.4 (2.3pt)

In the linear regime (i.e. consider that $p - P_{\text{cap}}$ is very small), express the relative drop in the flow rate, $\frac{\delta Q_{N-1}}{Q_{N-1}}$, in these thinnest vessels, as a function of the tumour volume ratio $v = V_T/V$ and K_N , N , p_c , δr_c , r_{N-1} , P_{cap} .

Official English text of the problem.

Answers• **A.1**

$$Q_i = \frac{\pi r_0^4 \Delta P}{8 \eta \ell_0 N 2^i}, \quad i = 0, \dots, N - 1.$$

• **A.2**

$$Q_0 = 1.4536 \dots \text{ ml/h} \simeq 1.5 \text{ ml/h}.$$

- **A.3**

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}.$$

Attenuation at sufficiently low nonzero frequencies requires

$$R^2 C \geq 2L \iff h \leq \frac{64\eta^2 \ell^2}{3\rho E r^3}.$$

The inequality is strict for attenuation at order ω^2 ; equality gives attenuation at order ω^4 .

- **A.4**

$$h_{\text{max}} = \frac{64\eta^2 \ell_0^2}{3\rho E r_0^3} \simeq 7.7 \times 10^{-5} \text{ m} = 77 \text{ }\mu\text{m}.$$

Level 0 imposes the strongest constraint.

- **B.1**

$$v = \frac{2\mu}{1 + \mu + \sqrt{(1 - \mu)^2 + 4\kappa\mu}}.$$

- **B.2** With T_b the normal body temperature,

$$T(0) = T_b + \frac{\mathcal{P} R_T^2}{2k}.$$

- **B.3**

$$\mathcal{P}_{\text{min}} = \frac{3k(43.0 - 37.0) \text{ K}}{R_T^2} \simeq 4.3 \times 10^3 \text{ W m}^{-3}.$$

This is the threshold; strictly exceeding 43.0°C everywhere requires power density above it.

- **B.4** Taking δQ_{N-1} to mean the positive decrease,

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq \frac{2\delta r_c}{N r_{N-1}(p_c - P_{\text{cap}})} \left(\frac{K_N v}{1 - v} - P_{\text{cap}} \right).$$

This applies just above the onset of compression, where $p > P_{\text{cap}}$.

Solution

A.1

The geometrical scaling gives

$$r_i = r_0 2^{-i/3}, \quad \ell_i = \ell_0 2^{-i/3}.$$

Consequently, the hydraulic resistance of one vessel at level i is

$$R_i = \frac{8\eta \ell_i}{\pi r_i^4} = \frac{8\eta \ell_0}{\pi r_0^4} 2^i = R_0 2^i.$$

Conservation of volume flow and symmetry imply that the flow divides equally at each bifurcation. Thus

$$Q_i = \frac{Q_0}{2^i}.$$

The pressure drop across any one vessel at level i is therefore

$$\Delta P_i = R_i Q_i = R_0 Q_0.$$

Every path from the inlet to the capillary bed passes through one vessel at each of the N levels. Adding the pressure drops along such a path,

$$\Delta P = N R_0 Q_0.$$

Hence

$$Q_i = \frac{Q_0}{2^i} = \frac{\pi r_0^4 \Delta P}{8 \eta \ell_0 N 2^i}.$$

Equivalently, each level contributes $R_i/2^i = R_0$ to the total network resistance, so $R_{\text{eq}} = N R_0$.

A.2

Using the supplied conversion between mmHg and pascals,

$$\Delta P = (55 - 30) \frac{1.013 \times 10^5}{760} \text{ Pa} = 3.3322 \times 10^3 \text{ Pa}.$$

Therefore,

$$Q_0 = \frac{\pi (6.0 \times 10^{-5})^4 (3.3322 \times 10^3)}{8 (3.5 \times 10^{-3}) (2.0 \times 10^{-3}) (6)} = 4.0379 \times 10^{-10} \text{ m}^3 \text{ s}^{-1}.$$

Since $1 \text{ m}^3 = 10^6 \text{ ml}$ and $1 \text{ h} = 3600 \text{ s}$,

$$Q_0 = (4.0379 \times 10^{-10}) (10^6) (3600) = 1.4536 \text{ ml/h}.$$

To two significant figures,

$$Q_0 \simeq 1.5 \text{ ml/h}.$$

A.3

In the circuit analogy, pressure corresponds to voltage and volume flow to current. The capacitor relation and the pressure balance around the circuit are

$$Q = C \frac{dp_{\text{out}}}{dt}, \quad p_{\text{in}} - p_{\text{out}} = RQ + L \frac{dQ}{dt}.$$

