

IPhO 2016

Theory problems and solutions · Zürich, Switzerland · 3 theory problems with complete solutions

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Problem 1: Two Problems in Mechanics

Please read the general instructions in the separate envelope before you start this problem.

Part A. The Hidden Disk (3.5 points)

We consider a solid wooden cylinder of radius r_1 and thickness h_1 . Somewhere inside the wooden cylinder, the wood has been replaced by a metal disk of radius r_2 and thickness h_2 . The metal disk is placed in such a way that its symmetry axis B is parallel to the symmetry axis S of the wooden cylinder, and is placed at the same distance from the top and bottom face of the wooden cylinder. We denote the distance between S and B by d . The density of wood is ρ_1 , the density of the metal is $\rho_2 > \rho_1$. The total mass of the wooden cylinder and the metal disk inside is M .

In this task, we place the wooden cylinder on the ground so that it can freely roll to the left and right. See Fig. 1 for a side view and a view from the top of the setup.

The goal of this task is to determine the size and the position of the metal disk.

In what follows, when asked to express the result in terms of known quantities, you may always assume that the following are known:

$$r_1, h_1, \rho_1, \rho_2, M. \quad (1)$$

The goal is to determine r_2 , h_2 and d , through indirect measurements.

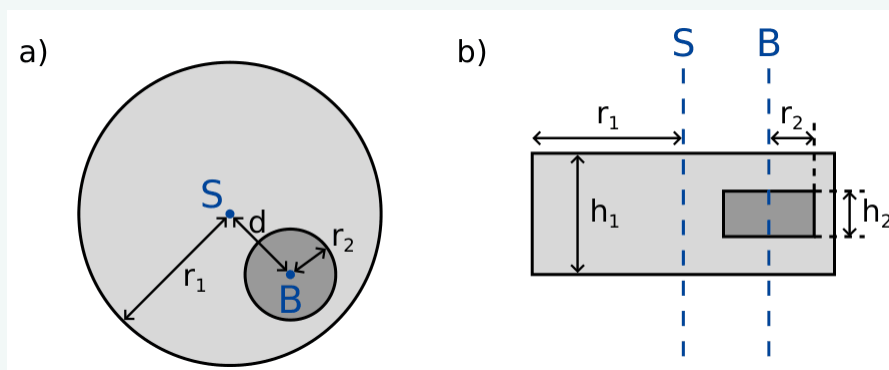


Figure 1: a) side view b) view from above

We denote b as the distance between the centre of mass C of the whole system and the symmetry axis S of the wooden cylinder. In order to determine this distance, we design the following experiment: We place the wooden cylinder on a horizontal base in such a way that it is in a stable equilibrium. Let us now slowly incline the base by an angle Θ (see Fig. 2). As a result of

the static friction, the wooden cylinder can roll freely without sliding. It will roll down the incline a little bit, but then come to rest in a stable equilibrium after rotating by an angle ϕ which we measure.

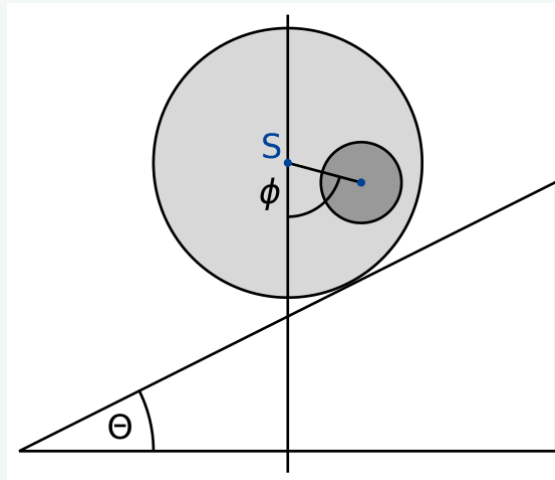


Figure 2: Cylinder on an inclined base.

A.1 — 0.8pt

Find an expression for b as a function of the quantities (1), the angle ϕ and the tilting angle Θ of the base.

From now on, we can assume that the value of b is known.

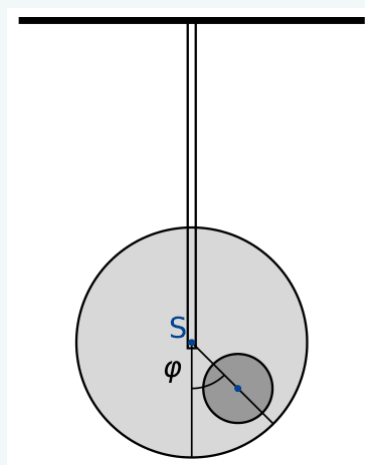


Figure 3: Suspended system.

Next we want to measure the moment of inertia I_S of the system with respect to the symmetry axis S . To this end, we suspend the wooden cylinder at its symmetry axis from a rigid rod. We then turn it away from its equilibrium position by a small angle φ , and let it go. See figure 3 for the setup. We find that φ describes a periodic motion with period T .

A.2 — 0.5pt

Find the equation of motion for φ . Express the moment of inertia I_S of the system around its symmetry axis S in terms of T , b and the known quantities (1). You may assume that we are only disturbing the equilibrium position by a small amount so that φ is always very small.

From the measurements in questions **A.1** and **A.2**, we now want to determine the geometry and the position of the metal disk inside the wooden cylinder.

A.3 — 0.4pt

Find an expression for the distance d as a function of b and the quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

A.4 — 0.7pt

Find an expression for the moment of inertia I_S in terms of b and the known quantities (1). You may also include r_2 and h_2 as variables in your expression, as they will be calculated in subtask **A.5**.

A.5 — 1.1pt

Using all the above results, write down an expression for h_2 and r_2 in terms of b , T and the known quantities (1). You may express h_2 as a function of r_2 .

Part B. Rotating Space Station (6.5 points)

Alice is an astronaut living on a space station. The space station is a gigantic wheel of radius R rotating around its axis, thereby providing artificial gravity for the astronauts. The astronauts live on the inner side of the rim of the wheel. The gravitational attraction of the space station and the curvature of the floor can be ignored.

B.1 — 0.5pt

At what angular frequency ω_{ss} does the space station rotate so that the astronauts experience the same gravity g_E as on the Earth's surface?

Alice and her astronaut friend Bob have an argument. Bob does not believe that they are in fact living in a space station and claims that they are on Earth. Alice wants to prove to Bob that they are living on a rotating space station by using physics. To this end, she attaches a mass m to a spring with spring constant k and lets it oscillate. The mass oscillates only in the vertical direction, and cannot move in the horizontal direction.

B.2 — 0.2pt

Assuming that on Earth gravity is constant with acceleration g_E , what would be the angular oscillation frequency ω_E that a person on Earth would measure?

B.3 — 0.6pt

What angular oscillation frequency ω does Alice measure on the space station?

Alice is convinced that her experiment proves that they are on a rotating space station. Bob remains sceptical. He claims that when taking into account the change in gravity above the surface of the Earth, one finds a similar effect. In the following tasks we investigate whether Bob is right.

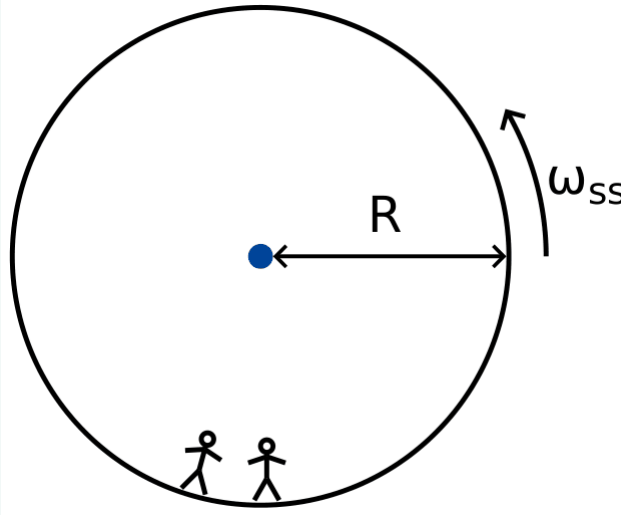


Figure 4: Space station

B.4 — 0.8pt

Derive an expression of the gravity $g_E(h)$ for small heights h above the surface of the Earth and compute the oscillation frequency $\tilde{\omega}_E$ of the oscillating mass (linear approximation is enough). Denote the radius of the Earth by R_E . Neglect the rotation of Earth.

Indeed, for this space station, Alice does find that the spring pendulum oscillates with the frequency that Bob predicted.

B.5 — 0.3pt

For what radius R of the space station does the oscillation frequency ω match the oscillation frequency $\tilde{\omega}_E$ on the Earth? Express your answer in terms of R_E .

Exasperated with Bob's stubbornness, Alice comes up with an experiment to prove her point. To this end she climbs on a tower of height H over the floor of the space station and drops a mass. This experiment can be understood in the rotating reference frame as well as in an inertial reference frame.

In a uniformly rotating reference frame, the astronauts perceive a fictitious force $\vec{F}_C$ called the Coriolis force. The force $\vec{F}_C$ acting on an object of mass m moving at velocity $\vec{v}$ in a rotating frame with constant angular frequency $\vec{\omega}_{ss}$ is given by

$$\vec{F}_C = 2m\vec{v} \times \vec{\omega}_{ss}. \quad (2)$$

In terms of the scalar quantities you may use

$$F_C = 2mv\omega_{ss} \sin \phi, \quad (3)$$

where ϕ is the angle between the velocity and the axis of rotation. The force is perpendicular to both the velocity v and the axis of rotation. The sign of the force can be determined from the right-hand rule, but in what follows you may choose it freely.

B.6 — 1.1pt

Calculate the horizontal velocity v_x and the horizontal displacement d_x (relative to the base of the tower, in the direction perpendicular to the tower) of the mass at the moment it hits the floor. You may assume that the height H of the tower is small, so that the acceleration as measured by the

astronauts is constant during the fall. Also, you may assume that $d_x \ll H$.

To get a good result, Alice decides to conduct this experiment from a much taller tower than before. To her surprise, the mass hits the floor at the base of the tower, so that $d_x = 0$.

B.7 — 1.3pt

Find a lower bound for the height of the tower for which it can happen that $d_x = 0$.

Alice is willing to make one last attempt at convincing Bob. She wants to use her spring oscillator to show the effect of the Coriolis force. To this end she changes the original setup: She attaches her spring to a ring which can slide freely on a horizontal rod in the x direction without any friction. The spring itself oscillates in the y direction. The rod is parallel to the floor and perpendicular to the axis of rotation of the space station. The xy plane is thus perpendicular to the axis of rotation, with the y direction pointing straight towards the center of rotation of the station.

