

IPhO 2017

Theory problems and solutions · Yogyakarta, Indonesia · 3 theory problems with complete solutions

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Problem 1: Dark Matter

The first formal inference for the existence of dark matter was provided by Fritz Zwicky based on his observation on the dynamics of the Coma galaxy cluster, a cluster of galaxies that consist of about a thousand of galaxies. Zwicky used the Virial theorem to estimate the mass of the galaxy cluster. In a simple sun-planet system, where the planet revolves around the sun in a circular orbit, the Virial theorem states that the kinetic energy of the planet is exactly related to its gravitational potential energy. While in a general case for a system of many particles bounded by some interaction, the Virial theorem, relates the time average total kinetic energy with its time average total potential energy.

In 1933, based on his observation on the velocity of the galaxies near the edge of the Coma galaxy cluster, Zwicky estimated that the cluster has more mass than what was visually observed (i.e. the galaxies). The gravitational attraction from the observable matter (the galaxies) was too small to account for the velocities of the galaxies. Thus there must be some hidden masses that account for such a large velocity. That hidden mass is the dark matter mass. In what follows, assume that the mass of each galaxy is the sum of its visible mass and the mass of the dark matter which moves together with that galaxy, and dark matter interacts with visible matter only gravitationally.

A. Cluster of Galaxies

Consider a cluster of galaxies consist of a large number N of galaxies and dark matter that distributed homogeneously in a sphere of radius R with the total mass (galaxies and dark matters) of the cluster M . Assume that the average total mass (visible and dark matter) of a galaxy is m .

A.1	Assuming a continuous distribution of matter in the cluster, find the total gravitational potential energy of the cluster, in terms of M and R .	1.0 pt.
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Due to cosmological expansion, any distance object will be moving away from an observer on Earth with a speed that depends on the distance from the observer to the object. A certain Lyman (a hydrogen emission spectral line) frequency from a type IA supernova on the i -th galaxy in the galaxy cluster is observed to be f_i , with $i = 1, \dots, N$, while the same corresponding Lyman frequency on Earth is f_0 .

A.2	Determine the average speed V_{cr} of the whole galaxy cluster moving away from the Earth in terms of f_i (with $i = 1, \dots, N$), f_0 and N . Note that a galaxy's speed is very small compared to the speed of light c .	0.5 pt.
A.3	By assuming that the galaxies velocities with respect to the center of the cluster are isotropic (the same in all direction), determine the root-mean square speed v_{rms} of the	1.5 pt.

galaxies with respect to the center of the cluster in terms of N , f_i (with $i = 1, \dots, N$), and f_0 . From this result determine the mean kinetic energy of a galaxy with respect to the center of the cluster in terms of v_{rms} and m .

To find the total mass of the cluster, one can use Virial theorem. The theorem stated that for a system of particles bounded by their conservative force,

$$\langle K \rangle_t = -\gamma \langle U \rangle_t,$$

where $\langle K \rangle_t$ is the time average total kinetic energy, $\langle U \rangle_t$ is the time average total potential energy, and γ is a constant. This theorem can be derived by assuming that for a system of particles bounded by its own interaction, the magnitudes of the position and momentum of each particle are finite, and thus the following quantity

$$\Gamma = \sum_i \vec{p}_i \cdot \vec{r}_i$$

is finite.

A.4	Using the fact that the time average over a long period of time of $d\Gamma/dt$ vanishes, $\langle d\Gamma/dt \rangle_t = 0$, determine γ in the Virial theorem above for the case of gravitational interaction. (Hint: Try to do the problem with the summation in Γ for a small finite number of galaxies).	1.7 pt.
A.5	From the previous results determine the total dark matter mass of the cluster in terms of N , m_g , R , and v_{rms} , where m_g is the average total visible mass of a galaxy. Note that the root-mean-square speed of the dark matter is the same as that of the galaxies.	0.5 pt.

B. Dark Matter in a Galaxy

Dark matter also exists inside and around a galaxy. Consider a spherical galaxy with a visible edge radius R_g (an approximate outermost distance where a large number of stars still visible, but note that a very small number of stars may still be distributed in the region beyond R_g). Assume that the stars in the galaxy are point particles with an average mass m_s . The stars in the galaxy, distributed homogeneously with a number density n , are assumed to move in circular orbits.

B.1	If the galaxy consists only of stars, find the speed $v(r)$ of a star as a function of its distance to the center of the galaxy and sketch $v(r)$ for $r < R_g$ and $r \geq R_g$.	0.8 pt.
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The existence of dark matter can be inferred from the galaxy rotation curve, which is a plot of $v(r)$ obtained from observations. The figure below shows a common pattern of the galaxy rotation curve. You may assume simplifyingly that $v(r)$ is a linear function for $r \leq R_g$ and a constant v_0 for $r > R_g$.

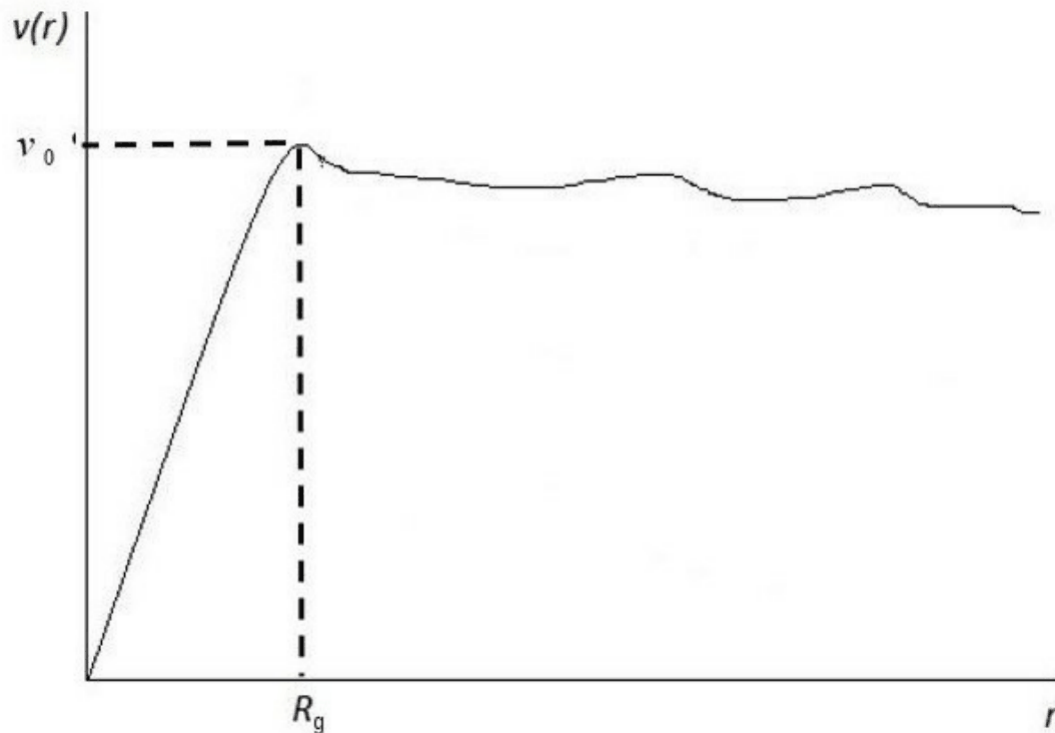


Fig. 1 Plot of galaxy rotation curve in a galaxy.

B.2	Find the total mass m_R of that part of the galaxy which lies within the sphere of radius R_g in terms of v_0 and R_g .	0.5 pt.
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The discrepancy between the figure in B.2. and the plot obtained in B.1 indicates the existence of dark matter.

B.3	Determine the dark matter mass density as a function of r , R_g , v_0 , n , and m_s for $r < R_g$ and $r \geq R_g$.	1.5 pt.
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C. Interstellar Gas and Dark Matter

Now consider a young galaxy whose mass is dominated by interstellar gas and dark matters (neglect the mass of the stars). The interstellar gas is assumed to consist of identical particles of mass m_p . The number density $n(r)$ and temperature $T(r)$ of the gas depend on the distance from the center of the galaxy r . Although many physical processes happen in the gas, we can assume the gas is in a hydrostatic equilibrium due to its pressure and the galaxy gravitational attraction.

C.1	Find the pressure gradient of the gas dP/dr , in terms of $m'(r)$, r and $n(r)$. Here, $m'(r)$ is the total mass of gas and dark matters within a sphere of radius r from the galaxy center.	0.5 pt.
C.2	Assuming the interstellar gas as an ideal gas, find $m'(r)$ in terms of $n(r)$, $T(r)$ and their derivatives with respect to r .	0.5 pt.

Next for simplicity assume that the gas is in isothermal distribution at temperature T_0 and the interstellar gas number density is given by

$$n(r) = \frac{\alpha}{r(\beta + r)^2},$$

where α and β are some constants.

C.3 Find the dark matter mass density as a function of r inside the galaxy. 1.0 pt.

Official English text of the problem.

Answers

Let

$$\bar{f} = \frac{1}{N} \sum_{i=1}^N f_i.$$

Here G is the gravitational constant and k_B is Boltzmann's constant. The Doppler results retain first order in speed divided by c .

• A.1

$$U = -\frac{3GM^2}{5R}.$$

• A.2

$$V_{cr} \simeq c \left(1 - \frac{\bar{f}}{f_0} \right).$$

• A.3

$$v_{rms} \simeq \frac{c}{f_0} \sqrt{\frac{3}{N} \sum_{i=1}^N (f_i - \bar{f})^2}, \quad \langle K_{gal} \rangle = \frac{1}{2} m v_{rms}^2.$$

• A.4

$$\gamma = \frac{1}{2}.$$

• A.5

$$M_{DM} = \frac{5Rv_{rms}^2}{3G} - Nm_g.$$

• B.1

$$v(r) = \begin{cases} r \sqrt{\frac{4\pi G n m_s}{3}}, & 0 \leq r < R_g, \\ \sqrt{\frac{4\pi G n m_s R_g^3}{3r}}, & r \geq R_g. \end{cases}$$

The graph rises linearly from the origin to its maximum at R_g , then decreases as $r^{-1/2}$, approaching zero.