Combining these equations gives

$$LC \frac{d^2 p_{\text{out}}}{dt^2} + RC \frac{dp_{\text{out}}}{dt} + p_{\text{out}} = p_{\text{in}}.$$

After transients have decayed, the input and output oscillate at the same angular frequency. Representing them by complex amplitudes,

$$(1 - \omega^2 LC + i\omega RC) \tilde{P}_{\text{out}} = \tilde{P}_{\text{in}}.$$

Taking absolute values,

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}}.$$

The square of the denominator is

$$1 + C(R^2 C - 2L)\omega^2 + L^2 C^2 \omega^4.$$

For sufficiently small nonzero ω , attenuation at leading order requires

$$R^2 C > 2L.$$

If $R^2 C < 2L$, the denominator is smaller than unity at sufficiently low frequencies and the pressure oscillation is amplified. At the boundary $R^2 C = 2L$, the denominator squared becomes $1 + L^2 C^2 \omega^4 > 1$, so the boundary also attenuates, although only at order ω^4 . The exact condition is therefore $R^2 C \geq 2L$.

Substituting the given vessel parameters,

$$R^2 C = \left(\frac{8\ell\eta}{\pi r^4} \right)^2 \frac{3\ell\pi r^3}{2Eh} = \frac{96\eta^2 \ell^3}{\pi E h r^5},$$

whereas

$$2L = \frac{9\ell\rho}{2\pi r^2}.$$

Thus

$$64\eta^2 \ell^2 \geq 3\rho E h r^3$$

or, equivalently,

$$h \leq h_{\text{crit}} = \frac{64\eta^2 \ell^2}{3\rho E r^3}.$$

For attenuation at order ω^2 , replace these inequalities by strict ones. At zero frequency, the pressure gain is exactly unity.

A.4

At level i ,

$$h_{\text{crit},i} = \frac{64\eta^2 \ell_i^2}{3\rho E r_i^3} = \frac{64\eta^2 \ell_0^2}{3\rho E r_0^3} 2^{i/3}.$$

This bound increases with level number. Since the same wall thickness must satisfy the condition at every level, the strongest constraint is imposed by level 0.

Using $E = 0.06 \times 10^6$ Pa,

$$h_{\text{max}} = \frac{64(3.5 \times 10^{-3})^2 (2.0 \times 10^{-3})^2}{3(1.05 \times 10^3)(0.06 \times 10^6)(6.0 \times 10^{-5})^3} = 7.6818 \times 10^{-5} \text{ m}.$$

Thus, within the given model,

$$h_{\text{max}} \simeq 7.7 \times 10^{-5} \text{ m} = 77 \mu\text{m}.$$

A thickness below this limiting value gives the strict leading-order attenuation condition throughout the network.

B.1

The total tissue volume V is fixed. Initially it contains normal tissue at density ρ_0 , so conservation of normal-cell mass gives

$$M_N = \rho_0 V = \rho_N (V - V_T).$$

Writing $v = V_T/V$ and using equation (3),

$$1 = \left(1 + \frac{p}{K_N}\right)(1 - v).$$

Hence the pressure is related to tumour volume by

$$p = \frac{K_N v}{1 - v}.$$

The tumour mass is

$$M_T = \rho_0 \left(1 + \frac{p}{K_T}\right) V_T.$$

Dividing by $M_N = \rho_0 V$,

$$\mu = v \left(1 + \frac{p}{K_T}\right) = v \left(1 + \frac{\kappa v}{1 - v}\right).$$

Rearranging gives a quadratic equation for v :

$$(1 - \kappa)v^2 - (1 + \mu)v + \mu = 0.$$

The physical root approaches zero when μ approaches zero. For $\kappa \neq 1$, it is

$$v = \frac{1 + \mu - \sqrt{(1 + \mu)^2 - 4(1 - \kappa)\mu}}{2(1 - \kappa)}.$$

Rationalizing yields a form that also applies directly when $\kappa = 1$:

$$v = \frac{2\mu}{1 + \mu + \sqrt{(1 - \mu)^2 + 4\kappa\mu}}.$$

As a check, for equal bulk moduli the two tissues have equal densities, giving $v = \mu/(1 + \mu)$, as this expression predicts.