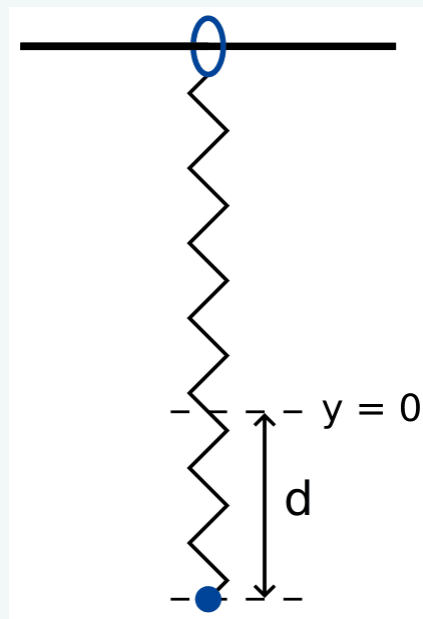


Figure 5: Setup.

B.8 — 1.7pt

Alice pulls the mass a distance d downwards from the equilibrium point $x = 0, y = 0$, and then lets it go (see figure 5).

- Give an algebraic expression of $x(t)$ and $y(t)$. You may assume that $\omega_{ss}d$ is small, and neglect the Coriolis force for motion along the y -axis.
- Sketch the trajectory $(x(t), y(t))$, marking all important features such as amplitude.

Alice and Bob continue to argue.

General Data Sheet

Speed of light in vacuum	c	$= 299\,792\,458 \text{ m} \cdot \text{s}^{-1}$
Vacuum permeability (magnetic constant)	μ_0	$= 4\pi \times 10^{-7} \text{ kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity (electrical constant)	ε_0	$= 8.854\,187\,817 \times 10^{-12} \text{ A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$

Elementary charge	e	$= 1.602\,176\,620\,8(98) \times 10^{-19} \text{ A} \cdot \text{s}$
Mass of the electron	m_e	$= 9.109\,383\,56(11) \times 10^{-31} \text{ kg}$
		$= 0.510\,998\,946\,1(31) \frac{\text{MeV}}{c^2}$
Mass of the proton	m_p	$= 1.672\,621\,898(21) \times 10^{-27} \text{ kg}$
		$= 938.272\,081\,3(58) \frac{\text{MeV}}{c^2}$
Mass of the neutron	m_n	$= 1.674\,927\,471(21) \times 10^{-27} \text{ kg}$
		$= 939.565\,413\,3(58) \frac{\text{MeV}}{c^2}$
Unified atomic mass unit	u	$= 1.660\,539\,040(20) \times 10^{-27} \text{ kg}$
Rydberg constant	R_∞	$= 10\,973\,731.568\,508(65) \text{ m}^{-1}$
Universal constant of gravitation	G	$= 6.674\,08(31) \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Acceleration due to gravity (in Zurich)	g	$= 9.81 \text{ m} \cdot \text{s}^{-2}$
Planck's constant	h	$= 6.626\,070\,040(81) \times 10^{-34} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Avogadro number	N_A	$= 6.022\,140\,857(74) \times 10^{23} \text{ mol}^{-1}$
Molar gas constant	R	$= 8.314\,4598(48) \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Molar mass constant	M_u	$= 1 \times 10^{-3} \text{ kg} \cdot \text{mol}^{-1}$
Boltzmann constant	k_B	$= 1.380\,648\,52(79) \times 10^{-23} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$
Stefan-Boltzmann constant	σ	$= 5.670\,367(13) \times 10^{-8} \text{ kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$

Official English text of the problem.

Answers

For Part A, define the mass of an entirely wooden cylinder and the excess mass introduced by the metal:

$$m_0 = \pi \rho_1 r_1^2 h_1, \quad \Delta m = M - m_0 = \pi(\rho_2 - \rho_1) r_2^2 h_2.$$

For Part B, write $\Omega = \omega_{ss}$, and choose $+x$ along the station's rotation and $+y$ towards its centre.

• A.1

$$b = \frac{r_1 \sin \Theta}{\sin \phi}.$$

• A.2

$$I_S \ddot{\varphi} = -Mgb \sin \varphi \simeq -Mgb\varphi, \quad I_S = \frac{MgbT^2}{4\pi^2}.$$

• A.3

$$d = \frac{Mb}{\Delta m} = \frac{Mb}{\pi(\rho_2 - \rho_1) r_2^2 h_2}.$$

• A.4

$$I_S = \frac{1}{2}m_0r_1^2 + \frac{1}{2}\Delta m r_2^2 + \frac{M^2b^2}{\Delta m}.$$

• A.5

$$r_2 = \sqrt{\frac{MgbT^2}{2\pi^2\Delta m} - \frac{m_0r_1^2}{\Delta m} - \frac{2M^2b^2}{(\Delta m)^2}}, \quad h_2 = \frac{\Delta m}{\pi(\rho_2 - \rho_1)r_2^2}.$$

• B.1

$$\omega_{ss} = \sqrt{\frac{g_E}{R}}.$$

• B.2

$$\omega_E = \sqrt{\frac{k}{m}}.$$

• B.3

$$\omega = \sqrt{\frac{k}{m} - \Omega^2} = \sqrt{\frac{k}{m} - \frac{g_E}{R}}.$$

• B.4

$$g_E(h) = \frac{g_E}{(1 + h/R_E)^2} \simeq g_E \left(1 - \frac{2h}{R_E}\right), \quad \tilde{\omega}_E = \sqrt{\frac{k}{m} - \frac{2g_E}{R_E}}.$$

• B.5

$$R = \frac{R_E}{2}.$$

• B.6

$$v_x = -2\Omega H, \quad d_x = -\frac{2}{3}\Omega H \sqrt{\frac{2H}{g_E}} = -\frac{2\sqrt{2}}{3}H \sqrt{\frac{H}{R}}.$$

Both are opposite to the station's rotation.

• B.7 A rigorous lower bound is

$$H > R \left(1 - \frac{1}{\sqrt{1 + 4\pi^2}}\right) \simeq 0.8428R.$$

The first exact return to the tower's base occurs at

$$H_{\min} \simeq 0.8716R.$$

- **B.8** With $a = 2\Omega d/\omega$,

$$\boxed{x(t) = a[\omega t - \sin(\omega t)]}, \quad \boxed{y(t) = -d \cos(\omega t)}.$$

The trajectory consists of successive cycloidal arches, with cusps at $(2\pi na, -d)$, upper turning points at $((2n+1)\pi a, d)$, and horizontal spacing $2\pi a$. The vertical amplitude is d .

Solution

A.1

Let P be the contact point with the inclined base. The normal force and static friction both act at P , so they exert no torque about P . Static equilibrium therefore requires the vertical line through the centre of mass C to pass through P .

The centre of mass lies along SB . Using the angles shown in Figure 2, the horizontal displacements of P and C from S are respectively

$$x_P - x_S = r_1 \sin \Theta, \quad x_C - x_S = b \sin \phi.$$

Consequently,

$$Mg(b \sin \phi - r_1 \sin \Theta) = 0,$$

and hence

$$\boxed{b = \frac{r_1 \sin \Theta}{\sin \phi}}.$$

The stable equilibrium is the branch with $\cos \phi > 0$. In particular, the inclination must be small enough that $r_1 \sin \Theta < b$; equality corresponds to the limiting, non-restoring configuration.

A.2

The suspension fixes the axis S . Gravity acts at C , a distance b from this axis, and produces the restoring torque

$$\tau_S = -Mgb \sin \varphi.$$

The rotational equation of motion is therefore

$$I_S \ddot{\varphi} = -Mgb \sin \varphi.$$

For $|\varphi| \ll 1$, use $\sin \varphi \simeq \varphi$:

$$\ddot{\varphi} + \frac{Mgb}{I_S} \varphi = 0.$$

This is simple harmonic motion with

$$\left(\frac{2\pi}{T}\right)^2 = \frac{Mgb}{I_S}.$$

Thus

$$\boxed{I_S = \frac{MgbT^2}{4\pi^2}},$$

where the supplied gravitational acceleration is $g = 9.81 \text{ m s}^{-2}$.

A.3

Represent the actual density distribution as the sum of:

1. A complete wooden cylinder, of mass

$$m_0 = \pi \rho_1 r_1^2 h_1,$$

centred on S .

2. An additional disk of density $\rho_2 - \rho_1$, centred on B , with mass

$$\Delta m = \pi(\rho_2 - \rho_1) r_2^2 h_2 = M - m_0.$$

This accounts for the wood that was removed before the metal was inserted.

Taking the first moment of mass about S ,

$$Mb = m_0(0) + \Delta m d.$$

It follows that

$$d = \frac{Mb}{\Delta m} = \frac{Mb}{M - \pi \rho_1 r_1^2 h_1} = \frac{Mb}{\pi(\rho_2 - \rho_1) r_2^2 h_2}.$$

A.4

The complete wooden cylinder contributes

$$I_{\text{wood}} = \frac{1}{2} m_0 r_1^2.$$

The additional disk has moment of inertia $\frac{1}{2} \Delta m r_2^2$ about its own symmetry axis. By the parallel-axis theorem, its contribution about S is

$$I_{\text{additional}} = \frac{1}{2} \Delta m r_2^2 + \Delta m d^2.$$

Therefore

$$I_S = \frac{1}{2} m_0 r_1^2 + \frac{1}{2} \Delta m r_2^2 + \Delta m d^2.$$

Substituting $d = Mb/\Delta m$ gives

$$I_S = \frac{1}{2} m_0 r_1^2 + \frac{1}{2} \Delta m r_2^2 + \frac{M^2 b^2}{\Delta m}.$$

A.5

Equating the expressions for I_S from A.2 and A.4,

$$\frac{MgbT^2}{4\pi^2} = \frac{1}{2} m_0 r_1^2 + \frac{1}{2} \Delta m r_2^2 + \frac{M^2 b^2}{\Delta m}.$$

Solving for r_2^2 ,

$$r_2^2 = \frac{MgbT^2}{2\pi^2 \Delta m} - \frac{m_0 r_1^2}{\Delta m} - \frac{2M^2 b^2}{(\Delta m)^2}.$$

Taking the positive root,

$$r_2 = \sqrt{\frac{MgbT^2}{2\pi^2\Delta m} - \frac{m_0r_1^2}{\Delta m} - \frac{2M^2b^2}{(\Delta m)^2}}.$$

The mass relation then determines the thickness:

$$h_2 = \frac{\Delta m}{\pi(\rho_2 - \rho_1)r_2^2}.$$

Here $m_0 = \pi\rho_1r_1^2h_1$ and $\Delta m = M - m_0$, so all quantities on the right are known or measured. Physically consistent measurements must give a positive radicand and satisfy $d + r_2 \leq r_1$ and $h_2 \leq h_1$.

B.1

An astronaut at rest relative to the rim follows a circle of radius R . The required centripetal acceleration is

$$a_c = \omega_{ss}^2 R.$$

The normal force supplying this acceleration is experienced as the astronaut's weight. Matching the acceleration on Earth requires

$$\omega_{ss}^2 R = g_E.$$

Thus

$$\omega_{ss} = \sqrt{\frac{g_E}{R}}.$$

Write $\Omega = \omega_{ss}$ below.

B.2

Let y denote upward displacement from the spring's equilibrium position. Constant gravity changes the equilibrium extension, but its constant force cancels when the equation is written about equilibrium:

$$m\ddot{y} = -ky.$$

Therefore

$$\omega_E = \sqrt{\frac{k}{m}}.$$

B.3

Use the rotating frame and let y point radially inward from equilibrium. If the equilibrium radius is r_0 , the instantaneous radius is $r_0 - y$.