• B.2

$$m_R = \frac{v_0^2 R_g}{G}.$$

• B.3

$$\rho_{\text{DM}}(r) = \begin{cases} \frac{3v_0^2}{4\pi G R_g^2} - nm_s, & r < R_g, \\ \frac{v_0^2}{4\pi G r^2}, & r \geq R_g. \end{cases}$$

• C.1

$$\frac{dP}{dr} = -\frac{Gm'(r)m_p n(r)}{r^2}.$$

• C.2

$$m'(r) = -\frac{k_B r^2}{G m_p} \left[\frac{dT}{dr} + \frac{T(r)}{n(r)} \frac{dn}{dr} \right].$$

• C.3

$$\rho_{\text{DM}}(r) = \frac{k_B T_0 (\beta^2 + 6\beta r + 3r^2)}{4\pi G m_p r^2 (\beta + r)^2} - \frac{\alpha m_p}{r(\beta + r)^2} \quad (r > 0).$$

Solution

A.1

Choose zero gravitational potential energy when all the mass is dispersed at infinity. The uniform mass density and the mass inside radius r are

$$\rho = \frac{3M}{4\pi R^3}, \quad M(r) = \frac{4\pi}{3} \rho r^3 = M \frac{r^3}{R^3}.$$

Assemble the sphere by bringing successive thin shells from infinity. A shell of mass

$$dM = 4\pi r^2 \rho dr = \frac{3Mr^2}{R^3} dr$$

has interaction energy with the previously assembled interior mass

$$dU = -\frac{GM(r)}{r} dM.$$

This construction counts every pair interaction once. Therefore,

$$U = -\int_0^R \frac{G}{r} \left(\frac{Mr^3}{R^3} \right) \left(\frac{3Mr^2}{R^3} \right) dr = -\frac{3GM^2}{R^6} \frac{R^5}{5}.$$

Thus,

$$U = -\frac{3GM^2}{5R}.$$

A.2

Let u_i be the line-of-sight velocity of galaxy i relative to Earth, positive for recession. For $u_i \ll c$, the Doppler relation to first order is

$$f_i \simeq f_0 \left(1 - \frac{u_i}{c}\right),$$

so

$$u_i \simeq c \frac{f_0 - f_i}{f_0}.$$

The internal velocities average to zero in the cluster rest frame. Hence the cluster recession speed is the mean of these measured line-of-sight velocities:

$$V_{cr} = \frac{1}{N} \sum_{i=1}^N u_i \simeq \frac{c}{N f_0} \sum_{i=1}^N (f_0 - f_i).$$

Consequently,

$$V_{cr} \simeq c \left(1 - \frac{1}{N f_0} \sum_{i=1}^N f_i\right) = c \left(1 - \frac{\bar{f}}{f_0}\right).$$

A.3

The line-of-sight component of galaxy i 's velocity relative to the cluster centre is

$$w_i = u_i - V_{cr} \simeq -\frac{c}{f_0} (f_i - \bar{f}).$$

For an isotropic velocity distribution, the mean squared velocity is shared equally among three mutually perpendicular directions:

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle.$$

Taking the z -direction along the line of sight gives

$$v_{rms}^2 = \langle v_x^2 + v_y^2 + v_z^2 \rangle = 3 \langle w^2 \rangle = \frac{3}{N} \sum_{i=1}^N w_i^2.$$

Thus,

$$v_{rms} \simeq \frac{c}{f_0} \sqrt{\frac{3}{N} \sum_{i=1}^N (f_i - \bar{f})^2}.$$

Equivalently,

$$v_{rms}^2 \simeq \frac{3c^2}{f_0^2} \left[\frac{1}{N} \sum_{i=1}^N f_i^2 - \left(\frac{1}{N} \sum_{i=1}^N f_i \right)^2 \right].$$

Using the representative total galaxy mass m , the mean kinetic energy in the cluster rest frame is

$$\langle K_{gal} \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m v_{rms}^2.$$

A.4

Work in the centre-of-mass frame. Differentiating the given quantity gives

$$\frac{d\Gamma}{dt} = \sum_i \left(\frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \vec{p}_i \cdot \frac{d\vec{r}_i}{dt} \right) = \sum_i \vec{F}_i \cdot \vec{r}_i + 2K.$$

Let $\vec{F}_{ij}$ be the force on particle i due to particle j . Newton's third law allows the force term to be grouped into unordered pairs:

$$\sum_i \vec{r}_i \cdot \vec{F}_i = \sum_{i < j} \left(\vec{r}_i \cdot \vec{F}_{ij} + \vec{r}_j \cdot \vec{F}_{ji} \right) = \sum_{i < j} (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{ij}.$$

For Newtonian gravity,

$$\vec{F}_{ij} = -Gm_i m_j \frac{\vec{r}_i - \vec{r}_j}{r_{ij}^3}, \quad r_{ij} = |\vec{r}_i - \vec{r}_j|.$$

Therefore,

$$\sum_i \vec{r}_i \cdot \vec{F}_i = - \sum_{i < j} \frac{Gm_i m_j}{r_{ij}} = U,$$

where U is the total pairwise gravitational potential energy. It follows that

$$\frac{d\Gamma}{dt} = 2K + U.$$

Since Γ remains bounded,

$$\left\langle \frac{d\Gamma}{dt} \right\rangle_t = \lim_{\tau \rightarrow \infty} \frac{\Gamma(t + \tau) - \Gamma(t)}{\tau} = 0.$$

Taking the time average consequently yields

$$2\langle K \rangle_t + \langle U \rangle_t = 0, \quad \langle K \rangle_t = -\frac{1}{2}\langle U \rangle_t.$$

Comparison with the stated theorem gives

$$\boxed{\gamma = \frac{1}{2}}.$$

A.5

Treat the cluster as being in stationary virial equilibrium, so its observed bulk properties represent the time averages. Because the dark matter and galaxies have the same rms speed, the total kinetic energy is

$$K = \frac{1}{2} M v_{rms}^2.$$

Combining the virial theorem with A.1,

$$2K = -U \implies M v_{rms}^2 = \frac{3GM^2}{5R}.$$

For $M > 0$,

$$M = \frac{5R v_{rms}^2}{3G}.$$

The total visible mass is Nm_g . Subtracting it from the total mass gives

$$M_{\text{DM}} = \frac{5Rv_{rms}^2}{3G} - Nm_g.$$

B.1

The stellar mass density is

$$\rho_* = nm_s.$$

Neglect the total mass of the sparse stars beyond R_g , while retaining them as test particles whose orbits can trace the outer gravitational field. The enclosed stellar mass is then

$$M_*(r) = \begin{cases} \frac{4\pi}{3}nm_sr^3, & r < R_g, \\ \frac{4\pi}{3}nm_sR_g^3, & r \geq R_g. \end{cases}$$

By the shell theorem, only enclosed mass contributes to the gravitational acceleration at r . Equating this acceleration to the centripetal acceleration gives

$$\frac{v(r)^2}{r} = \frac{GM_*(r)}{r^2}, \quad v(r)^2 = \frac{GM_*(r)}{r}.$$

Thus,

$$v(r) = \begin{cases} r\sqrt{\frac{4\pi Gnm_s}{3}}, & 0 \leq r < R_g, \\ \sqrt{\frac{4\pi Gnm_sR_g^3}{3r}}, & r \geq R_g. \end{cases}$$

For the sketch, define

$$v_g = R_g\sqrt{\frac{4\pi Gnm_s}{3}}.$$

The curve is a straight line from $(0, 0)$ to (R_g, v_g) . It is continuous and has its maximum at that point. Beyond R_g , it follows

$$v(r) = v_g\sqrt{\frac{R_g}{r}},$$

decreasing toward the horizontal asymptote $v = 0$.

B.2

At $r = R_g$, the observed circular speed is v_0 . The circular-orbit condition gives

$$\frac{v_0^2}{R_g} = \frac{Gm_R}{R_g^2}.$$

Therefore, the total enclosed mass, including dark matter, is

$$m_R = \frac{v_0^2 R_g}{G}.$$

B.3

For any spherically symmetric mass distribution traced by circular orbits,

$$M_{\text{enc}}(r) = \frac{rv(r)^2}{G}.$$

The stated approximation to the observed rotation curve is

$$v(r) = \begin{cases} v_0 r / R_g, & r < R_g, \\ v_0, & r \geq R_g. \end{cases}$$

Hence,

$$M_{\text{enc}}(r) = \begin{cases} \frac{v_0^2 r^3}{GR_g^2}, & r < R_g, \\ \frac{v_0^2 r}{G}, & r \geq R_g. \end{cases}$$

A spherical shell has mass $dM_{\text{enc}} = 4\pi r^2 \rho_{\text{tot}} dr$, so

$$\rho_{\text{tot}}(r) = \frac{1}{4\pi r^2} \frac{dM_{\text{enc}}}{dr} = \begin{cases} \frac{3v_0^2}{4\pi GR_g^2}, & r < R_g, \\ \frac{v_0^2}{4\pi Gr^2}, & r \geq R_g. \end{cases}$$

Subtracting the stellar density nm_s inside the visible edge, and neglecting the stellar density outside it, gives

$$\rho_{\text{DM}}(r) = \begin{cases} \frac{3v_0^2}{4\pi GR_g^2} - nm_s, & r < R_g, \\ \frac{v_0^2}{4\pi Gr^2}, & r \geq R_g. \end{cases}$$

The enclosed mass is continuous at R_g , although the density can jump there in this idealized model.

C.1

Here $m'(r)$ denotes the enclosed mass specified in the problem; the prime is part of its notation, not a derivative.

Consider a small gas element of radial thickness dr and cross-sectional area dA . Its mass is

$$dm = m_p n(r) dA dr.$$

The net outward pressure force is

$$dF_P = -\frac{dP}{dr} dA dr.$$

The inward gravitational force has magnitude

$$dF_G = \frac{Gm'(r)}{r^2} m_p n(r) dA dr.$$

Hydrostatic equilibrium requires these forces to balance. Therefore,

$$\boxed{\frac{dP}{dr} = -\frac{Gm'(r)m_p n(r)}{r^2}}.$$

The negative sign expresses the outward decrease of pressure needed to support the gas against inward gravity.