B.2

Let T_b be the body temperature far from the tumour. In the stationary state, the outward heat current through a sphere equals the total power generated inside it. With outward heat flow taken as positive, Fourier's law is

$$\dot{H}(r) = -4\pi r^2 k \frac{dT}{dr}.$$

Inside the tumour, $0 \leq r \leq R_T$, uniform heat generation gives

$$-4\pi r^2 k \frac{dT}{dr} = \frac{4\pi r^3}{3} \mathcal{P},$$

so

$$\frac{dT}{dr} = -\frac{\mathcal{P}r}{3k}.$$

Outside the tumour, the entire generated power crosses every spherical surface:

$$-4\pi r^2 k \frac{dT}{dr} = \frac{4\pi R_T^3}{3} \mathcal{P}.$$

Integrating with $T(r) \rightarrow T_b$ as $r \rightarrow \infty$,

$$T(r) = T_b + \frac{\mathcal{P}R_T^3}{3kr}, \quad r \geq R_T.$$

Thus the temperature at the tumour surface is

$$T(R_T) = T_b + \frac{\mathcal{P}R_T^2}{3k}.$$

Temperature is continuous at the surface. Integrating the interior gradient inward from that surface gives

$$T(r) = T(R_T) + \frac{\mathcal{P}}{6k}(R_T^2 - r^2) = T_b + \frac{\mathcal{P}}{6k}(3R_T^2 - r^2).$$

Consequently,

$$\boxed{T(0) = T_b + \frac{\mathcal{P}R_T^2}{2k}}.$$

The temperature decreases quadratically inside the tumour and approaches T_b as $1/r$ outside it.

B.3

The temperature decreases with distance from the centre, so the coldest tumour cells are at its surface. Heating every tumour cell above 43.0°C therefore requires

$$T(R_T) = T_b + \frac{\mathcal{P}R_T^2}{3k} > 43.0^\circ\text{C}.$$

The threshold power density is

$$\mathcal{P}_{\min} = \frac{3k(43.0 - 37.0) \text{ K}}{R_T^2}.$$

Substituting $R_T = 5.0 \times 10^{-2} \text{ m}$,

$$\mathcal{P}_{\min} = \frac{3(0.60)(6.0)}{(5.0 \times 10^{-2})^2} = 4.32 \times 10^3 \text{ W m}^{-3}.$$

Thus

$$\boxed{\mathcal{P}_{\min} \simeq 4.3 \times 10^3 \text{ W m}^{-3}}.$$

At this threshold the surface is exactly at 43.0°C , while the centre is at 46.0°C . Strictly exceeding the specified temperature everywhere requires $\mathcal{P} > 4.32 \times 10^3 \text{ W m}^{-3}$.

B.4

First linearize the pressure–radius relation. Equation (4) can be written as

$$p - P_{\text{cap}} = (p_c - P_{\text{cap}}) \left[2 \frac{\delta r}{\delta r_c} - \left(\frac{\delta r}{\delta r_c} \right)^2 \right].$$

On the branch connected to the uncompressed state, sufficiently close to $p = P_{\text{cap}}$,

$$\delta r \simeq \frac{\delta r_c}{2(p_c - P_{\text{cap}})} (p - P_{\text{cap}}).$$

Only the terminal vessels change radius. Set

$$\epsilon = \frac{\delta r}{r_{N-1}} \ll 1.$$

Their lengths remain unchanged, so Poiseuille's law gives

$$R'_{N-1} = R_{N-1}(1 - \epsilon)^{-4} \simeq R_{N-1}(1 + 4\epsilon).$$

From **A.1**, the 2^{N-1} terminal vessels together initially contribute R_0 to the network resistance. The other $N - 1$ levels remain unchanged. Thus

$$R'_{\text{eq}} = (N - 1)R_0 + R_0(1 - \epsilon)^{-4} \simeq NR_0 + 4R_0\epsilon.$$

For fixed inlet and capillary pressures, the total flow is inversely proportional to R_{eq} . Symmetry still makes the terminal flow equal to the total flow divided by 2^{N-1} , so its fractional decrease is the same as that of the total flow:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq \frac{R'_{\text{eq}} - R_{\text{eq}}}{R_{\text{eq}}} = \frac{4}{N} \frac{\delta r}{r_{N-1}}.$$

The factor $1/N$ accounts for the fact that only one of the N equal resistance contributions changes.

Substituting the linearized radius change,

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq \frac{2\delta r_c}{Nr_{N-1}} \frac{p - P_{\text{cap}}}{p_c - P_{\text{cap}}}.$$

Finally, normal-cell mass conservation from **B.1** gives $p = K_N v / (1 - v)$. Therefore,

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq \frac{2\delta r_c}{Nr_{N-1}(p_c - P_{\text{cap}})} \left(\frac{K_N v}{1 - v} - P_{\text{cap}} \right).$$

This expression applies just above the onset volume fraction

$$v_* = \frac{P_{\text{cap}}}{K_N + P_{\text{cap}}},$$

with $(p - P_{\text{cap}})/(p_c - P_{\text{cap}}) \ll 1$ and $\delta r/r_{N-1} \ll 1$. Below the onset, the stated model does not prescribe vessel narrowing.

The solution was checked against the official solution.