The centrifugal force acts outward and has magnitude

$$m\Omega^2(r_0 - y).$$

The inward spring force decreases by ky when the mass moves inward by y . At equilibrium, its value F_0 satisfies

$$F_0 = m\Omega^2 r_0.$$

The inward equation of motion is therefore

$$m\ddot{y} = (F_0 - ky) - m\Omega^2(r_0 - y) = -(k - m\Omega^2)y.$$

The horizontal constraint balances the Coriolis force, so it does not alter this radial equation. Hence

$$\omega = \sqrt{\frac{k}{m} - \Omega^2} = \sqrt{\frac{k}{m} - \frac{g_E}{R}}.$$

Oscillations require $k/m > \Omega^2$.

B.4

Let M_E be Earth's mass. Newton's gravitational law gives

$$g_E(h) = \frac{GM_E}{(R_E + h)^2}.$$

Since $g_E = GM_E/R_E^2$,

$$g_E(h) = g_E \left(1 + \frac{h}{R_E}\right)^{-2}.$$

For $h \ll R_E$, the first-order expansion is

$$g_E(h) \simeq g_E \left(1 - \frac{2h}{R_E}\right).$$

Let the equilibrium height be h_0 , and let y be upward displacement from equilibrium. After subtracting the equilibrium force balance,

$$m\ddot{y} = -ky - m[g_E(h_0 + y) - g_E(h_0)].$$

Within the linear approximation,

$$g_E(h_0 + y) - g_E(h_0) \simeq -\frac{2g_E}{R_E}y.$$

Thus

$$m\ddot{y} = -\left(k - \frac{2mg_E}{R_E}\right)y,$$

giving

$$\tilde{\omega}_E = \sqrt{\frac{k}{m} - \frac{2g_E}{R_E}}.$$

Gravity decreases with height, so an upward displacement reduces the downward gravitational force. This weakens the net restoring force and lowers the frequency.

B.5

Equating the squared frequencies from B.3 and B.4,

$$\frac{k}{m} - \frac{g_E}{R} = \frac{k}{m} - \frac{2g_E}{R_E}.$$

Therefore

$$R = \frac{R_E}{2}.$$

B.6

Choose $+x$ along the station's rotation and $+y$ inward, towards its centre. With this convention, the Coriolis acceleration components are

$$a_{C,x} = 2\Omega\dot{y}, \quad a_{C,y} = -2\Omega\dot{x}.$$

For a short tower, approximate the downward acceleration by the constant $g_E = \Omega^2 R$. The small horizontal deflection permits us to neglect its feedback on the vertical fall. Released from rest,

$$y(t) = H - \frac{1}{2}g_E t^2, \quad \dot{y} = -g_E t.$$

The impact time is

$$t_f = \sqrt{\frac{2H}{g_E}}.$$

The horizontal equation becomes

$$\ddot{x} = 2\Omega\dot{y} = -2\Omega g_E t.$$

Using $x(0) = \dot{x}(0) = 0$, integration gives

$$\dot{x} = -\Omega g_E t^2, \quad x = -\frac{1}{3}\Omega g_E t^3.$$

At impact,

$$v_x = -\Omega g_E t_f^2 = -2\Omega H,$$

and

$$d_x = -\frac{1}{3}\Omega g_E t_f^3 = -\frac{2}{3}\Omega H \sqrt{\frac{2H}{g_E}}.$$

Using $\Omega^2 = g_E/R$,

$$d_x = -\frac{2\sqrt{2}}{3}H \sqrt{\frac{H}{R}}.$$

The mass lands behind the tower in the direction opposite to rotation. The consistency condition is

$$\frac{|d_x|}{H} = \frac{2\sqrt{2}}{3} \sqrt{\frac{H}{R}} \ll 1,$$

as expected for $H \ll R$.

B.7

For a tall tower, use an inertial frame and the full circular geometry. The constant-acceleration approximation of B.6 no longer applies.

Let

$$r = R - H$$

be the distance of the release point from the station's centre, with $0 < r < R$. Choose inertial coordinates so that release occurs at $(r, 0)$, with tangential velocity $(0, \Omega r)$.

After release, no real force acts on the mass, so its trajectory is a straight line:

$$X(t) = r, \quad Y(t) = \Omega r t.$$

It first reaches the rim when

$$X^2 + Y^2 = R^2.$$

Writing $u = \Omega t_f$, this gives

$$r^2(1 + u^2) = R^2, \quad u = \sqrt{\frac{R^2}{r^2} - 1}.$$

At impact, the angular position of the mass is

$$\alpha = \arctan u, \quad 0 < \alpha < \frac{\pi}{2},$$

while the tower's base has rotated through the angle u .

For the mass to hit the tower's base, their angles must agree modulo 2π :

$$u - \arctan u = 2\pi n.$$

For $u > 0$, $u > \arctan u$, so $n = 0$ cannot give a nonzero tower height. Therefore $n \geq 1$, implying

$$u = 2\pi n + \arctan u > 2\pi.$$

Consequently,

$$\frac{R^2}{r^2} - 1 > 4\pi^2, \quad r < \frac{R}{\sqrt{1 + 4\pi^2}}.$$

Since $H = R - r$,

$$H > R \left(1 - \frac{1}{\sqrt{1 + 4\pi^2}} \right) \simeq 0.8428R.$$

For the station identified in B.5, this is $H > 0.4214R_E$.

One can also obtain the smallest exact return height. The function

$$f(u) = u - \arctan u$$

is strictly increasing for $u > 0$, because

$$f'(u) = \frac{u^2}{1 + u^2} > 0.$$

Thus the smallest return corresponds to $n = 1$. Solving

$$u_1 - \arctan u_1 = 2\pi$$

gives

$$u_1 \simeq 7.72525.$$

Therefore

$$H_{\min} = R \left(1 - \frac{1}{\sqrt{1 + u_1^2}} \right) \simeq 0.871625R.$$

This sharper value is consistent with the requested lower bound.

B.8

Continue with $+x$ along rotation and $+y$ inward. The freely sliding support supplies no horizontal restoring force.

Use the local flat-floor approximation and, as instructed, neglect the y -directed Coriolis force generated by horizontal motion. The vertical equation remains

$$\ddot{y} + \omega^2 y = 0, \quad \omega^2 = \frac{k}{m} - \Omega^2.$$

The initial conditions are

$$y(0) = -d, \quad \dot{y}(0) = 0, \quad x(0) = 0, \quad \dot{x}(0) = 0.$$

Hence

$$y(t) = -d \cos(\omega t).$$

The horizontal motion is driven by the Coriolis force associated with the vertical velocity:

$$\ddot{x} = 2\Omega \dot{y}.$$

Integrating once,

$$\dot{x} = 2\Omega [y(t) - y(0)] = 2\Omega d [1 - \cos(\omega t)].$$

Integrating again,

$$x(t) = 2\Omega d \left[t - \frac{\sin(\omega t)}{\omega} \right].$$

To describe the sketch, define

$$a = \frac{2\Omega d}{\omega}, \quad \tau = \omega t, \quad P = \frac{2\pi}{\omega}.$$

Then

$$x = a(\tau - \sin \tau), \quad y = -d \cos \tau.$$

This is a cycloid with independently scaled horizontal and vertical coordinates.

One complete arch has the following points:

Time	x/a	y/d	Feature
0	0	-1	Initial cusp; mass at rest
$P/4$	$\pi/2 - 1$	0	Upward crossing of equilibrium height
$P/2$	π	1	Highest point; horizontal tangent
$3P/4$	$3\pi/2 + 1$	0	Downward crossing of equilibrium height
P	2π	-1	Next cusp; mass momentarily at rest

The complete sketch consists of these arches repeated towards increasing x :

- The vertical amplitude is d , with y ranging from $-d$ to $+d$.
- Cusps occur at $(2\pi na, -d)$, with vertical limiting tangents.
- Upper turning points occur at $((2n + 1)\pi a, d)$, with horizontal tangents.
- The distance between successive cusps is

$$L = 2\pi a = \frac{4\pi\Omega d}{\omega}.$$

- Since

$$\dot{x} = 2\Omega d(1 - \cos \omega t) \geq 0,$$

the trajectory progresses continuously in $+x$, without loops.

- The mean horizontal drift velocity is $2\Omega d$; the horizontal oscillation about that uniform drift has amplitude a .

These expressions apply while the stated neglect of the vertical Coriolis force and the local flat-floor approximation remain valid.

The solution was checked against the official solution.

Problem 2: Nonlinear Dynamics in Electric Circuits

Please read the general instructions in the separate envelope before you start this problem.

Introduction

Bistable non-linear semiconducting elements (e.g. thyristors) are widely used in electronics as switches and generators of electromagnetic oscillations. The primary field of applications of thyristors is controlling alternating currents in power electronics, for instance rectification of AC current to DC at the megawatt scale. Bistable elements may also serve as model systems for self-organization phenomena in physics (this topic is covered in part B of the problem), biology (see part C) and other fields of modern nonlinear science.

Goals

To study instabilities and nontrivial dynamics of circuits including elements with non-linear $I - V$ characteristics. To discover possible applications of such circuits in engineering and in modeling of biological systems.

Part A. Stationary states and instabilities (3 points)

Fig. 1 shows the so-called **S-shaped** $I - V$ characteristics of a non-linear element X . In the voltage range between $U_h = 4.00 \text{ V}$ (the holding voltage) and $U_{th} = 10.0 \text{ V}$ (the threshold voltage) this $I - V$ characteristics is multivalued. For simplicity, the graph on Fig. 1 is chosen to be piece-wise linear (each branch is a segment of a straight line). In particular, the line in the upper branch touches the origin if it is extended. This approximation gives a good description of real thyristors.

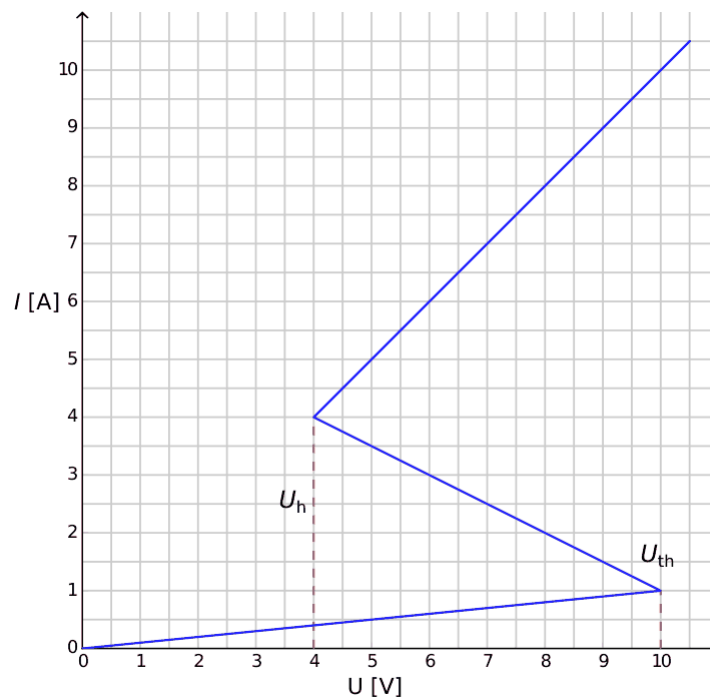


Figure 1: $I - V$ characteristics of the non-linear element X .