C.2

For an ideal gas,

$$P(r) = n(r)k_B T(r).$$

Differentiating,

$$\frac{dP}{dr} = k_B \left[T(r) \frac{dn}{dr} + n(r) \frac{dT}{dr} \right].$$

Substituting into the hydrostatic equilibrium equation and solving for the enclosed mass gives

$$m'(r) = -\frac{k_B r^2}{Gm_p n(r)} \left[T(r) \frac{dn}{dr} + n(r) \frac{dT}{dr} \right].$$

Thus,

$$\boxed{m'(r) = -\frac{k_B r^2}{Gm_p} \left[\frac{dT}{dr} + \frac{T(r)}{n(r)} \frac{dn}{dr} \right]}.$$

An equivalent form is

$$m'(r) = -\frac{k_B T(r) r}{Gm_p} \left[\frac{d \ln n}{d \ln r} + \frac{d \ln T}{d \ln r} \right].$$

C.3

For the isothermal gas, $T(r) = T_0$, so C.2 becomes

$$m'(r) = -\frac{k_B T_0 r^2}{Gm_p} \frac{d \ln n}{dr}.$$

The prescribed number density satisfies

$$\ln n = \ln \alpha - \ln r - 2 \ln(\beta + r),$$

and therefore

$$\frac{d \ln n}{dr} = -\frac{1}{r} - \frac{2}{\beta + r}.$$

The total enclosed mass is consequently

$$m'(r) = \frac{k_B T_0}{Gm_p} \left(r + \frac{2r^2}{\beta + r} \right) = \frac{k_B T_0}{Gm_p} \frac{r(\beta + 3r)}{\beta + r}.$$

Differentiate this enclosed mass:

$$\frac{dm'(r)}{dr} = \frac{k_B T_0}{Gm_p} \frac{\beta^2 + 6\beta r + 3r^2}{(\beta + r)^2}.$$

Thus the total mass density is

$$\rho_{\text{tot}}(r) = \frac{1}{4\pi r^2} \frac{dm'(r)}{dr} = \frac{k_B T_0 (\beta^2 + 6\beta r + 3r^2)}{4\pi Gm_p r^2 (\beta + r)^2}.$$

The gas mass density is

$$\rho_{\text{gas}}(r) = m_p n(r) = \frac{\alpha m_p}{r(\beta + r)^2}.$$

Subtracting it from the total density yields

$$\rho_{\text{DM}}(r) = \frac{k_B T_0 (\beta^2 + 6\beta r + 3r^2)}{4\pi G m_p r^2 (\beta + r)^2} - \frac{\alpha m_p}{r(\beta + r)^2} \quad (r > 0).$$

The solution was checked against the official solution.

Problem 2: Earthquake, Volcano and Tsunami

Indonesia is the supermarket of natural hazards. Almost all of the natural hazards have occurred in Indonesia, such as volcano eruptions, earthquakes and tsunamis.

A. Merapi Volcano Eruption

Merapi volcano in Yogyakarta is one of the most active volcano in Java. Pyroclastic flows are well-known eruption characteristics of the volcano. The pyroclastic flow is a hot mixture of gas and rock which travels away from a volcano. In October 26th, 2010, Merapi showed his explosive character by producing an ash plume that reached 12 km altitude (Fig. 1) and pyroclastic currents displacing more than 20,000 people around the volcano.



(Fig. 1: Pyroclastic cloud during Merapi eruption, Courtesy of Volcanological Office of Yogyakarta, BPPTKG)

(Fig. 1: Pyroclastic cloud during Merapi eruption, Courtesy of Volcanological Office of Yogyakarta, BPPTKG)

Let us look at the causes of the largest eruption of Merapi in 2010. It is known by geophysicists that the influence of the external water into the magma plays an important role to the explosive behavior of volcanic eruptions (hydro magmatic eruptions). Let's assume that we dealt with a volcano as a system that consists of mixture of magmatic particles and water. The volcano vents structures and atmosphere are being boundary of the system. The explosive eruption is considered to be happening in two stages, (1) an instantaneous magma-water interaction, and (2) a system expansion. In the first stage, a mass of magma (m_m) at an absolute temperature (T_m) mixes with a mass of external water (m_w) at an absolute temperature (T_w). The thermal equilibrium is reached almost instantaneously. This interaction can be perceived as a nearly-constant volume process. Latent heat of evaporation of water and latent heat of melting of magma can be neglected.

A.1	Find the equilibrium temperature at the first stage in terms of the masses and heat capacity per unit mass of water c_{Vw} and magma c_{Vm} .	0.5 pt.
A.2	Determine its equilibrium pressure at the first stage by assuming that the mixture can be modeled as ideal gas. Assume that the volume per unit mole of the mixture is v_e .	0.3 pt.

The system expansion (the second stage) can occur through several possibilities, one of which is thermal detonation. Although such process is quite complicated, we can empirically measure the relative velocity of the erupted mixture. The velocity of gas during the eruption depend on the pressure p , the total mass m and the volume V of the mixture in the conduit of a volcano.

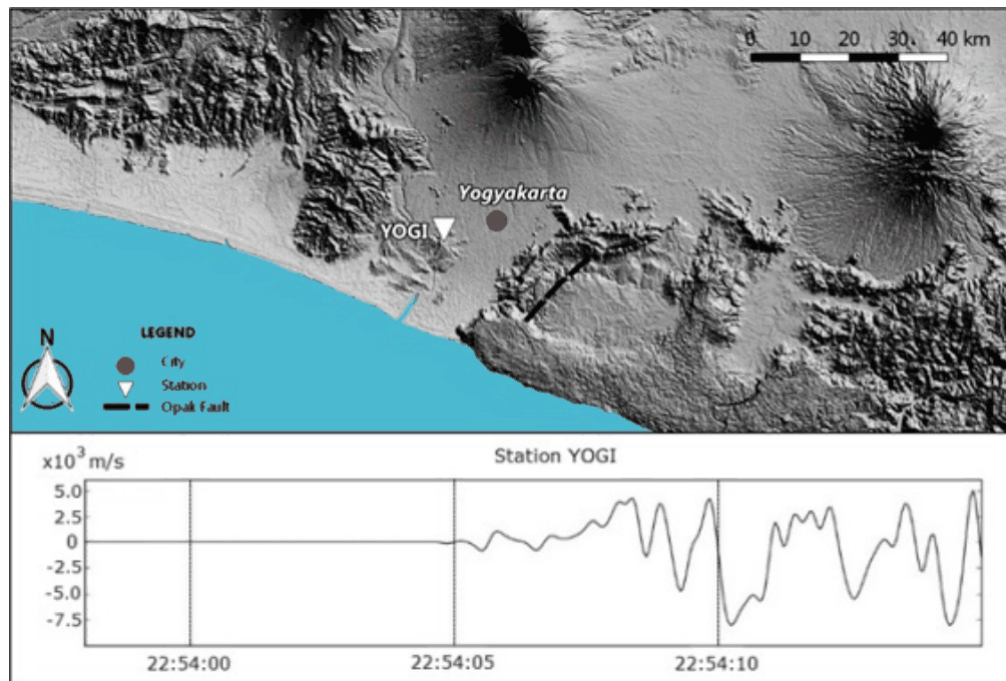
A.3	Express the velocity of gas during the eruption in terms of p , m , and V up to a proportional constant κ .	0.5 pt.
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The observed pressure is around the order of 100 MPa. This makes the eruption (relative) velocity can be as large as ballistic velocity.

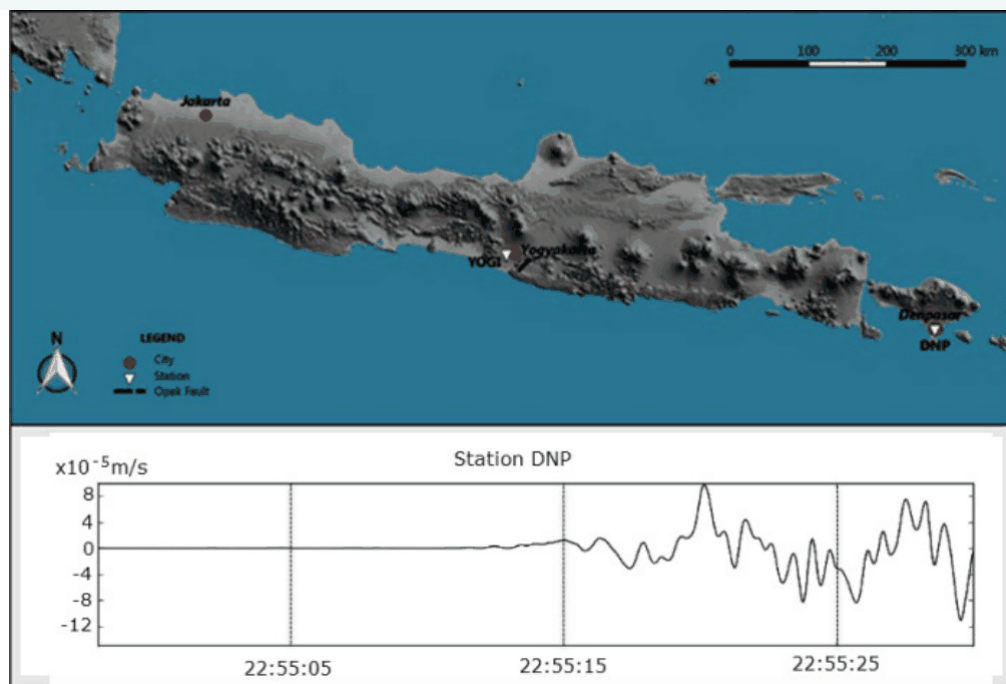
B. The Yogyakarta Earthquake

The 2006 Yogyakarta earthquake of magnitude $M_w = 6.4$, which destroyed many buildings in the Bantul and Yogyakarta area, occurred at 05:54:00.00 local time or 22:54:00.00 UTC. The earthquake was caused by a sudden displacement of the Opak fault segment (see Fig. 2). The hypocenter was located 15 km below the surface.