A.1 — 0.4pt

Using the graph, determine the resistance R_{on} of the element X on the upper branch of the $I - V$ characteristics, and R_{off} on the lower branch, respectively. The middle branch is described by the equation

$$I = I_0 - \frac{U}{R_{\text{int}}}. \quad (1)$$

Find the values of the parameters I_0 and R_{int} .

The element X is connected in series (see Fig.2) with a resistor R , an inductor L and an ideal voltage source $\mathcal{E}$. One says that the circuit is in a stationary state if the current is constant in time, $I(t) = \text{const.}$

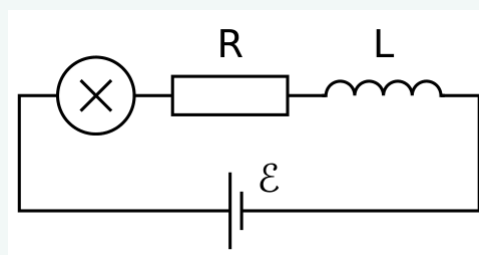


Figure 2: Circuit with element X , resistor R , inductor L and voltage source $\mathcal{E}$.

A.2 — 1pt

What are the possible numbers of stationary states that the circuit of Fig. 2 may have for a fixed value of $\mathcal{E}$ and for $R = 3.00 \, \Omega$? How does the answer change for $R = 1.00 \, \Omega$?

A.3 — 0.6pt

Let $R = 3.00\ \Omega$, $L = 1.00\ \mu\text{H}$ and $\mathcal{E} = 15.0\ \text{V}$ in the circuit shown in Fig. 2. Determine the values of the current $I_{\text{stationary}}$ and the voltage $V_{\text{stationary}}$ on the non-linear element X in the stationary state.

The circuit in Fig. 2 is in the stationary state with $I(t) = I_{\text{stationary}}$. This stationary state is said to be stable if after a small displacement (increase or decrease in the current), the current returns towards the stationary state. And if the system keeps moving away from the stationary state, it is said to be unstable.

A.4 — 1pt

Use numerical values of the question A.3 and study the stability of the stationary state with $I(t) = I_{\text{stationary}}$. Is it stable or unstable?

Part B. Bistable non-linear elements in physics: radio transmitter (5 points)

We now investigate a new circuit configuration (see Fig. 3). This time, the non-linear element X is connected in parallel to a capacitor of capacitance $C = 1.00\ \mu\text{F}$. This block is then connected in series to a resistor of resistance $R = 3.00\ \Omega$ and an ideal constant voltage source of voltage $\mathcal{E} = 15.0\ \text{V}$. It turns out that this circuit undergoes oscillations with the non-linear element X jumping from one branch of the $I - V$ characteristics to another over the course of one cycle.

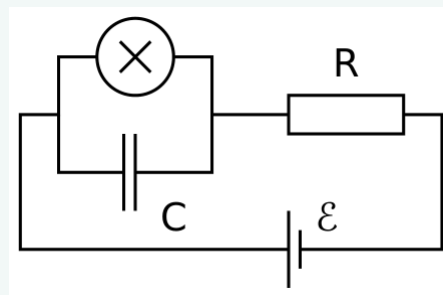


Figure 3: Circuit with element X , capacitor C , resistor R and voltage source $\mathcal{E}$.

B.1 — 1.8pt

Draw the oscillation cycle on the $I - V$ graph, including its direction (clockwise or anticlockwise). Justify your answer with equations and sketches.

B.2 — 1.9pt

Find expressions for the times t_1 and t_2 that the system spends on each branch of the $I - V$ graph during the oscillation cycle. Determine their numerical values. Find the numerical value of the oscillation period T assuming that the time needed for jumps between the branches of the $I - V$ graph is negligible.

B.3 — 0.7pt

Estimate the average power P dissipated by the non-linear element over the course of one oscillation. An order of magnitude is sufficient.

The circuit in Fig. 3 is used to build a radio transmitter. For this purpose, the element X is attached to one end of a linear antenna (a long straight wire) of length s . The other end of the wire is free. In the antenna, an electromagnetic standing wave is formed. The speed of electromagnetic

waves along the antenna is the same as in vacuum. The transmitter is using the main harmonic of the system, which has period T of question B.2.

B.4 — 0.6pt

What is the optimal value of s assuming that it cannot exceed 1 km?

Part C. Bistable non-linear elements in biology: neuristor (2 points)

In this part of the problem, we consider an application of bistable non-linear elements to modeling of biological processes. A neuron in a human brain has the following property: when excited by an external signal, it makes one single oscillation and then returns to its initial state. This feature is called excitability. Due to this property, pulses can propagate in the network of coupled neurons constituting the nerve systems. A semiconductor chip designed to mimic excitability and pulse propagation is called a *neuristor* (from neuron and transistor).

We attempt to model a simple neuristor using a circuit that includes the non-linear element X that we investigated previously. To this end, the voltage $\mathcal{E}$ in the circuit of Fig. 3 is decreased to the value $\mathcal{E}' = 12.0 \text{ V}$. The oscillations stop, and the system reaches its stationary state. Then, the voltage is rapidly increased back to the value $\mathcal{E} = 15.0 \text{ V}$, and after a period of time τ (with $\tau < T$) is set again to the value $\mathcal{E}'$ (see Fig. 4). It turns out that there is a certain critical value τ_{crit} , and the system shows qualitatively different behavior for $\tau < \tau_{\text{crit}}$ and for $\tau > \tau_{\text{crit}}$.

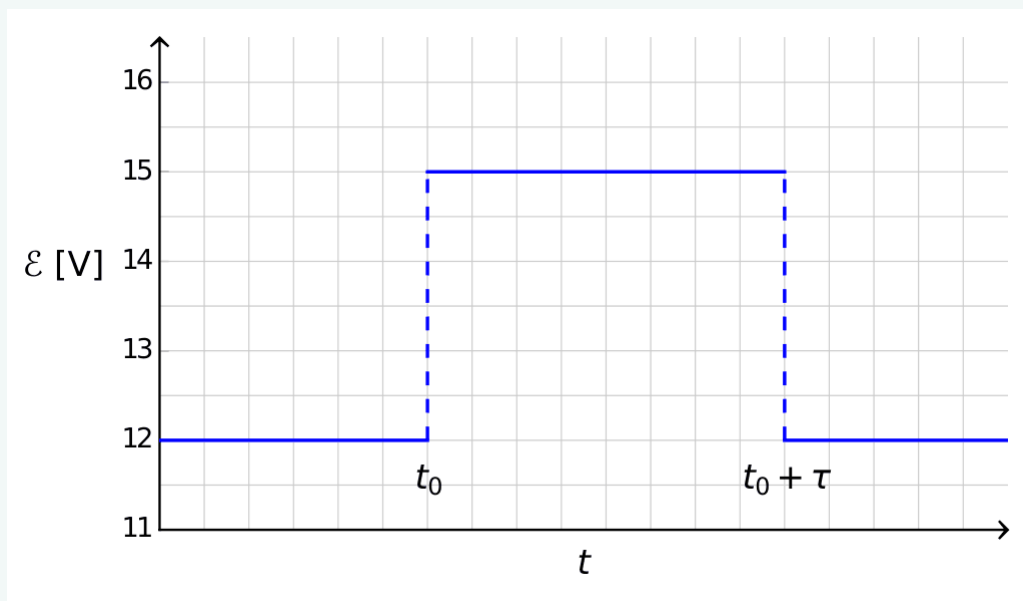


Figure 4: Voltage of the voltage source as a function of time.

C.1 — 1.2pt

Sketch the graphs of the time dependence of the current $I_X(t)$ on the non-linear element X for $\tau < \tau_{\text{crit}}$ and for $\tau > \tau_{\text{crit}}$.

C.2 — 0.6pt

Find the expression and the numerical value of the critical time τ_{crit} for which the scenario switches.

C.3 — 0.2pt

Is the circuit with $\tau = 1.00 \times 10^{-6} \text{ s}$ a neuristor?

General Data Sheet

Speed of light in vacuum	c	=	$299\,792\,458\text{ m} \cdot \text{s}^{-1}$
Vacuum permeability (magnetic constant)	μ_0	=	$4\pi \times 10^{-7}\text{ kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity (electrical constant)	ε_0	=	$8.854\,187\,817 \times 10^{-12}\text{ A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$
Elementary charge	e	=	$1.602\,176\,620\,8(98) \times 10^{-19}\text{ A} \cdot \text{s}$
Mass of the electron	m_e	=	$9.109\,383\,56(11) \times 10^{-31}\text{ kg}$
		=	$0.510\,998\,946\,1(31) \frac{\text{MeV}}{c^2}$
Mass of the proton	m_p	=	$1.672\,621\,898(21) \times 10^{-27}\text{ kg}$
		=	$938.272\,081\,3(58) \frac{\text{MeV}}{c^2}$
Mass of the neutron	m_n	=	$1.674\,927\,471(21) \times 10^{-27}\text{ kg}$
		=	$939.565\,413\,3(58) \frac{\text{MeV}}{c^2}$
Unified atomic mass unit	u	=	$1.660\,539\,040(20) \times 10^{-27}\text{ kg}$
Rydberg constant	R_∞	=	$10\,973\,731.568\,508(65)\text{ m}^{-1}$
Universal constant of gravitation	G	=	$6.674\,08(31) \times 10^{-11}\text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Acceleration due to gravity (in Zurich)	g	=	$9.81\text{ m} \cdot \text{s}^{-2}$
Planck's constant	h	=	$6.626\,070\,040(81) \times 10^{-34}\text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Avogadro number	N_A	=	$6.022\,140\,857(74) \times 10^{23}\text{ mol}^{-1}$
Molar gas constant	R	=	$8.314\,4598(48)\text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Molar mass constant	M_u	=	$1 \times 10^{-3}\text{ kg} \cdot \text{mol}^{-1}$
Boltzmann constant	k_B	=	$1.380\,648\,52(79) \times 10^{-23}\text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$
Stefan-Boltzmann constant	σ	=	$5.670\,367(13) \times 10^{-8}\text{ kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$

Official English text of the problem.