The seismic wave that propagates on the earth crust can be recorded using seismometer. The diagram from seismometer is called seismogram (Fig. 2 and 3, Lower graph). These seismograms represent the vertical ground velocity as a function of time recorded by the seismic station at Gamping Station Yogyakarta (YOGI) (Fig. 2) and Denpasar, Bali (DNP) (Fig. 3). In general, seismic wave consists of three wave types: the longitudinal or primary (P -wave), the transversal or secondary (S -wave), and the surface wave. The P -wave and S -wave travel in the subsurface while the surface wave travels along the Earth surface. Seismic waves traveling through subsurfaces to the stations can be divided into those that propagate in a straight line, those that are reflected by a layer's boundary, and those that are refracted into the next layer. The longitudinal wave or the primary wave has the highest velocity, while the surface wave has the lowest velocity, around 60% of the P -wave.



(Fig. 2: The maps location of YOGI)



(Fig. 3: The maps location of DNP (Denpasar))

The distance between the epicenter (the projection of hypocenter on the Earth surface) and the YOGI and DNP stations respectively are 22.5 km and 500 km. The depth of the Earth crust layer in Java, Indonesia, is 30 km. Beneath the Earth's crust is the Earth's mantle layer. Just like other wave phenomena, seismic wave also satisfies the Snell's law. The seismic wave can also be reflected by the mantle layer. In this problem we assume that the earth curvature is neglected.

B.1	Fig. 2 shows the seismogram at the YOGI station. Use the data to find the velocity of the P -wave in the Earth crust.	0.5 pt.
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B.2	Find the travel time of the direct P wave and reflected wave due to the Yogyakarta earthquake that arrived at the DNP station in Denpasar.	0.6 pt.
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By assuming that the Earth is composed of only two layers: the crust and the mantle, the primary wave propagates in the crust and in the mantle with different constant velocities. The velocity in the mantle is faster than in the crust. Note that P -wave refracted into the mantle at the right angle (90°) is being partly refracted back into the crust along its entire path of propagation along the crust-mantle boundary.

B.3	Find the velocity of the P -wave in the mantle.	1.2 pt.
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For a more realistic Earth structure, the crust can be divided into a number of thin layers so that the velocity of seismic wave is a function of the depth z according to $v(z) = v_0 + az$ where a is a constant and the hypocenter is approximated on the surface. In this model, the wave ray is curving.

B.4	Let us define the ray parameter $p = \sin \theta(z)/v(z)$, where $\theta(z)$ is the angle between the ray and the normal. Suppose a seismic wave arrives to the station with ray parameter p ; express the distance to the hypocenter in terms of p , v_0 , and a . Assume that the hypocenter is very close to the ground surface.	1.4 pt.
B.5	Find the travel time T from hypocenter to any station, in form of integral over z .	1.0 pt.

The earth consists of a stack of homogeneous layers with the velocity of each layer is v_i and the thickness of δz_i .

B.6	From the result of the previous problem, approximate the travel time (T) from the hypocenter to DNP station by assuming that the crust consists of only three discretized layers, ($i = 1, 2, 3$), characterized by $v_1 = 6.65$ km/sec, $v_2 = 6.97$ km/sec, $v_3 = 6.99$ km/sec, $p = 0.143$ sec/km, $\delta z_1 = 6.0$ km, $\delta z_2 = 9.0$ km, $\delta z_3 = 15$ km.	1.0 pt.
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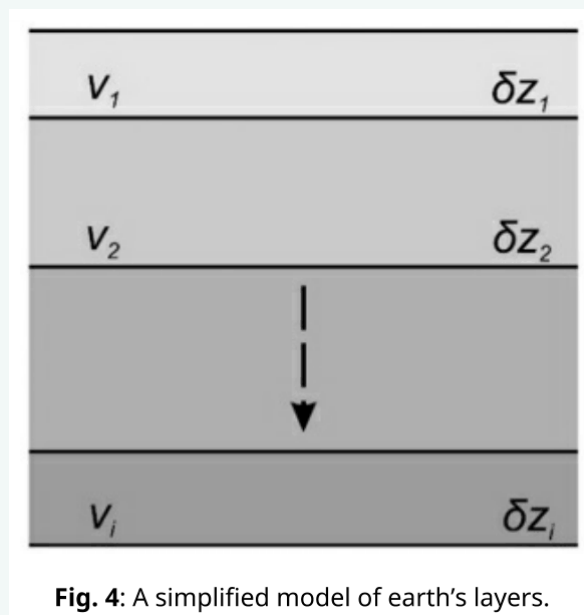


Fig. 4: A simplified model of earth's layers.

Fig. 4: A simplified model of earth's layers.

C. Java Tsunami

The 2006 Pangandaran earthquake and tsunami occurred on July 17 at 15:19:27 local time off the coast of west and central Java. During the earthquake where the epicenter fault is on the ocean floor, the fault may be displaced producing a remarkably large water wave called tsunami. In other words, a tsunami is a shallow-water wave which is initiated by a tiny amplitude, but with an extremely large wavelength. Consider a sudden fault causing a lifting of some ocean floor as shown in Fig. 5. Assume that the energy of the earthquake is transformed to the potential energy of this raised ocean water. For simple model we approximate that the raised water has a geometry of a box with its area of $\lambda L/2$ (where $L \gg \lambda$) and height of h .

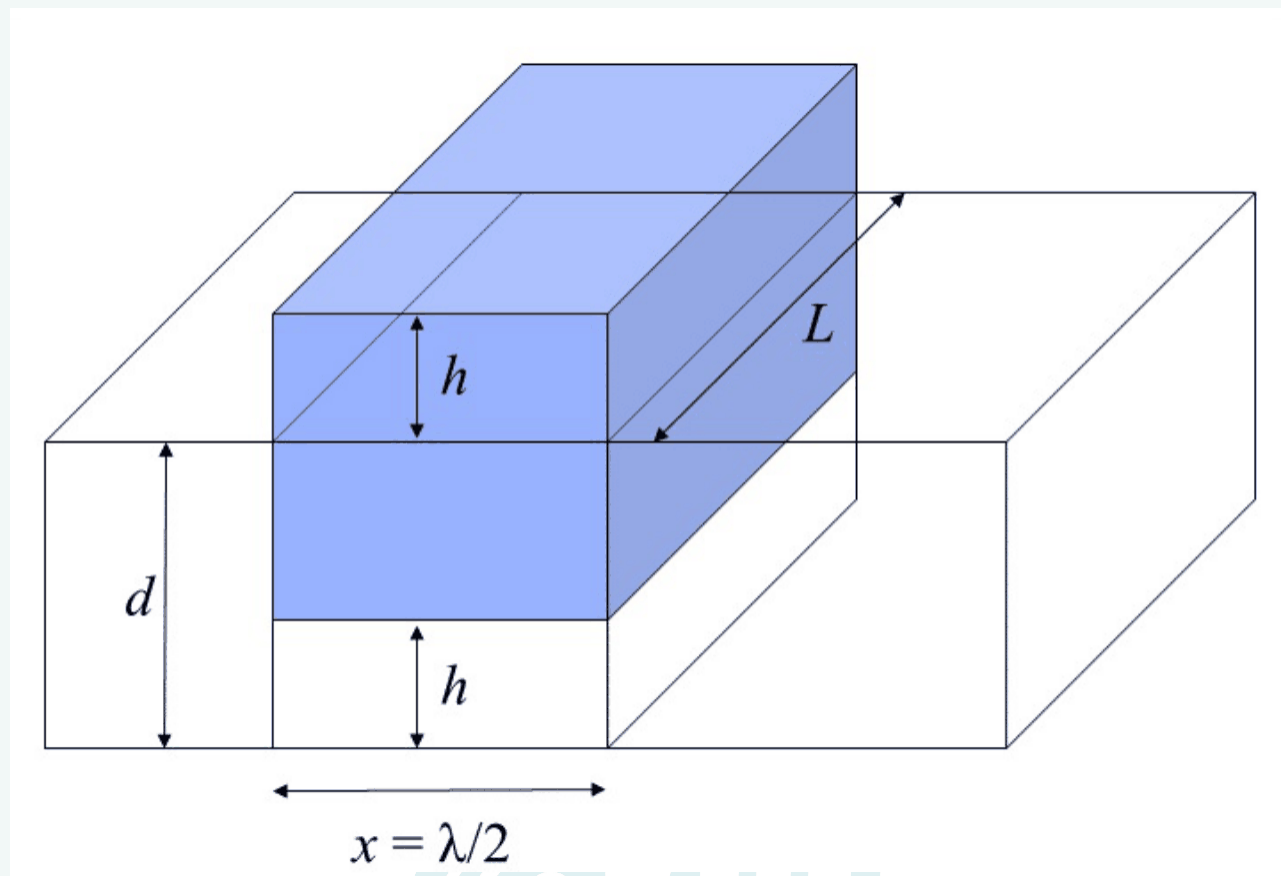


Fig. 5: Illustration for the tsunami wave d is the depth of the ocean.

C.1	Find the potential energy stored in the raised ocean water due to the earthquake with respect to ocean surface. Assume that the density of sea water is ρ .	0.5 pt.
C.2	Find the speed of tsunami wave up to dimensionless factor.	1.2 pt.
C.3	Using energy argument, determine the amplitude of the tsunami wave as a function of the depth, assuming that the depth varies slowly and also knowing that at a depth of d_0 the amplitude is A_0 .	1.3 pt.

Official English text of the problem.

Answers

The seismogram readings used below are approximately 5 s after the earthquake at YOGI and 72 s at DNP. Let v_c and v_m denote the crust and mantle speeds.

- **A.1:** $T_e = \frac{m_m c_{Vm} T_m + m_w c_{Vw} T_w}{m_m c_{Vm} + m_w c_{Vw}}.$
- **A.2:** $p_e = \frac{RT_e}{v_e}.$
- **A.3:** $u = \kappa \sqrt{\frac{pV}{m}}.$
- **B.1:** $v_c \simeq 5.4 \text{ km s}^{-1}.$
- **B.2:** $t_{\text{direct}} \simeq 92.5 \text{ s}, \quad t_{\text{reflected}} \simeq 92.8 \text{ s},$ both approximately 93 s at the precision of the graph.
- **B.3:** $v_m \simeq 7.6 \text{ km s}^{-1}.$
- **B.4:** $X = \frac{2}{ap} \sqrt{1 - p^2 v_0^2},$ assuming $a > 0$ and $0 < p v_0 < 1.$
- **B.5:** $T = 2 \int_0^{z_t} \frac{dz}{v(z) \sqrt{1 - p^2 v(z)^2}}, \quad z_t = \frac{p^{-1} - v_0}{a}.$
- **B.6:** Direct discretization gives $T \simeq 1.84 \times 10^2 \text{ s}.$ This layer sum has a geometric inconsistency with the stated 500 km offset, explained below.
- **C.1:** $U = \frac{1}{4} \rho g \lambda L h^2.$
- **C.2:** $c = C \sqrt{gd},$ where C is dimensionless; linear shallow-water theory gives $C = 1.$
- **C.3:** $A(d) = A_0 \left(\frac{d_0}{d} \right)^{1/4}.$

Solution

A.1 — Equilibrium temperature

During the rapid mixing, take heat exchange with the surroundings and boundary work to be negligible. Neglecting the specified latent heats and treating the heat capacities as constant, conservation of internal energy gives

$$m_m c_{Vm} (T_e - T_m) + m_w c_{Vw} (T_e - T_w) = 0.$$

Thus

$$T_e = \frac{m_m c_{Vm} T_m + m_w c_{Vw} T_w}{m_m c_{Vm} + m_w c_{Vw}}.$$

This is the initial-temperature average weighted by the two heat capacities.

A.2 — Equilibrium pressure

For n moles of the ideal-gas mixture,

$$p_e V = n R T_e.$$

Since its molar volume is $v_e = V/n,$

$$p_e = \frac{RT_e}{v_e} = \frac{R}{v_e} \frac{m_m c_{Vm} T_m + m_w c_{Vw} T_w}{m_m c_{Vm} + m_w c_{Vw}}.$$

Here R is the universal gas constant.

A.3 — Eruption velocity

Assume, as specified, that the velocity depends only on p, m, V . Write

$$u = \kappa p^\alpha m^\beta V^\gamma.$$

Using

$$[p] = ML^{-1}T^{-2}, \quad [m] = M, \quad [V] = L^3,$$

dimensional consistency requires

$$\alpha + \beta = 0, \quad -\alpha + 3\gamma = 1, \quad -2\alpha = -1.$$

Therefore

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}, \quad \gamma = \frac{1}{2},$$

and

$$u = \kappa \sqrt{\frac{pV}{m}}.$$

Equivalently, $u = \kappa \sqrt{p/\rho_{\text{mix}}}$, where $\rho_{\text{mix}} = m/V$. Dimensional analysis does not determine κ .

B.1 — P -wave speed in the crust

Let the source depth be $z_s = 15$ km. The horizontal distance to YOGI is 22.5 km, so the direct source-to-station distance is

$$s_Y = \sqrt{22.5^2 + 15^2} = 27.04 \text{ km}.$$

The first visible departure from the baseline in Fig. 2 is approximately at 22:54:05 UTC. Relative to the earthquake time 22:54:00, this gives

$$t_Y \simeq 5 \text{ s}.$$

Hence

$$v_c = \frac{s_Y}{t_Y} \simeq \frac{27.04}{5} \simeq 5.4 \text{ km s}^{-1}.$$

The numerical calculations below retain 5.4083 km s^{-1} internally, although the graph does not justify that many significant figures.

B.2 — Direct and reflected travel times to DNP

Write the horizontal source-to-DNP distance as $X = 500$ km, and the crust thickness as $H = 30$ km.

For the direct wave,

$$s_{\text{direct}} = \sqrt{X^2 + z_s^2},$$

so

$$t_{\text{direct}} = \frac{\sqrt{500^2 + 15^2}}{v_c} \simeq 92.5 \text{ s}.$$

For reflection at the horizontal crust–mantle boundary, reflect the source across the plane $z = H$. Its image is at depth

$$z_{\text{image}} = 2H - z_s = 45 \text{ km}.$$

Unfolding the reflected ray gives a straight line from this image to DNP. Consequently,

$$s_{\text{reflected}} = \sqrt{X^2 + (2H - z_s)^2},$$

and

$$t_{\text{reflected}} = \frac{\sqrt{500^2 + 45^2}}{v_c} \simeq 92.8 \text{ s}.$$

Both travel times are about 93 s; their calculated difference is approximately 0.33 s.

B.3 — *P*-wave speed in the mantle

The first departure from the flat baseline in Fig. 3 is approximately at 22:55:12 UTC, giving

$$t_D \simeq 72 \text{ s}.$$

This precedes the direct and reflected crustal arrivals calculated above. We identify it with the critically refracted wave that travels along the mantle boundary and radiates back into the crust.

Let θ_c be the critical angle measured from the vertical. Snell's law gives

$$\frac{\sin \theta_c}{v_c} = \frac{1}{v_m}, \quad \sin \theta_c = \frac{v_c}{v_m}.$$

The downward crustal leg spans depth $H - z_s$, and the upward leg spans depth H . Their combined vertical displacement in magnitude is

$$K = (H - z_s) + H = 2H - z_s = 45 \text{ km}.$$

Their combined length and horizontal projection are therefore

$$s_{\text{crust}} = \frac{K}{\cos \theta_c}, \quad X_{\text{crust}} = K \tan \theta_c.$$

The distance travelled along the mantle boundary is

$$X_{\text{mantle}} = X - K \tan \theta_c.$$

Thus

$$t_D = \frac{K}{v_c \cos \theta_c} + \frac{X - K \tan \theta_c}{v_m}.$$

Using $\sin \theta_c = v_c/v_m$, this simplifies to

$$t_D = \frac{X}{v_m} + \frac{K}{v_c} \sqrt{1 - \frac{v_c^2}{v_m^2}}.$$

Substituting the graph reading,

$$72 = \frac{500}{v_m} + \frac{45}{5.4083} \sqrt{1 - \left(\frac{5.4083}{v_m} \right)^2},$$

with distances in kilometres and speeds in kilometres per second. Solving gives

$$v_m \simeq 7.554 \text{ km s}^{-1},$$

or, at appropriate precision,

$$\boxed{v_m \simeq 7.6 \text{ km s}^{-1}}.$$

As a check, $\theta_c \simeq 45.7^\circ$, and the mantle segment is approximately 454 km long. The crustal legs take about 11.9 s, and the mantle segment about 60.1 s.

B.4 — Distance for the continuously varying velocity

Take z positive downward and x horizontal. Snell's law across the thin horizontal layers implies that

$$p = \frac{\sin \theta(z)}{v(z)}$$

is constant along a ray. Therefore

$$\sin \theta(z) = pv(z), \quad \cos \theta(z) = \sqrt{1 - p^2 v(z)^2}.$$

On the downward leg,

$$\frac{dx}{dz} = \tan \theta(z) = \frac{pv(z)}{\sqrt{1 - p^2 v(z)^2}}.$$

For a returning ray in a medium with $a > 0$, the deepest point occurs where the ray becomes horizontal:

$$pv(z_t) = 1.$$

Since $v(z) = v_0 + az$,

$$\boxed{z_t = \frac{p^{-1} - v_0}{a}}.$$

With both source and receiver approximated as lying on the surface, the downward and upward portions are symmetric. The horizontal distance is

$$X = 2 \int_0^{z_t} \frac{p(v_0 + az)}{\sqrt{1 - p^2(v_0 + az)^2}} dz.$$

Set $u = p(v_0 + az)$, so $du = pa \, dz$. Then

$$X = \frac{2}{ap} \int_{pv_0}^1 \frac{u}{\sqrt{1 - u^2}} du.$$

Since the antiderivative is $-\sqrt{1 - u^2}$,

$$\boxed{X = \frac{2}{ap} \sqrt{1 - p^2 v_0^2} = \frac{2}{a} \sqrt{\frac{1}{p^2} - v_0^2}}.$$

The returning-ray solution requires $0 < pv_0 < 1$.

B.5 — Travel-time integral

A ray segment of length ds takes time

$$dt = \frac{ds}{v(z)}.$$

Since $|dz| = ds \cos \theta(z)$,

$$dt = \frac{|dz|}{v(z)\sqrt{1 - p^2 v(z)^2}}.$$

Including both the downward and upward legs gives

$$T = 2 \int_0^{z_t} \frac{dz}{v(z)\sqrt{1 - p^2 v(z)^2}}.$$

For the linear velocity profile, this is

$$T = 2 \int_0^{(p^{-1} - v_0)/a} \frac{dz}{(v_0 + az)\sqrt{1 - p^2(v_0 + az)^2}}.$$

The singularity at the turning point is integrable. Evaluating the integral also gives

$$T = \frac{2}{a} \ln \left(\frac{1 + \sqrt{1 - p^2 v_0^2}}{p v_0} \right).$$

An equivalent form, useful for examining the numerical approximation in B.6, follows by subtracting pX :

$$\begin{aligned} T - pX &= 2 \int_0^{z_t} \left[\frac{1}{v\sqrt{1 - p^2 v^2}} - \frac{p^2 v}{\sqrt{1 - p^2 v^2}} \right] dz \\ &= 2 \int_0^{z_t} \sqrt{\frac{1}{v(z)^2} - p^2} dz. \end{aligned}$$

Hence, for a geometrically consistent ray,

$$T = pX + 2 \int_0^{z_t} \sqrt{\frac{1}{v(z)^2} - p^2} dz.$$

B.6 — Three-layer approximation

Retaining the surface-source approximation from B.4–B.5, discretize the first travel-time integral:

$$T \approx 2 \sum_{i=1}^3 \frac{\delta z_i}{v_i \sqrt{1 - p^2 v_i^2}}.$$

The factor of two accounts for passage through each layer on the downward and upward legs.