Answers

- **A.1:** $R_{\text{on}} = 1.00\ \Omega$, $R_{\text{off}} = 10.0\ \Omega$, $I_0 = 6.00\text{ A}$, $R_{\text{int}} = 2.00\ \Omega$.
- **A.2:** For $R = 3.00\ \Omega$, there is **one** stationary state. For $R = 1.00\ \Omega$, there are **three** when $8.00\text{ V} < \mathcal{E} < 11.0\text{ V}$, **two** at either endpoint, and **one** otherwise, for the nonnegative source voltage shown.
- **A.3:** $I_{\text{stationary}} = 3.00\text{ A}$, $V_{\text{stationary}} = 6.00\text{ V}$.
- **A.4: Stable;** small current perturbations decay exponentially with time constant $1.00\ \mu\text{s}$.
- **B.1: Anticlockwise.** In coordinates $(U/\text{V}, I_X/\text{A})$, the cycle is

$$(4, 0.4) \rightarrow (10, 1) \rightarrow (10, 10) \rightarrow (4, 4) \rightarrow (4, 0.4).$$
- **B.2:** Defining t_1 on the lower branch and t_2 on the upper branch,

$$t_1 = \frac{30}{13} \ln \frac{49}{10} \mu\text{s} = 3.67 \mu\text{s}, \quad t_2 = \frac{3}{4} \ln 25 \mu\text{s} = 2.41 \mu\text{s},$$

$$T = 6.08 \mu\text{s}.$$

- **B.3:** P is of order 10^1 W. Integration within the ideal model gives $P = 17.3$ W.
- **B.4:** $s = cT/4 = 456$ m.
- **C.1:** For $\tau < \tau_{\text{crit}}$, the current makes a small, continuous excursion from 0.923 A, remaining below 1.00 A, and returns to 0.923 A. For $\tau > \tau_{\text{crit}}$, it produces one large spike: a jump 1.00 $\rightarrow$ 10.0 A, decay to 4.00 A, a jump to 0.400 A, and recovery to 0.923 A.
- **C.2:**

$$\tau_{\text{crit}} = \frac{30}{13} \ln \frac{3}{2} \mu\text{s} = 0.936 \mu\text{s}.$$

- **C.3: Yes**, because $1.00 \mu\text{s} > \tau_{\text{crit}}$, the pulse triggers one complete excursion.

Solution

A.1

The upper branch passes through the origin when extended and through $(U, I) = (4.00 \text{ V}, 4.00 \text{ A})$. Hence

$$R_{\text{on}} = \frac{U}{I} = \frac{4.00}{4.00} \Omega = \boxed{1.00 \Omega}.$$

The lower branch passes through the origin and $(10.0 \text{ V}, 1.00 \text{ A})$, giving

$$R_{\text{off}} = \boxed{10.0 \Omega}.$$

The middle branch joins $(4.00 \text{ V}, 4.00 \text{ A})$ to $(10.0 \text{ V}, 1.00 \text{ A})$, so

$$-\frac{1}{R_{\text{int}}} = \frac{1.00 - 4.00}{10.0 - 4.00} \frac{\text{A}}{\text{V}} = -0.500 \Omega^{-1}.$$

Therefore

$$\boxed{R_{\text{int}} = 2.00 \Omega}, \quad I_0 = 4.00 \text{ A} + \frac{4.00 \text{ V}}{2.00 \Omega} = \boxed{6.00 \text{ A}}.$$

Here R_{int} is positive; the middle branch's differential resistance is $-R_{\text{int}}$.

A.2

Let U denote the voltage across X . Kirchhoff's voltage law gives

$$\mathcal{E} = RI + U + L \frac{dI}{dt}.$$

In a stationary state,

$$\mathcal{E} = RI + U.$$

Thus stationary states are intersections of the characteristic with the load line $I = (\mathcal{E} - U)/R$.

Equivalently, using the characteristic as $U(I)$,

$$\mathcal{E}(I) = \begin{cases} (R + 10.0 \Omega)I, & 0 \leq I \leq 1.00 \text{ A}, \\ 12.0 \text{ V} + (R - 2.00 \Omega)I, & 1.00 \text{ A} \leq I \leq 4.00 \text{ A}, \\ (R + 1.00 \Omega)I, & I \geq 4.00 \text{ A}. \end{cases}$$

For $R = 3.00 \Omega$, all three pieces increase with I . They join continuously at $\mathcal{E} = 13.0 \text{ V}$ and 16.0 V . Consequently, every nonnegative source voltage has exactly **one stationary state**.

For $R = 1.00 \, \Omega$, the middle piece decreases with I . Its endpoints occur at

$$\mathcal{E}(1.00 \, \text{A}) = 11.0 \, \text{V}, \quad \mathcal{E}(4.00 \, \text{A}) = 8.00 \, \text{V}.$$

The number of distinct stationary states is therefore

Source voltage	Number of stationary states
$0 \leq \mathcal{E} < 8.00 \, \text{V}$	1
$\mathcal{E} = 8.00 \, \text{V}$	2
$8.00 \, \text{V} < \mathcal{E} < 11.0 \, \text{V}$	3
$\mathcal{E} = 11.0 \, \text{V}$	2
$\mathcal{E} > 11.0 \, \text{V}$	1

At either boundary, two branch intersections coincide at a corner of the characteristic and count as one state.

A.3

For $R = 3.00 \, \Omega$ and $\mathcal{E} = 15.0 \, \text{V}$, the stationary state lies on the middle branch because $13.0 < 15.0 < 16.0 \, \text{V}$. Hence

$$15.0 \, \text{V} = 12.0 \, \text{V} + (3.00 - 2.00) \, \Omega I_{\text{stationary}}.$$

Therefore

$$I_{\text{stationary}} = 3.00 \, \text{A},$$

and

$$V_{\text{stationary}} = \mathcal{E} - RI_{\text{stationary}} = 6.00 \, \text{V}.$$

A.4

Write $I = I_{\text{stationary}} + \delta I$. A sufficiently small perturbation remains on the middle branch, where

$$U = R_{\text{int}}(I_0 - I).$$

Substituting into Kirchhoff's law and subtracting the stationary equation gives

$$L \frac{d(\delta I)}{dt} = -(R - R_{\text{int}})\delta I.$$

Thus

$$\delta I(t) = \delta I(0) \exp \left[-\frac{R - R_{\text{int}}}{L} t \right].$$

Since $R - R_{\text{int}} = 1.00 \, \Omega > 0$, the perturbation decays. The stationary state is therefore **stable**, with relaxation time

$$\frac{L}{R - R_{\text{int}}} = \frac{1.00 \, \mu\text{H}}{1.00 \, \Omega} = 1.00 \, \mu\text{s}.$$

The external resistor's positive differential resistance exceeds the magnitude of the element's negative differential resistance.

B.1

The capacitor and X have the same voltage U . Kirchhoff's current law gives

$$\frac{\mathcal{E} - U}{R} = I_X + C \frac{dU}{dt},$$

or

$$C \frac{dU}{dt} = \frac{\mathcal{E} - U}{R} - I_X(U).$$

On a branch of resistance r , $I_X = U/r$, so

$$C \frac{dU}{dt} = \frac{\mathcal{E}}{R} - \left(\frac{1}{R} + \frac{1}{r} \right) U.$$

On the **lower branch**, the voltage towards which the circuit would relax is

$$U_{\infty, \text{off}} = \frac{\mathcal{E} R_{\text{off}}}{R + R_{\text{off}}} = \frac{150}{13} \text{ V} = 11.538 \dots \text{ V}.$$

This exceeds $U_{\text{th}} = 10.0 \text{ V}$. Consequently, throughout the lower-branch part of the cycle, $dU/dt > 0$: the capacitor charges from 4.00 to 10.0 V.

At U_{th} , the lower branch ends. The capacitor voltage cannot jump, so X switches vertically on the $I - V$ plot:

$$(U, I_X) = (10.0 \text{ V}, 1.00 \text{ A}) \longrightarrow (10.0 \text{ V}, 10.0 \text{ A}).$$

On the **upper branch**, the corresponding limiting voltage is

$$U_{\infty, \text{on}} = \frac{\mathcal{E} R_{\text{on}}}{R + R_{\text{on}}} = 3.75 \text{ V}.$$

This lies below $U_{\text{h}} = 4.00 \text{ V}$. Thus $dU/dt < 0$: the capacitor discharges from 10.0 to 4.00 V. At the holding voltage, the current jumps from 4.00 to 0.400 A, again at constant voltage.

The requested sketch therefore has these successive segments:

Segment	Motion in the $I - V$ plane
Lower branch	From (4.00 V, 0.400 A) rightwards and slightly upwards to (10.0 V, 1.00 A)
Threshold switch	Vertically upwards to (10.0 V, 10.0 A)
Upper branch	Leftwards and downwards to (4.00 V, 4.00 A)
Holding-voltage switch	Vertically downwards to (4.00 V, 0.400 A)

The direction is **anticlockwise**.

The middle-branch equilibrium is unstable in this capacitor circuit. Indeed, near $U = 6.00 \text{ V}$,

$$C \frac{d(\delta U)}{dt} = \left(\frac{1}{R_{\text{int}}} - \frac{1}{R} \right) \delta U = \frac{\delta U}{6.00 \Omega}.$$

Voltage perturbations grow, so the middle branch is not a sustained part of the oscillation cycle.

B.2

For a branch of resistance r , define

$$\theta_r = C \frac{Rr}{R + r}, \quad U_{\infty, r} = \frac{\mathcal{E}r}{R + r}.$$

The differential equation from B.1 becomes

$$\frac{dU}{dt} = \frac{U_{\infty,r} - U}{\theta_r},$$

with solution

$$U(t) = U_{\infty,r} + (U(0) - U_{\infty,r})e^{-t/\theta_r}.$$

Define t_1 as the time on the lower branch and t_2 as the time on the upper branch.

For charging from U_h to U_{th} ,

$$t_1 = \theta_{\text{off}} \ln \left(\frac{U_{\infty,\text{off}} - U_h}{U_{\infty,\text{off}} - U_{th}} \right).$$

Here

$$\theta_{\text{off}} = \frac{30}{13} \mu\text{s},$$

so

$$t_1 = \frac{30}{13} \mu\text{s} \ln \left(\frac{150/13 - 4}{150/13 - 10} \right) = \frac{30}{13} \ln \frac{49}{10} \mu\text{s} = \boxed{3.67 \mu\text{s}}.$$

For discharging from U_{th} to U_h ,

$$t_2 = \theta_{\text{on}} \ln \left(\frac{U_{th} - U_{\infty,\text{on}}}{U_h - U_{\infty,\text{on}}} \right).$$

Since

$$\theta_{\text{on}} = 0.750 \mu\text{s},$$

we obtain

$$t_2 = 0.750 \mu\text{s} \ln \left(\frac{10.0 - 3.75}{4.00 - 3.75} \right) = \frac{3}{4} \ln 25 \mu\text{s} = \boxed{2.41 \mu\text{s}}.$$

Neglecting the switching durations,

$$\boxed{T = t_1 + t_2 = 6.08 \mu\text{s}}.$$

B.3

On either conducting branch, the instantaneous power dissipated in X is

$$p_X = UI_X = \frac{U^2}{r}.$$

We neglect energy dissipated during the ideal instantaneous switches. Using the voltage evolution derived above, the energy dissipated on each branch is

$$W_{\text{off}} = \frac{\theta_{\text{off}}}{R_{\text{off}}} \int_{U_h}^{U_{th}} \frac{U^2}{U_{\infty,\text{off}} - U} dU = 23.2 \mu\text{J},$$

and

$$W_{\text{on}} = \frac{\theta_{\text{on}}}{R_{\text{on}}} \int_{U_h}^{U_{th}} \frac{U^2}{U - U_{\infty,\text{on}}} dU = 82.3 \mu\text{J}.$$

Thus

$$P = \frac{W_{\text{off}} + W_{\text{on}}}{T} = \frac{105.5 \mu\text{J}}{6.0816 \mu\text{s}} = \boxed{17.3 \text{ W}}.$$

The requested order-of-magnitude estimate is therefore

$$\boxed{P \sim 10^1 \text{ W}}.$$

B.4

The emitted fundamental has wavelength

$$\lambda = cT = (299\,792\,458 \text{ m s}^{-1})(6.08162 \times 10^{-6} \text{ s}) = 1.82322 \times 10^3 \text{ m}.$$

In the ideal antenna model, the free end is a current node, while efficient excitation places a current antinode at the driven end. Their separation is an odd multiple of a quarter wavelength:

$$s = (2n + 1) \frac{\lambda}{4}, \quad n = 0, 1, 2, \dots$$

The shortest resonant length is

$$\boxed{s = \frac{\lambda}{4} = \frac{cT}{4} = 456 \text{ m}}.$$

The next possibility is $3\lambda/4 = 1.37 \text{ km}$, which exceeds the limit. Thus the quarter-wave length is the permitted optimum.