Using the numbers exactly as printed gives:

Layer i	pv_i	Two-leg contribution $\frac{2\delta z_i}{v_i \sqrt{1 - p^2 v_i^2}}$
1	0.95095	5.833 s
2	0.99671	31.863 s
3	0.99957	146.366 s

Thus the direct layer quadrature gives

$$T \approx 184.06 \text{ s} \simeq 1.84 \times 10^2 \text{ s}.$$

There is a consistency limitation in interpreting this number as the travel time to DNP. The horizontal distance of the same discrete ray would be

$$X_{\text{layers}} = 2 \sum_{i=1}^3 \frac{\delta z_i pv_i}{\sqrt{1 - p^2 v_i^2}} \simeq 1280.9 \text{ km},$$

which differs substantially from 500 km.

If instead one inserts the prescribed 500 km offset into the second form of B.5 and approximates its integral, one obtains the formal estimate

$$\begin{aligned} T &\approx p(500) + 2 \sum_{i=1}^3 \delta z_i \sqrt{\frac{1}{v_i^2} - p^2} \\ &= 71.5 + 0.8934 \\ &\approx 72.39 \text{ s}. \end{aligned}$$

The two continuum formulas are equivalent, but these coarse approximations fail to preserve their required distance relation. Thus 184 s is the direct travel-time layer sum; 72.4 s is a fixed-offset estimate. They cannot both describe the same literal three-layer ray with the supplied parameters.

C.1 — Potential energy of the raised water

Let the horizontal area of the raised box be

$$S = \frac{\lambda L}{2}.$$

Taking the undisturbed ocean surface as zero gravitational potential, a horizontal slab of thickness dz at height z has mass $\rho S dz$. Consequently,

$$U = \int_0^h \rho S g z dz = \frac{1}{2} \rho S g h^2.$$

Therefore

$$U = \frac{1}{4} \rho g \lambda L h^2.$$

Equivalently, the water above the original surface has mass $\rho S h$, with its centre of mass at height $h/2$.

C.2 — Tsunami speed

Use the shallow-water, small-amplitude approximations

$$\lambda \gg d, \quad h \ll d.$$

The horizontal fluid velocity is then approximately uniform through the depth, and the vertical pressure distribution is approximately hydrostatic.

Let $\eta(x, t)$ be the surface displacement and $u(x, t)$ the horizontal fluid velocity. Conservation of water volume, linearized about depth d , gives

$$\frac{\partial \eta}{\partial t} + d \frac{\partial u}{\partial x} = 0.$$

The horizontal pressure gradient is $\rho g \partial \eta / \partial x$. Newton's second law therefore gives

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}.$$

Differentiating the first equation with respect to time and substituting the second,

$$\frac{\partial^2 \eta}{\partial t^2} = gd \frac{\partial^2 \eta}{\partial x^2}.$$

This is the wave equation, so

$$c = \sqrt{gd}.$$

At the level requested in the question,

$$c = C \sqrt{gd},$$

with a dimensionless factor C ; the stated linear shallow-water assumptions give $C = 1$. The density cancels because it multiplies both inertia and the gravitational restoring force.

C.3 — Amplitude as the depth changes

Assume slow depth variation, negligible reflection and dissipation, and constant wavefront width. Since $L \gg \lambda$, the wave can be treated locally as a plane wave.

For a sinusoidal surface displacement of amplitude A ,

$$\eta = A \cos(kx - \omega t), \quad \langle \eta^2 \rangle = \frac{A^2}{2}.$$

The mean gravitational potential energy per unit horizontal area is

$$\bar{\mathcal{E}}_{\text{pot}} = \frac{1}{2} \rho g \langle \eta^2 \rangle = \frac{1}{4} \rho g A^2.$$

For a progressive shallow-water wave, continuity gives

$$u = \frac{c}{d} \eta.$$

Its mean kinetic energy per unit horizontal area is therefore

$$\bar{\mathcal{E}}_{\text{kin}} = \frac{1}{2} \rho d \langle u^2 \rangle = \frac{\rho c^2 A^2}{4d} = \frac{1}{4} \rho g A^2.$$

Thus the total mean energy density is

$$\bar{\mathcal{E}} = \frac{1}{2} \rho g A^2.$$

Shallow-water waves are nondispersive, so their group velocity equals $c = \sqrt{gd}$. Conservation of energy flux per unit wavefront width gives

$$\bar{\mathcal{E}} c = \text{constant},$$

or

$$A^2 \sqrt{d} = \text{constant}.$$

Using the reference amplitude A_0 at depth d_0 ,

$$A^2 \sqrt{d} = A_0^2 \sqrt{d_0}.$$

Therefore

$$A(d) = A_0 \left(\frac{d_0}{d} \right)^{1/4}.$$

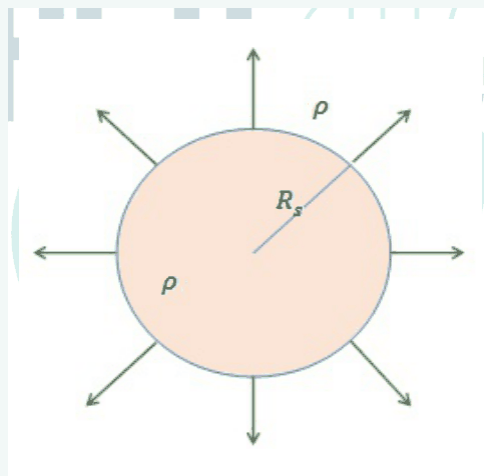
This is Green's law. Its validity requires small amplitude relative to depth and sufficiently slow depth variation; it does not describe the eventual breaking of the wave.

The solution was checked against the official solution.

Problem 3: Cosmic Inflation

Due to the relative movement of galaxies observed from the earth, the wavelength of visible spectrum of a particular galaxy differs from its original wavelength, which is known as the electromagnetic Doppler effect. One expects, for a collection of galaxies, to a random distributions of wavelength shifts: some positive (red shift) and some negative (blue shift). However, observations show that all, except for a nearby group of galaxies, are red shifted. This must be true even if the observation take place on different point in the universe. As a conclusion, our universe must be expanding. On the other hand local irregularity of the universe can be neglected on scales of more than 100 Mpc, in which 1 pc = 3.26 light-years. Averaged over large scales, the clumpy distribution of galaxies becomes more and more isotropic (independent of direction) and homogeneous (independent of position). Therefore we can assume the universe as a matter having a uniform mass density ρ and is expanding.

A. Expansion of Universe



For a simple model of our universe, let us consider an expanding uniform-density sphere embedded in a medium of a much larger sphere with the same density. Let say at some time, the radius of the smaller sphere is R_s . To express the expansion of the sphere, the time dependency of the radius $R(t)$ can be expressed by scale factor $a(t)$, that is $R(t) = a(t)R_s$.

Using Newton's law of gravity to evaluate velocity of a mass element on the sphere boundary according the model of our universe, one can obtain a set of Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = A_1 \rho(t) - \frac{kc^2}{R_s^2 a^2(t)} \quad (1)$$

where k a dimensionless constant, and c is velocity of light.

Part	Question	Points
A.1	Determine the constant A_1 in the equation (1)	1.3 pt.

The discussion so far is non-relativistic. But in fact, it can be extended to a relativistic system by reinterpreting $\rho(t)c^2$ as total energy density (excluding the gravitational potential energy). In this relativistic system derives the 2nd Friedmann equation:

$$\dot{\rho} + A_2 \left(\rho + \left(\frac{p}{c^2} \right) \right) \frac{\dot{a}}{a} = 0 \quad (2)$$

using the 1st law thermodynamics of an adiabatic system, where p denotes the pressure on the sphere.

Part	Question	Points
A.2	Determine the constants A_2 in the equation (2)	0.9 pt.

To solve Eqs. (1) and (2), one should assume a relation $p = p(\rho)$, such as $p(t)/c^2 = w\rho(t)$, where w is a constant. There is also a factor $H = \dot{a}/a$ being called Hubble parameter. The present values of parameters are usually symbolized by subscript 0 such as t_0, ρ_0, H_0, a_0 and so on. For simplicity, we take $a_0 = 1$.

Universe is believed to start from a big explosion called Big-Bang that produces radiation of relativistic particles. During its expansion, the universe is cooling down and the particles in it become non-relativistic. However, the recent observations clarify that the present universe is dominated by cosmological constant energy density. For the case of photon, as the universe is expanding, the photon's wavelength expands proportionally to the scale factor.

Part	Question	Points
A.3	For each of the following three cases determine the resulting value of w : (i) a universe filled only with radiation (i.e. photon energy), (ii) a universe filled only with non-relativistic matter and (iii) a universe with constant energy density.	1.2 pt.
A.4	In the case of $k = 0$, find $a(t)$ for each case of (i) to (iii) being mentioned in A.3. Use the initial condition $a(t = 0) = 0$ for case (i) and (ii), and use the condition $a_0 = 1$ for case (iii).	1.2 pt.

Constant k in Eq. (1) refers to classification of spatial geometry of the universe. Its value can be $k = +1$ for positive-curvature universe (closed), $k = 0$ for flat universe (infinite), and $k = -1$ for negative-curvature universe (open, infinite). Let define a density ratio $\Omega = \rho/\rho_c$, where $\rho_c c^2 = H^2/A_1$ is critical energy density. Note that A_1 is obtained from problem A.1.

Part	Question	Points
A.5	Express k in Eq.(1) in terms of Ω , H , a and R_0 .	0.1 pt.
A.6	Find a range for Ω that corresponds to each value of $k = +1$, $k = 0$ and $k = -1$.	0.3 pt.

B. Motivation To Introduce Inflation Phase and Its General Conditions

The observation of cosmic microwave background radiation (CMB) suggests that our present universe is approximately flat. The problem is that if this is true the present universe should start from exactly flat early universe, otherwise any deviation from the flatness will eventually grow over time and spoil the present flatness.

Part	Question	Points
B.1	Find $(\Omega(t) - 1)$ as a function of time for the universe when it is dominated by radiation or non-relativistic matter (see problem A.3).	0.5 pt.

To solve the problem, at some early time in its history, the universe should undergo a constant energy density domination period which leads to an exponential expansion so called inflation period.