C.1

First find the initial stationary state at $\mathcal{E}' = 12.0 \text{ V}$. It lies on the lower branch:

$$I_r = \frac{\mathcal{E}'}{R + R_{\text{off}}} = \frac{12}{13} \text{ A} = \boxed{0.923 \text{ A}},$$

$$U_r = R_{\text{off}} I_r = \frac{120}{13} \text{ V} = \boxed{9.23 \text{ V}}.$$

Let $u = t - t_0$. Before the pulse, $I_X = I_r$. While the source is at 15.0 V and the element remains on the lower branch,

$$I_X(u) = \frac{\mathcal{E}}{R + R_{\text{off}}} + \left(I_r - \frac{\mathcal{E}}{R + R_{\text{off}}} \right) e^{-u/\theta_{\text{off}}},$$

or numerically,

$$I_X(u) = \frac{15 - 3e^{-u/\theta_{\text{off}}}}{13} \text{ A}.$$

The critical duration is the time required for this current to reach 1.00 A , corresponding to $U = U_{\text{th}}$.

For $\tau < \tau_{\text{crit}}$, the graph is a small continuous bump:

- Before t_0 , the current is constant at 0.923 A .
- During the pulse, it rises exponentially, with decreasing slope, but remains below 1.00 A .
- At $t_0 + \tau$, the current is continuous and its slope changes sign.
- Afterwards it decays exponentially back to 0.923 A :

$$I_X(u) = I_r + [I_X(\tau) - I_r] e^{-(u-\tau)/\theta_{\text{off}}}, \quad u \geq \tau.$$

There is no branch switch.

For $\tau > \tau_{\text{crit}}$, the graph contains one large spike:

1. The current initially follows the same gradual rise from 0.923 to 1.00 A.
2. At $t = t_0 + \tau_{\text{crit}}$, it jumps vertically from 1.00 to 10.0 A.
3. It then decreases on the upper branch. While the source remains at 15.0 V,

$$I_X(u) = 3.75 \text{ A} + 6.25 \text{ A} e^{-(u-\tau_{\text{crit}})/\theta_{\text{on}}}.$$

4. On reaching 4.00 A, it jumps vertically to 0.400 A.
5. It subsequently rises on the lower branch and ultimately approaches 0.923 A from below.

If the source returns to 12.0 V while the element is still on the upper branch, the current remains continuous at that instant, but its decay becomes faster. Its subsequent expression is

$$I_X(u) = 3.00 \text{ A} + [I_X(\tau) - 3.00 \text{ A}] e^{-(u-\tau)/\theta_{\text{on}}}.$$

The asymptote 3.00 A lies below the holding current, so the current necessarily reaches 4.00 A and switches off. If this occurs at $u = u_d$, the following recovery is

$$I_X(u) = I_r + (0.400 \text{ A} - I_r) e^{-(u-u_d)/\theta_{\text{off}}}.$$

For a longer pulse, the upper-to-lower switch can occur before the pulse ends. If the source stays at 15.0 V, that switch occurs at

$$u = \tau_{\text{crit}} + t_2 = 3.35 \mu\text{s}.$$

The current then recovers on the lower branch under the 15.0 V source until $u = \tau$, after which it approaches the 12.0 V rest state. The source changes produce slope changes, not instantaneous changes in I_X .

There cannot be a second spike during the prescribed pulse: with 15.0 V maintained, the second threshold crossing would occur at $u = \tau_{\text{crit}} + T > T > \tau$.

C.2

At the critical duration, the capacitor has just charged from U_r to U_{th} :

$$U_{\text{th}} = U_{\infty, \text{off}} + (U_r - U_{\infty, \text{off}}) e^{-\tau_{\text{crit}}/\theta_{\text{off}}}.$$

Solving,

$$\tau_{\text{crit}} = \theta_{\text{off}} \ln \left(\frac{U_{\infty, \text{off}} - U_r}{U_{\infty, \text{off}} - U_{\text{th}}} \right).$$

In terms of the circuit parameters,

$$\tau_{\text{crit}} = \frac{CRR_{\text{off}}}{R + R_{\text{off}}} \ln \left[\frac{(\mathcal{E} - \mathcal{E}')R_{\text{off}}}{\mathcal{E}R_{\text{off}} - (R + R_{\text{off}})U_{\text{th}}} \right].$$

Numerically,

$$\tau_{\text{crit}} = \frac{30}{13} \mu\text{s} \ln \left(\frac{150/13 - 120/13}{150/13 - 10} \right) = \frac{30}{13} \ln \frac{3}{2} \mu\text{s}.$$

Therefore

$$\tau_{\text{crit}} = 0.936 \mu\text{s}.$$

C.3

The specified pulse satisfies

$$\tau = 1.00 \mu\text{s} > \tau_{\text{crit}} = 0.936 \mu\text{s}, \quad \tau < T.$$

It therefore triggers the switch to the upper branch. The circuit completes one large current excursion and then returns to its original stationary state under the 12.0 V source.

Yes, the circuit behaves as a neuristor.

The solution was checked against the official solution.

Problem 3: Large Hadron Collider

Please read the general instructions in the separate envelope before you start this problem.

In this task, the physics of the particle accelerator LHC (Large Hadron Collider) at CERN is discussed. CERN is the world's largest particle physics laboratory. Its main goal is to get insight into the fundamental laws of nature. Two beams of particles are accelerated to high energies, guided around the accelerator ring by a strong magnetic field and then made to collide with each other. The protons are not spread uniformly around the circumference of the accelerator, but they are clustered in so-called bunches. The resulting particles generated by collisions are observed with large detectors. Some parameters of the LHC can be found in table 1.

LHC ring	
Circumference of ring	26659 m
Number of bunches per proton beam	2808
Number of protons per bunch	1.15×10^{11}
Proton beams	
Energy of protons	7.00 TeV
Centre of mass energy	14.0 TeV

Table 1: Typical numerical values of relevant LHC parameters.

Particle physicists use convenient units for the energy, momentum and mass: The energy is measured in electron volts [eV]. By definition, 1 eV is the amount of energy gained by a particle with elementary charge, e , moved through a potential difference of one volt ($1 \text{ eV} = 1.602 \cdot 10^{-19} \text{ kg m}^2\text{s}^{-2}$).

The momentum is measured in units of eV/c and the mass in units of eV/c^2 , where c is the speed of light in vacuum. Since 1 eV is a very small quantity of energy, particle physicists often use MeV ($1 \text{ MeV} = 10^6 \text{ eV}$), GeV ($1 \text{ GeV} = 10^9 \text{ eV}$) or TeV ($1 \text{ TeV} = 10^{12} \text{ eV}$).

Part A deals with the acceleration of protons or electrons. Part B is concerned with the identification of particles produced in the collisions at CERN.

Part A. LHC accelerator (6 points)

Acceleration:

Assume that the protons have been accelerated by a voltage V such that their velocity is very close to the speed of light and neglect any energy loss due to radiation or collisions with other particles.

A.1 — 0.7pt

Find the exact expression for the final velocity v of the protons as a function of the accelerating voltage V , and physical constants.

A design for a future experiment at CERN plans to use the protons from the LHC and to collide them with electrons which have an energy of 60.0 GeV.

A.2 — 0.8pt

For particles with high energy and low mass the relative deviation $\Delta = (c - v)/c$ of the final velocity v from the speed of light is very small. Find a first order approximation for Δ and calculate Δ for electrons with an energy of 60.0 GeV using the accelerating voltage V and physical constants.

We now return to the protons in the LHC. Assume that the beam pipe has a circular shape.

A.3 — 1.0pt

Derive an expression for the uniform magnetic flux density B necessary to keep the proton beam on a circular track. The expression should only contain the energy of the protons E , the circumference L , fundamental constants and numbers. You may use suitable approximations if their effect is smaller than precision given by the least number of significant digits.

Calculate the magnetic flux density B for a proton energy of $E = 7.00$ TeV, neglecting interactions between the protons.

Radiated Power:

An accelerated charged particle radiates energy in the form of electromagnetic waves. The radiated power P_{rad} of a charged particle that circulates with a constant angular velocity depends only on its acceleration a , its charge q , the speed of light c and the permittivity of free space ϵ_0 .

A.4 — 1.0pt

Use dimensional analysis to find an expression for the radiated power P_{rad} .

The real formula for the radiated power contains a factor $1/(6\pi)$; moreover, a full relativistic derivation gives an additional multiplicative factor γ^4 , with $\gamma = (1 - v^2/c^2)^{-1/2}$.

A.5 — 1.0pt

Calculate P_{tot} , the total radiated power of the LHC, for a proton energy of $E = 7.00$ TeV (Note table 1). You may use suitable approximations.

Linear Acceleration:

At CERN, protons at rest are accelerated by a linear accelerator of length $d = 30.0 \text{ m}$ through a potential difference of $V = 500 \text{ MV}$. Assume that the electrical field is homogeneous. A linear accelerator consists of two plates as sketched in Figure 1.

A.6 — 1.5pt

Determine the time T that the protons take to pass through this field.

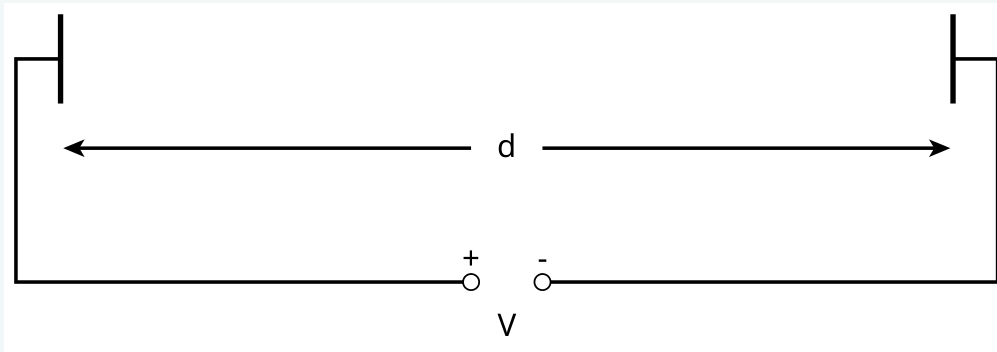


Figure 1: Sketch of an accelerator module.