Part	Question	Points
B.2	For this constant energy density domination period, find $(\Omega(t) - 1)$ as a function of time. Assume that $(\Omega(t) - 1) \ll 1$.	0.3 pt.
B.3	Show that condition for inflation implies several following conditions: negative pressure, accelerated expansion ($\ddot{a} > 0$), and decreasing Hubble radius ($d(aH)^{-1}/dt < 0$).	0.9 pt.
B.4	Show that the condition of decreasing Hubble radius can be expressed in terms of parameter $\epsilon = -\dot{H}/H^2$ as $\epsilon < 1$.	0.2 pt.

Inflation occurs as long as $\epsilon < 1$ and then ends when $\epsilon = 1$. We can define e-folding number N , such that $dN = d \ln a = H dt$ and $N = 0$ at the end of inflation.

C. Inflation Generated by Homogenously Distributed Matter

As an example of simple physical system that can generate period of inflation is a universe dominated by homogenously distributed matter. This kind of matter is called inflaton and can be characterized by a function $\phi(t)$.

The dynamical equation of the matter can be expressed as

$$\ddot{\phi} + 3H\dot{\phi} = -V', \quad (3)$$

where $V = V(\phi)$ is a potential function and $V' = \frac{\partial V}{\partial \phi}$. The Hubble parameter satisfies

$$H^2 = \frac{1}{3M_{pl}^2} \left[\frac{1}{2} \dot{\phi}^2 + V \right]. \quad (4)$$

with M_{pl} is a constant called the reduced Planck mass. Inflation phase occurs during domination of potential energy V over kinetic energy $\dot{\phi}^2/2$ for sufficient time such that $\ddot{\phi}$ term in equation (3) can be neglected. This condition is called slow-roll approximation.

The quantities ϵ and $\eta_V = \delta + \epsilon$, where $\delta = -\ddot{\phi}/(H\dot{\phi})$, are called 'slow-roll' parameters.

Part	Question	Points
C.1	Estimate parameter ϵ , parameter η_V , $dN/d\phi$ in terms of potential $V(\phi)$ and its first and second derivative (V' and V'').	1.7 pt.

D. Inflation with A Simple Potential

Predictions of any inflation model should be compared to observational constraints from CMB as follow $n_s = 0.968 \pm 0.006$ and $r < 0.12$, where $r = 16\epsilon$ and $n_s = 1 + 2\eta_V - 6\epsilon$ are evaluated at $\phi = \phi_{start}$ for inflation model being generated by a dominant matter. Assume that potential of matter takes a simple form $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{pl}} \right)^n$ where n is any integer and Λ is a constant.

Part	Question	Points
D.1	Calculate ϕ_{end} at the end of inflation.	0.5 pt.
D.2	Express r and n_s in terms of e-folding number N and integer n . Estimate the value of n that is closest to observational values r and n_s . Take $N = 60$ in your calculation.	0.9 pt.

Official English text of the problem.

Answers

The results below use the dimensionally consistent critical density $\rho_c = H^2/A_1$. Write $\Delta = \Omega - 1$. For D.2, let $\mathcal{N} = \ln(a_{end}/a_{start}) = -N_{start} > 0$; the requested 60 e-folds means $\mathcal{N} = 60$.

- **A.1:** $A_1 = \frac{8\pi G}{3}$.
- **A.2:** $A_2 = 3$.
- **A.3:** $w_{\text{radiation}} = \frac{1}{3}, \quad w_{\text{matter}} = 0, \quad w_{\text{constant density}} = -1$.
- **A.4:** On the expanding branch,

$$\begin{aligned} \text{(i)} \quad & a(t) = (2H_0 t)^{1/2} = (t/t_0)^{1/2}, \\ \text{(ii)} \quad & a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3} = (t/t_0)^{2/3}, \\ \text{(iii)} \quad & a(t) = e^{H_0(t-t_0)}. \end{aligned}$$

Here $H_0^2 = 8\pi G\rho_0/3$, with $H_0 t_0 = 1/2$ in (i) and $2/3$ in (ii).

- **A.5:** $k = (\Omega - 1) \left(\frac{aHR_0}{c} \right)^2$.
- **A.6:** $k = +1 : \Omega > 1; \quad k = 0 : \Omega = 1; \quad k = -1 : \Omega < 1$. For nonnegative density, the last range is $0 \leq \Omega < 1$.
- **B.1:** To leading order near flatness, relative to a reference time t_* ,

$$\Delta(t) \simeq \begin{cases} \Delta_*(t/t_*), & \text{radiation,} \\ \Delta_*(t/t_*)^{2/3}, & \text{non-relativistic matter.} \end{cases}$$

- **B.2:** With $H_\Lambda = \sqrt{8\pi G\rho_\Lambda/3}$,

$$\Delta(t) \simeq \Delta_i e^{-2H_\Lambda(t-t_i)}.$$

- **B.3:** Constant density gives

$$p = -\rho c^2 < 0, \quad \frac{\ddot{a}}{a} = \frac{8\pi G\rho}{3} > 0, \quad \frac{d}{dt}(aH)^{-1} < 0.$$

More generally, accelerated expansion requires $p < -\rho c^2/3$.

- **B.4:**

$$\frac{d}{dt}(aH)^{-1} = \frac{\epsilon - 1}{a} < 0 \quad \Longleftrightarrow \quad \epsilon < 1.$$

- **C.1:** In the slow-roll approximation,

$$\epsilon \simeq \frac{M_{pl}^2}{2} \left(\frac{V'}{V} \right)^2, \quad \eta_V \simeq M_{pl}^2 \frac{V''}{V}, \quad \frac{dN}{d\phi} \simeq -\frac{V}{M_{pl}^2 V'}.$$

- **D.1:** For $n > 0$, on the positive-field branch,

$$\phi_{\text{end}} \simeq \frac{n}{\sqrt{2}} M_{pl}.$$

The formal threshold is $|\phi| \simeq |n| M_{pl} / \sqrt{2}$. Nonpositive n does not give the assumed end of slow-roll inflation in this model.

- **D.2:**

$$r \simeq \frac{16n}{4\mathcal{N} + n}, \quad n_s \simeq 1 - \frac{2(n+2)}{4\mathcal{N} + n}.$$

Equivalently, the denominators are $n - 4N_{\text{start}}$ in the statement's signed convention. At $\mathcal{N} = 60$,

n	r	n_s
1	0.0664	0.9751
2	0.1322	0.9669

If $r < 0.12$ is enforced, $n = 1$ is the closest positive-integer choice. No positive integer satisfies both quoted observational bounds.

Solution

A.1

Consider a mass element on the boundary of a comoving sphere of radius $R = aR_s$. By Newton's shell theorem, its acceleration depends only on the enclosed mass,

$$M = \frac{4\pi}{3} \rho R^3.$$

In the non-relativistic model, this enclosed mass is constant. Newton's equation is

$$\ddot{R} = -\frac{GM}{R^2}.$$

Multiplying by $\dot{R}$ and integrating gives the conserved specific mechanical energy

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = \mathcal{E}.$$

Therefore

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2GM}{R^3} + \frac{2\mathcal{E}}{R^2} = \frac{8\pi G}{3}\rho + \frac{2\mathcal{E}}{a^2 R_s^2}.$$

Since $\dot{R}/R = \dot{a}/a$, comparison with (1), identifying $2\mathcal{E} = -kc^2$, yields

$$\boxed{A_1 = \frac{8\pi G}{3}}.$$

A.2

Let the physical volume of the sphere be

$$\mathcal{V} = \frac{4\pi}{3}(aR_s)^3.$$

Its total internal energy is $U = \rho c^2 \mathcal{V}$. The first law for an adiabatic system gives

$$dU = -p d\mathcal{V}.$$

Thus

$$c^2 \mathcal{V} d\rho + (\rho c^2 + p) d\mathcal{V} = 0.$$

Using $\dot{\mathcal{V}}/\mathcal{V} = 3\dot{a}/a$,

$$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0.$$

Hence

$$\boxed{A_2 = 3}.$$

A.3

For constant w , the continuity equation becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a},$$

so, using $a_0 = 1$,

$$\boxed{\rho = \rho_0 a^{-3(1+w)}}.$$

For radiation, the photon number density decreases as a^{-3} . Each photon has energy hc/λ , and the given wavelength scaling $\lambda \propto a$ makes its energy proportional to a^{-1} . Therefore

$$\rho c^2 \propto a^{-4},$$

which gives

$$3(1+w) = 4, \quad \boxed{w = \frac{1}{3}}.$$

For non-relativistic matter, the energy is dominated by approximately constant rest energy per particle. The number density decreases as a^{-3} , giving

$$\rho \propto a^{-3}, \quad \boxed{w = 0}.$$

This is the pressureless-matter approximation.

For constant energy density, $\dot{\rho} = 0$. In an expanding universe, the continuity equation requires

$$\rho + \frac{p}{c^2} = 0,$$

and hence

$$\boxed{w = -1}.$$

A.4

For $k = 0$,

$$H^2 = \frac{8\pi G}{3}\rho = H_0^2 a^{-3(1+w)}.$$

Choosing the expanding branch,

$$\dot{a} = H_0 a^{1-\frac{3}{2}(1+w)}.$$

For $w \neq -1$, integration with $a(0) = 0$ gives

$$a^{\frac{3}{2}(1+w)} = \frac{3}{2}(1+w)H_0 t.$$

Consequently, for radiation,

$$\boxed{a(t) = (2H_0 t)^{1/2}}.$$

The normalization $a(t_0) = 1$ implies $2H_0 t_0 = 1$, so

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}.$$

For non-relativistic matter,

$$\boxed{a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3}}.$$

Here $3H_0 t_0/2 = 1$, giving

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}.$$

For constant energy density, $H = H_0$ is constant. Integrating $\dot{a}/a = H_0$ and imposing $a(t_0) = 1$,

$$\boxed{a(t) = e^{H_0(t-t_0)}}.$$

A.5

Setting $k = 0$ in (1) determines the critical **mass-equivalent density**:

$$\rho_c = \frac{H^2}{A_1} = \frac{3H^2}{8\pi G}.$$

Thus the printed definition $\rho_c c^2 = H^2/A_1$ is dimensionally inconsistent. The corresponding critical energy density is

$$\rho_c c^2 = \frac{H^2 c^2}{A_1}.$$

Using the consistent definition,

$$\Omega = \frac{A_1 \rho}{H^2}.$$

Divide (1) by H^2 :

$$1 = \Omega - \frac{kc^2}{a^2 R_s^2 H^2}.$$

Since $a_0 = 1$, the present radius is $R_0 = R_s$. Therefore

$$k = (\Omega - 1) \frac{a^2 H^2 R_0^2}{c^2}.$$

A.6

The factor multiplying $\Omega - 1$ in A.5 is positive. Hence k and $\Omega - 1$ have the same sign:

Spatial curvature	Density ratio
$k = +1$, closed	$\Omega > 1$
$k = 0$, flat	$\Omega = 1$
$k = -1$, open	$\Omega < 1$

With nonnegative energy density, the open-universe range is $0 \leq \Omega < 1$.