Part B. Particle Identification (4 points)

Time of flight:

It is important to identify the high energy particles that are generated in the collision in order to interpret the interaction process. A simple method is to measure the time (t) that a particle with known momentum needs to pass a length l in a so-called Time-of-Flight (ToF) detector. Typical particles which are identified in the detector, together with their masses, are listed in table 2.

Particle	Mass [MeV/c^2]
Deuteron	1876
Proton	938
charged Kaon	494
charged Pion	140
Electron	0.511

Table 2: Particles and their masses.

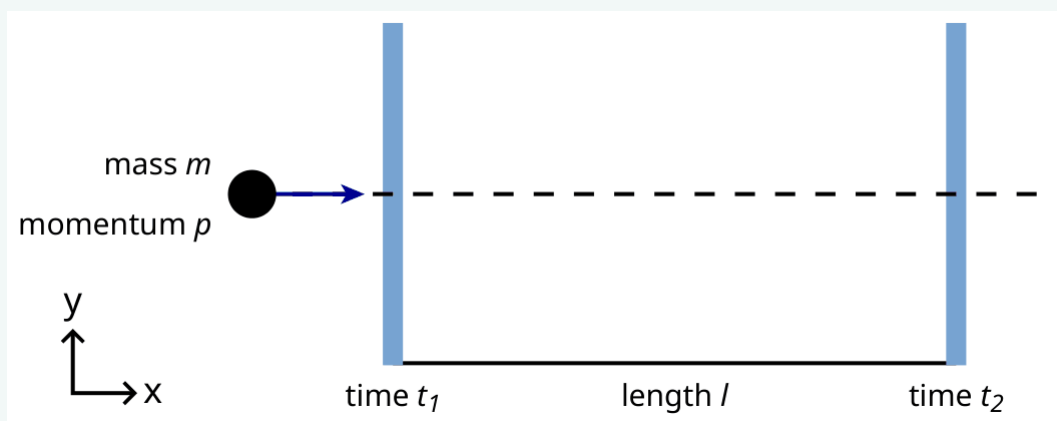


Figure 2: Schematic view of a time-of-flight detector.

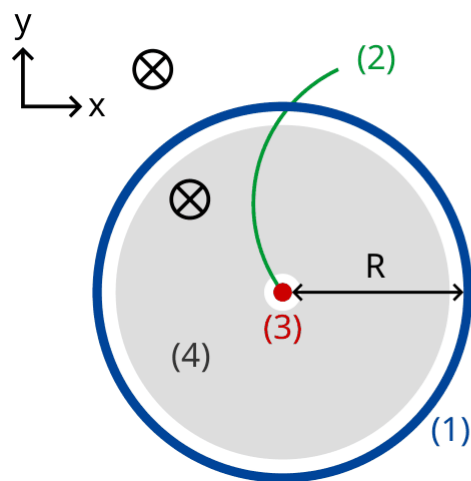
B.1 — 0.8pt

Express the particle mass m in terms of the momentum p , the flight length l and the flight time t , assuming that particles have elementary charge e and travel with velocity close to c on straight tracks in the ToF detector and that they travel perpendicular to the two detection planes (see figure 2).

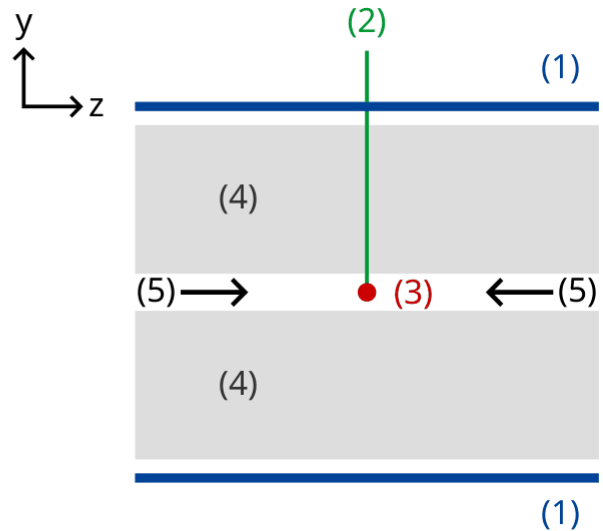
B.2 — 0.7pt

Calculate the minimal length l of a ToF detector that allows to safely distinguish a charged kaon from a charged pion, given both their momenta are measured to be $1.00 \text{ GeV}/c$. For a good separation it is required that the difference in the time-of-flight is larger than three times the time resolution of the detector. The typical resolution of a ToF detector is 150 ps ($1 \text{ ps} = 10^{-12} \text{ s}$).

In the following, particles produced in a typical LHC detector are identified in a two stage detector consisting of a tracking detector and a ToF detector. Figure 3 shows the setup in the plane transverse and longitudinal to the proton beams. Both detectors are tubes surrounding the interaction region with the beam passing in the middle of the tubes. The tracking detector measures the trajectory of a charged particle which passes through a magnetic field whose direction is parallel to the proton beams. The radius r of the trajectory allows one to determine the transverse momentum p_T of the particle. Since the collision time is known the ToF detector only needs one tube to measure the flight time (time between the collision and the detection in the ToF tube). This ToF tube is situated just outside the tracking chamber. For this task you may assume that all particles created by the collision travel perpendicular to the proton beams, which means that the created particles have no momentum along the direction of the proton beams.



transverse plane



cross section of the
longitudinal view at the center
of the tube along the beamline

- (1) - ToF tube
- (2) - track
- (3) - collision point
- (4) - tracking tube
- (5) - proton beams
- ⊗ - magnetic field

Figure 3 : Experimental setup for particle identification with a tracking chamber and a ToF detector. Both detectors are tubes surrounding the collision point in the middle. Left : transverse view perpendicular to the beamline. Right : longitudinal view parallel to the beam line. The particle is travelling perpendicular to the beam line.

B.3 — 1.7pt

Express the particle mass in terms of the magnetic flux density B , the radius R of the ToF tube, fundamental constants and the measured quantities: radius r of the track and time-of-flight t .

We detected four particles and want to identify them. The magnetic flux density in the tracking detector was $B = 0.500 \text{ T}$. The radius R of the ToF tube was 3.70 m . Here are the measurements ($1 \text{ ns} = 10^{-9} \text{ s}$):

Particle	Radius of the trajectory r [m]	Time of flight t [ns]
A	5.10	20
B	2.94	14
C	6.06	18
D	2.31	25

B.4 — 0.8pt

Identify the four particles by calculating their mass.

General Data Sheet

Speed of light in vacuum	c	$= 299\,792\,458\,\text{m} \cdot \text{s}^{-1}$
Vacuum permeability (magnetic constant)	μ_0	$= 4\pi \times 10^{-7}\,\text{kg} \cdot \text{m} \cdot \text{A}^{-2} \cdot \text{s}^{-2}$
Vacuum permittivity (electrical constant)	ε_0	$= 8.854\,187\,817 \times 10^{-12}\,\text{A}^2 \cdot \text{s}^4 \cdot \text{kg}^{-1} \cdot \text{m}^{-3}$
Elementary charge	e	$= 1.602\,176\,620\,8(98) \times 10^{-19}\,\text{A} \cdot \text{s}$
Mass of the electron	m_e	$= 9.109\,383\,56(11) \times 10^{-31}\,\text{kg}$
		$= 0.510\,998\,946\,1(31) \frac{\text{MeV}}{c^2}$
Mass of the proton	m_p	$= 1.672\,621\,898(21) \times 10^{-27}\,\text{kg}$
		$= 938.272\,081\,3(58) \frac{\text{MeV}}{c^2}$
Mass of the neutron	m_n	$= 1.674\,927\,471(21) \times 10^{-27}\,\text{kg}$
		$= 939.565\,413\,3(58) \frac{\text{MeV}}{c^2}$
Unified atomic mass unit	u	$= 1.660\,539\,040(20) \times 10^{-27}\,\text{kg}$
Rydberg constant	R_∞	$= 10\,973\,731.568\,508(65)\,\text{m}^{-1}$
Universal constant of gravitation	G	$= 6.674\,08(31) \times 10^{-11}\,\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$
Acceleration due to gravity (in Zurich)	g	$= 9.81\,\text{m} \cdot \text{s}^{-2}$
Planck's constant	h	$= 6.626\,070\,040(81) \times 10^{-34}\,\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$
Avogadro number	N_A	$= 6.022\,140\,857(74) \times 10^{23}\,\text{mol}^{-1}$
Molar gas constant	R	$= 8.314\,4598(48)\,\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{mol}^{-1} \cdot \text{K}^{-1}$
Molar mass constant	M_u	$= 1 \times 10^{-3}\,\text{kg} \cdot \text{mol}^{-1}$
Boltzmann constant	k_B	$= 1.380\,648\,52(79) \times 10^{-23}\,\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-2} \cdot \text{K}^{-1}$
Stefan-Boltzmann constant	σ	$= 5.670\,367(13) \times 10^{-8}\,\text{kg} \cdot \text{s}^{-3} \cdot \text{K}^{-4}$

Official English text of the problem.

Answers

- **A.1:** For acceleration from rest,

$$v = c \sqrt{1 - \left(1 + \frac{eV}{m_p c^2}\right)^{-2}}.$$

- **A.2:**

$$\Delta \simeq \frac{1}{2} \left(1 + \frac{eV}{mc^2}\right)^{-2} \simeq \frac{m^2 c^4}{2e^2 V^2}.$$

For the 60.0 GeV electrons,

$$\Delta = 3.63 \times 10^{-11}.$$

- **A.3:**

$$B = \frac{2\pi}{ecL} \sqrt{E^2 - m_p^2 c^4} \simeq \frac{2\pi E}{ecL} = 5.50 \text{ T.}$$

- **A.4:**

$$P_{\text{rad}} = C \frac{q^2 a^2}{\varepsilon_0 c^3},$$

where dimensional analysis cannot determine the dimensionless constant C .

- **A.5:** Including both proton beams,

$$P_{\text{tot}} = 5.12 \text{ kW.}$$

- **A.6:**

$$T = \frac{d}{c} \sqrt{1 + \frac{2m_p c^2}{eV}} = 2.18 \times 10^{-7} \text{ s} = 0.218 \mu\text{s.}$$

- **B.1:**

$$m = \frac{p}{c} \sqrt{\left(\frac{ct}{l}\right)^2 - 1}.$$

- **B.2:** The threshold length is

$$l_{\text{crit}} = 1.27739 \dots \text{ m.}$$

The stated strict separation criterion requires $l > l_{\text{crit}}$, giving a minimum practical length of approximately **1.28 m**.

- **B.3:**

$$m = \frac{eBr}{c} \sqrt{\left[\frac{ct}{2r \arcsin(R/(2r))} \right]^2 - 1}.$$

- **B.4:** A: proton, $m \approx 0.94 \text{ GeV}/c^2$; B: charged pion, $m \approx 0.14 \text{ GeV}/c^2$; C: proton, $m \approx 0.94 \text{ GeV}/c^2$; D: charged kaon, $m \approx 0.50 \text{ GeV}/c^2$.

Solution

A.1

For a proton accelerated from rest, conservation of energy gives

$$eV = K = (\gamma - 1)m_p c^2, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Thus

$$\gamma = 1 + \frac{eV}{m_p c^2}.$$

Solving for v ,

$$v = c \sqrt{1 - \left(1 + \frac{eV}{m_p c^2}\right)^{-2}}.$$

This expression is exact under the assumption of acceleration from rest without energy losses.

A.2

For a particle of mass m ,

$$\Delta = 1 - \frac{v}{c} = 1 - \sqrt{1 - \gamma^{-2}}.$$

At high energy, $\gamma^{-2} \ll 1$. Using the first-order expansion

$$\sqrt{1 - x} = 1 - \frac{x}{2} + O(x^2),$$

we obtain

$$\Delta = \frac{1}{2\gamma^2} + O(\gamma^{-4}).$$

In terms of the accelerating voltage,

$$\Delta \simeq \frac{1}{2} \left(1 + \frac{eV}{mc^2}\right)^{-2}.$$

When $eV \gg mc^2$, this further reduces to

$$\Delta \simeq \frac{m^2 c^4}{2e^2 V^2}.$$

Equivalently, using the total energy $E = \gamma mc^2$,

$$\Delta \simeq \frac{1}{2} \left(\frac{mc^2}{E}\right)^2.$$

For the electrons,

$$m_e c^2 = 0.5109989461 \text{ MeV}, \quad E = 60.0 \text{ GeV} = 6.00 \times 10^4 \text{ MeV},$$

so

$$\Delta \simeq \frac{1}{2} \left(\frac{0.5109989461}{6.00 \times 10^4}\right)^2 = \boxed{3.63 \times 10^{-11}}.$$

A.3

Let the circular orbit radius be

$$\rho = \frac{L}{2\pi}.$$

The magnetic force changes the direction of the relativistic momentum while leaving its magnitude constant. For uniform circular motion,

$$\left| \frac{d\mathbf{p}}{dt} \right| = \frac{pv}{\rho}.$$

Equating this to the magnetic force,

$$evB = \frac{pv}{\rho},$$

gives

$$B = \frac{p}{e\rho}.$$

The relativistic energy–momentum relation is

$$E^2 = p^2c^2 + m_p^2c^4,$$

where E is the total proton energy. Therefore,

$$B = \frac{2\pi}{ecL} \sqrt{E^2 - m_p^2c^4}.$$

At $E = 7.00 \text{ TeV}$,

$$\frac{m_p c^2}{E} = \frac{938.2720813 \text{ MeV}}{7.00 \times 10^6 \text{ MeV}} \approx 1.34 \times 10^{-4}.$$

Replacing pc by E produces a relative correction of only

$$\frac{1}{2} \left(\frac{m_p c^2}{E} \right)^2 \approx 8.98 \times 10^{-9},$$

far below the precision of the given energy. Hence

$$B \simeq \frac{2\pi E}{ecL}.$$

Using $E = 7.00 \times 10^{12} e \text{ J}$,

$$B \simeq \frac{2\pi(7.00 \times 10^{12})}{(299792458)(26659)} \text{ T} = \boxed{5.50 \text{ T}}.$$

A.4

Write

$$P_{\text{rad}} = Cq^\alpha a^\beta c^\delta \varepsilon_0^\eta,$$

where C is dimensionless. Using mass, length, time and charge as base dimensions,

$$[P] = ML^2T^{-3}, \quad [q] = Q, \quad [a] = LT^{-2}, \quad [c] = LT^{-1},$$

and

$$[\varepsilon_0] = Q^2T^2M^{-1}L^{-3}.$$

Matching powers gives

$$\begin{aligned} -\eta &= 1, & \alpha + 2\eta &= 0, \\ \beta + \delta - 3\eta &= 2, & -2\beta - \delta + 2\eta &= -3. \end{aligned}$$

Thus

$$\eta = -1, \quad \alpha = 2, \quad \beta = 2, \quad \delta = -3.$$

Consequently,

$$P_{\text{rad}} = C \frac{q^2 a^2}{\varepsilon_0 c^3}.$$

Dimensional analysis determines the dependence on the physical quantities but cannot determine C .

A.5

Including the numerical factor and relativistic correction specified in the problem, one proton radiates

$$P_1 = \frac{e^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3}.$$

For circular motion,

$$a = \frac{v^2}{\rho} \simeq \frac{c^2}{\rho}, \quad \rho = \frac{L}{2\pi}, \quad \gamma = \frac{E}{m_p c^2}.$$

Hence

$$P_1 \simeq \frac{e^2 c}{6\pi\epsilon_0 \rho^2} \left(\frac{E}{m_p c^2} \right)^4.$$

Numerically,

$$\rho = \frac{26659}{2\pi} \text{ m} = 4242.91 \text{ m}, \quad \gamma = \frac{7.00 \times 10^6}{938.2720813} = 7460.52.$$

Substitution yields

$$P_1 \simeq 7.93 \times 10^{-12} \text{ W}.$$

There are two beams, each containing 2808 bunches with 1.15×10^{11} protons per bunch. Thus the total number of protons is

$$N = 2(2808)(1.15 \times 10^{11}) = 6.4584 \times 10^{14}.$$

Adding the individual radiated powers,

$$P_{\text{tot}} = NP_1 \simeq 5.12467 \times 10^3 \text{ W}.$$

Therefore,

$$\boxed{P_{\text{tot}} = 5.12 \text{ kW}.}$$

A.6

The homogeneous electric field has magnitude V/d , so the force on a proton is constant:

$$F = \frac{eV}{d}.$$

Relativistically, the equation of motion is

$$\frac{dp}{dt} = F.$$

Since the proton starts from rest,

$$p_f = FT, \quad T = \frac{p_f}{F}.$$

The work done over the accelerator is eV , giving final total energy

$$E_f = m_p c^2 + eV.$$

Therefore,

$$p_f c = \sqrt{E_f^2 - m_p^2 c^4} = \sqrt{(eV)^2 + 2eV m_p c^2}.$$

Combining these expressions,

$$T = \frac{d}{eVc} \sqrt{(eV)^2 + 2eVm_p c^2},$$

or

$$T = \frac{d}{c} \sqrt{1 + \frac{2m_p c^2}{eV}}.$$

Here $eV = 500 \text{ MeV}$. Thus

$$T = \frac{30.0}{299792458} \sqrt{1 + \frac{2(938.2720813)}{500}} \text{ s} = \boxed{2.18 \times 10^{-7} \text{ s}}.$$

The use of constant force, rather than constant acceleration, accounts for the relativistic change in velocity.

B.1

For the straight path,

$$v = \frac{l}{t}.$$

Relativistic momentum satisfies

$$p = \gamma m v = \frac{m v}{\sqrt{1 - v^2/c^2}}.$$

Solving for m ,

$$m = \frac{p}{v} \sqrt{1 - \frac{v^2}{c^2}} = p \sqrt{\frac{t^2}{l^2} - \frac{1}{c^2}}.$$

Therefore,

$$m = \frac{p}{c} \sqrt{\left(\frac{ct}{l}\right)^2 - 1}.$$

This expression is exact; no expansion about $v = c$ is needed.

B.2

Solving the result of B.1 for flight time gives

$$t(m) = \frac{l}{c} \sqrt{1 + \left(\frac{mc}{p}\right)^2}.$$

At equal momentum, the heavier kaon arrives later. Their time difference is

$$t_K - t_\pi = \frac{l}{c} \left[\sqrt{1 + \left(\frac{m_K c}{p}\right)^2} - \sqrt{1 + \left(\frac{m_\pi c}{p}\right)^2} \right].$$

The required separation is

$$t_K - t_\pi > 3(150 \text{ ps}) = 450 \text{ ps}.$$

The threshold length is therefore

$$l_{\text{crit}} = \frac{3\sigma_t c}{\sqrt{1 + (m_K c/p)^2} - \sqrt{1 + (m_\pi c/p)^2}}, \quad \sigma_t = 150 \text{ ps.}$$

Using $p = 1000 \text{ MeV}/c$,

$$l_{\text{crit}} = \frac{(450 \times 10^{-12})(299792458)}{\sqrt{1 + (494/1000)^2} - \sqrt{1 + (140/1000)^2}} \text{ m.}$$

The dimensionless denominator is 0.10561117, yielding

$$l_{\text{crit}} = 1.27739 \dots \text{ m.}$$

Thus l must exceed this threshold; approximately $\boxed{1.28 \text{ m}}$ is required. The exact relativistic expression is used because the kaon's mass is not negligible compared with p/c .

B.3

Because the particle has no momentum parallel to the beamline, its entire momentum is transverse to the magnetic field. As in A.3,

$$p = eBr.$$

The magnetic field does no work, so the speed is constant along the circular track.

Let θ be the angle swept out about the centre of the trajectory before the particle first reaches the ToF tube. The collision point and detection point are separated by the tube radius R . They are also the endpoints of a chord of the trajectory circle, whose radius is r . Therefore,

$$R = 2r \sin \frac{\theta}{2}.$$

For the first encounter with the tube,

$$\theta = 2 \arcsin \frac{R}{2r},$$

and the flight path length is

$$s = r\theta = 2r \arcsin \frac{R}{2r}.$$

This requires $r \geq R/2$; a smaller orbit never reaches the tube.

The speed is

$$v = \frac{s}{t}.$$

Using the mass formula from B.1 with the actual path length s , together with $p = eBr$,

$$m = \frac{eBr}{c} \sqrt{\left(\frac{ct}{s}\right)^2 - 1}.$$

Consequently,

$$m = \frac{eBr}{c} \sqrt{\left[\frac{ct}{2r \arcsin(R/(2r))}\right]^2 - 1}.$$

The inverse sine is evaluated in radians.

B.4

For each particle, use

$$s = 2r \arcsin \frac{3.70 \text{ m}}{2r}, \quad p = e(0.500 \text{ T})r,$$

and

$$m = \frac{p}{c} \sqrt{\left(\frac{ct}{s}\right)^2 - 1}.$$

The momentum conversion follows directly from $p = eBr$:

$$p [\text{GeV}/c] = 0.299792458 B [\text{T}] r [\text{m}].$$

Retaining guard digits to compare the calculated masses with Table 2 gives:

Particle	Path length s [m]	Momentum p [MeV/c]	Calculated mass [MeV/ c^2]	Identification
A	3.7864	764.47	938.65	Proton
B	4.0019	440.69	139.32	Charged pion
C	3.7600	908.37	935.10	Proton
D	4.2908	346.26	495.90	Charged kaon

The small differences from the tabulated masses are consistent with the rounded measurement values.

To sensible precision,

$A :$	$m \approx 0.94 \text{ GeV}/c^2$	proton,
$B :$	$m \approx 0.14 \text{ GeV}/c^2$	charged pion,
$C :$	$m \approx 0.94 \text{ GeV}/c^2$	proton,
$D :$	$m \approx 0.50 \text{ GeV}/c^2$	charged kaon.

The solution was checked against the official solution.