B.1

From A.5,

$$\Delta(t) \equiv \Omega(t) - 1 = \frac{kc^2}{R_0^2 a^2 H^2} = \frac{kc^2}{R_0^2 \dot{a}^2}.$$

This identity is exact.

To study a small departure from flatness, evaluate $\dot{a}$ using the flat-universe solutions of A.4 to leading order. If

$$a \propto t^\alpha,$$

then

$$\dot{a}^2 \propto t^{2\alpha-2}, \quad \Delta \propto t^{2-2\alpha}.$$

During radiation domination, $\alpha = 1/2$, so

$$\Delta(t) \simeq \Delta_* \frac{t}{t_*}.$$

During matter domination, $\alpha = 2/3$, so

$$\Delta(t) \simeq \Delta_* \left(\frac{t}{t_*} \right)^{2/3}.$$

Thus the magnitude of any small, nonzero departure from flatness grows during either era. These power laws apply while curvature remains a small correction to the expansion.

B.2

Let the constant density be ρ_Λ , and define

$$H_\Lambda^2 = \frac{8\pi G}{3} \rho_\Lambda.$$

For $|\Omega - 1| \ll 1$, the curvature contribution to (1) is small, so

$$H \simeq H_\Lambda, \quad a(t) \simeq a_i e^{H_\Lambda(t-t_i)}.$$

Substitution into

$$\Delta = \frac{kc^2}{R_0^2 a^2 H^2}$$

gives

$$\Delta(t) \simeq \Delta_i e^{-2H_\Lambda(t-t_i)}.$$

Inflation therefore suppresses departures from flatness exponentially.

B.3

Differentiate (1), and use the continuity equation:

$$2H\dot{H} = A_1\dot{\rho} + \frac{2kc^2 H}{R_0^2 a^2},$$

so

$$\dot{H} = -\frac{3A_1}{2} \left(\rho + \frac{p}{c^2} \right) + \frac{kc^2}{R_0^2 a^2}.$$

Adding H^2 from (1) eliminates the curvature term:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{A_1}{2} \left(\rho + \frac{3p}{c^2} \right).$$

Using A.1,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2} \right).$$

For positive density, accelerated expansion requires

$$p < -\frac{1}{3} \rho c^2,$$

which in particular requires negative pressure.

For the constant-density period considered here, A.3 gives $p = -\rho c^2$. Hence

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3} \rho > 0.$$

Finally, $aH = \dot{a}$, so

$$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt} \frac{1}{\dot{a}} = -\frac{\ddot{a}}{\dot{a}^2} < 0.$$

This proves all three requested properties.

The radius appearing in this condition is the **comoving** Hubble radius, $c/(aH)$; the constant c does not affect the derivative's sign.

B.4

Direct differentiation gives

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{a^2H^2} = -\frac{1}{a} \left(1 + \frac{\dot{H}}{H^2} \right).$$

Since $\epsilon = -\dot{H}/H^2$,

$$\boxed{\frac{d}{dt}(aH)^{-1} = \frac{\epsilon - 1}{a}}.$$

Because $a > 0$, the radius decreases exactly when

$$\boxed{\epsilon < 1}.$$

Equivalently,

$$\frac{\ddot{a}}{a} = H^2(1 - \epsilon).$$

C.1

First differentiate (4):

$$2H\dot{H} = \frac{1}{3M_{pl}^2} (\dot{\phi}\ddot{\phi} + V'\dot{\phi}).$$

Equation (3) gives $\ddot{\phi} + V' = -3H\dot{\phi}$, so

$$2H\dot{H} = -\frac{H\dot{\phi}^2}{M_{pl}^2}.$$

Thus

$$\boxed{\dot{H} = -\frac{\dot{\phi}^2}{2M_{pl}^2}}, \quad \epsilon = \frac{\dot{\phi}^2}{2M_{pl}^2H^2}.$$

The slow-roll assumptions give

$$H^2 \simeq \frac{V}{3M_{pl}^2}, \quad 3H\dot{\phi} \simeq -V'.$$

Substituting these into ϵ ,

$$\epsilon \simeq \frac{V'^2}{18M_{pl}^2H^4} = \boxed{\frac{M_{pl}^2}{2} \left(\frac{V'}{V} \right)^2}.$$

To obtain η_V , differentiate the slow-roll field equation:

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} \simeq -V''\dot{\phi}.$$

Dividing by $3H^2\dot{\phi}$,

$$\frac{\dot{H}}{H^2} + \frac{\ddot{\phi}}{H\dot{\phi}} \simeq -\frac{V''}{3H^2}.$$

The left side is $-\epsilon - \delta$. Therefore

$$\eta_V = \delta + \epsilon \simeq \frac{V''}{3H^2} = \boxed{M_{pl}^2 \frac{V''}{V}}.$$

Finally, using the statement's definition $dN = Hdt$,

$$\frac{dN}{d\phi} = \frac{H}{\dot{\phi}} \simeq -\frac{3H^2}{V'} = \boxed{-\frac{V}{M_{pl}^2 V'}}.$$

This expression applies along a rolling, monotonic field trajectory.

D.1

For

$$V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{pl}} \right)^n,$$

the derivative ratios are

$$\frac{V'}{V} = \frac{n}{\phi}, \quad \frac{V''}{V} = \frac{n(n-1)}{\phi^2}.$$

Consequently,

$$\epsilon \simeq \frac{n^2 M_{pl}^2}{2\phi^2}.$$

Extrapolating this slow-roll expression to the end condition $\epsilon = 1$ gives

$$|\phi_{\text{end}}| \simeq \frac{|n|}{\sqrt{2}} M_{pl}.$$

For $n > 0$, take the positive-field branch. The field rolls toward smaller ϕ , so ϵ increases and reaches unity at

$$\boxed{\phi_{\text{end}} \simeq \frac{n}{\sqrt{2}} M_{pl}}.$$

The field has the same units as M_{pl} .

The statement permits any integer n , but the assumed exit restricts the relevant cases:

- For $n = 0$, the potential is constant and $\epsilon \simeq 0$; this model provides no slow-roll exit.
- For $n < 0$ and $\phi > 0$, $V' < 0$, so $\dot{\phi} \simeq -V'/(3H) > 0$. Thus $\epsilon \propto \phi^{-2}$ decreases. The formal $\epsilon = 1$ threshold is an entry into inflation.

The end-field result is an estimate because slow roll ceases to be a controlled approximation as ϵ approaches unity.

D.2

For the monomial potential, C.1 gives

$$\frac{dN}{d\phi} \simeq -\frac{\phi}{nM_{pl}^2}.$$

Integrating from the end of inflation, where $N = 0$,

$$N(\phi) \simeq -\frac{\phi^2 - \phi_{\text{end}}^2}{2nM_{pl}^2}.$$

For $n > 0$, the initial field is larger than the final field, so the statement's N_{start} is negative. Define the positive duration

$$\mathcal{N} \equiv -N_{\text{start}} = \ln \frac{a_{\text{end}}}{a_{\text{start}}}.$$

Then

$$\phi_{\text{start}}^2 \simeq \phi_{\text{end}}^2 + 2nM_{pl}^2\mathcal{N} = \frac{n}{2}(4\mathcal{N} + n)M_{pl}^2.$$

The slow-roll parameters at the start are therefore

$$\epsilon_{\text{start}} \simeq \frac{n}{4\mathcal{N} + n}, \quad \eta_{V,\text{start}} \simeq \frac{2(n-1)}{4\mathcal{N} + n}.$$

Thus

$$r \simeq \frac{16n}{4\mathcal{N} + n}$$

and

$$n_s \simeq 1 + \frac{4(n-1) - 6n}{4\mathcal{N} + n} = 1 - \frac{2(n+2)}{4\mathcal{N} + n}.$$

In the statement's signed convention these become

$$r \simeq \frac{16n}{n - 4N_{\text{start}}}, \quad n_s \simeq 1 - \frac{2(n+2)}{n - 4N_{\text{start}}}.$$

For $\mathcal{N} \gg n$, the familiar leading forms are

$$r \simeq \frac{4n}{\mathcal{N}}, \quad n_s \simeq 1 - \frac{n+2}{2\mathcal{N}}.$$

Interpreting the requested $N = 60$ as a positive duration $\mathcal{N} = 60$, the tensor bound gives

$$\frac{16n}{240 + n} < 0.12 \implies n < \frac{28.8}{15.88} \simeq 1.814.$$

Therefore the only positive integer satisfying this bound is $n = 1$.

The numerical predictions, all dimensionless, are:

n	r	n_s	Comparison with the stated constraints
1	0.0664	0.9751	Satisfies $r < 0.12$; n_s is slightly above 0.974
2	0.1322	0.9669	Satisfies the quoted n_s interval; exceeds the r bound

n	r	n_s	Comparison with the stated constraints
3	0.1975	0.9588	Outside both quoted bounds

Hence

$$n = 1 \quad \text{if the tensor upper bound is enforced.}$$

Its predicted spectral index is about 0.0011 above the upper edge of the quoted interval. The integer $n = 2$ is closer to the central spectral index, but violates the tensor bound. **No positive integer satisfies both quoted bounds simultaneously.**

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.