

IPhO 2015

Theory problems and solutions · Mumbai, India · 3 theory problems with complete solutions

eulersolve.org/ipho/2015/

Problem 1: Particles from the Sun

Photons from the surface of the Sun and neutrinos from its core can tell us about solar temperatures and also confirm that the Sun shines because of nuclear reactions.

Throughout this problem, take the mass of the Sun to be $M_{\odot} = 2.00 \times 10^{30}$ kg, its radius, $R_{\odot} = 7.00 \times 10^8$ m, its luminosity (radiation energy emitted per unit time), $L_{\odot} = 3.85 \times 10^{26}$ W, and the Earth-Sun distance, $d_{\odot} = 1.50 \times 10^{11}$ m.

Note:

$$(i) \quad \int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2} \right) e^{ax} + \text{constant}$$

$$(ii) \quad \int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right) e^{ax} + \text{constant}$$

$$(iii) \quad \int x^3 e^{ax} dx = \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4} \right) e^{ax} + \text{constant}$$

A Radiation from the sun :

Part	Question	Marks
A1	Assume that the Sun radiates like a perfect blackbody. Use this fact to calculate the temperature, T_s , of the solar surface.	0.3

The spectrum of solar radiation can be approximated well by the Wien distribution law. Accordingly, the solar energy incident on any surface on the Earth per unit time per unit frequency interval, $u(\nu)$, is given by

$$u(\nu) = A \frac{R_{\odot}^2}{d_{\odot}^2} \frac{2\pi h}{c^2} \nu^3 \exp(-h\nu/k_B T_s),$$

where ν is the frequency and A is the area of the surface normal to the direction of the incident radiation.

Now, consider a solar cell which consists of a thin disc of semiconducting material of area, A , placed perpendicular to the direction of the Sun's rays.

Part	Question	Marks
A2	Using the Wien approximation, express the total radiated solar power, P_{in} , incident on the surface of the solar cell, in terms of A , $R_{\odot}$, $d_{\odot}$, T_s and the fundamental constants c , h , k_B .	0.3
A3	Express the number of photons, $n_{\gamma}(\nu)$, per unit time per unit frequency interval incident on the surface of the solar cell in terms of A , $R_{\odot}$, $d_{\odot}$, T_s , ν and the fundamental	0.2

Part	Question	Marks
	constants c, h, k_B .	

The semiconducting material of the solar cell has a “band gap” of energy, E_g . We assume the following model. Every photon of energy $E \geq E_g$ excites an electron across the band gap. This electron contributes an energy, E_g , as the useful output energy, and any extra energy is dissipated as heat (not converted to useful energy).

Part	Question	Marks
A4	Define $x_g = h\nu_g/k_B T_s$ where $E_g = h\nu_g$. Express the useful output power of the cell, P_{out} , in terms of $x_g, A, R_\odot, d_\odot, T_s$ and the fundamental constants c, h, k_B .	1.0
A5	Express the efficiency, η , of this solar cell in terms of x_g .	0.2
A6	Make a qualitative sketch of η versus x_g . The values at $x_g = 0$ and $x_g \rightarrow \infty$ should be clearly shown. What is the slope of $\eta(x_g)$ at $x_g = 0$ and $x_g \rightarrow \infty$?	1.0
A7	Let x_0 be the value of x_g for which η is maximum. Obtain the cubic equation that gives x_0 . Estimate the value of x_0 within an accuracy of ± 0.25 . Hence calculate $\eta(x_0)$.	1.0
A8	The band gap of pure silicon is $E_g = 1.11$ eV. Calculate the efficiency, η_{Si} , of a silicon solar cell using this value.	0.2

¹ *Amol Dighe (TIFR), Anwesh Mazumdar (HBCSE-TIFR) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

In the late nineteenth century, Kelvin and Helmholtz (KH) proposed a hypothesis to explain how the Sun shines. They postulated that starting as a very large cloud of matter of mass, $M_\odot$, and negligible density, the Sun has been shrinking continuously. The shining of the Sun would then be due to the release of gravitational potential energy through this slow contraction.

Part	Question	Marks
A9	Let us assume that the density of matter is uniform inside the Sun. Find the total gravitational potential energy, Ω , of the Sun at present, in terms of $G, M_\odot$ and $R_\odot$.	0.3
A10	Estimate the maximum possible time, τ_{KH} (in years), for which the Sun could have been shining, according to the KH hypothesis. Assume that the luminosity of the Sun has been constant throughout this period.	0.5

The τ_{KH} calculated above does not match the age of the solar system estimated from studies of meteorites. This shows that the energy source of the Sun cannot be purely gravitational.

B Neutrinos from the Sun :

In 1938, Hans Bethe proposed that nuclear fusion of hydrogen into helium in the core of the Sun is the source of its energy. The net nuclear reaction is:



The “electron neutrinos”, ν_e , produced in this reaction may be taken to be massless. They escape the Sun and their detection on the Earth confirms the occurrence of nuclear reactions inside the Sun. Energy carried away by the neutrinos can be neglected in this problem.

Part	Question	Marks
B1	Calculate the flux density, Φ_ν , of the number of neutrinos arriving at the Earth, in units of $\text{m}^{-2} \text{s}^{-1}$. The energy released in the above reaction is $\Delta E = 4.0 \times 10^{-12} \text{ J}$. Assume that the energy radiated by the Sun is entirely due to this reaction.	0.6

Travelling from the core of the Sun to the Earth, some of the electron neutrinos, ν_e , are converted to other types of neutrinos, ν_x . The efficiency of the detector for detecting ν_x is $1/6$ of its efficiency for detecting ν_e . If there is no neutrino conversion, we expect to detect an average of N_1 neutrinos in a year. However, due to the conversion, an average of N_2 neutrinos (ν_e and ν_x combined) are actually detected per year.

Part	Question	Marks
B2	In terms of N_1 and N_2 , calculate what fraction, f , of ν_e is converted to ν_x .	0.4

In order to detect neutrinos, large detectors filled with water are constructed. Although the interactions of neutrinos with matter are very rare, occasionally they knock out electrons from water molecules in the detector. These energetic electrons move through water at high speeds, emitting electromagnetic radiation in the process. As long as the speed of such an electron is greater than the speed of light in water (refractive index, n), this radiation, called Cherenkov radiation, is emitted in the shape of a cone.

Part	Question	Marks
B3	Assume that an electron knocked out by a neutrino loses energy at a constant rate of α per unit time, while it travels through water. If this electron emits Cherenkov radiation for a time, Δt , determine the energy imparted to this electron (E_{imparted}) by the neutrino, in terms of α , Δt , n , m_e and c . (Assume the electron to be at rest before its interaction with the neutrino.)	2.0

The fusion of H into He inside the Sun takes place in several steps. Nucleus of ${}^7\text{Be}$ (rest mass, m_{Be}) is produced in one of these intermediate steps. Subsequently, it can absorb an electron, producing a ${}^7\text{Li}$ nucleus (rest mass, $m_{\text{Li}} < m_{\text{Be}}$) and emitting a ν_e . The corresponding nuclear reaction is:



When a Be nucleus ($m_{\text{Be}} = 11.65 \times 10^{-27} \text{ kg}$) is at rest and absorbs an electron also at rest, the emitted neutrino has energy $E_\nu = 1.44 \times 10^{-13} \text{ J}$. However, the Be nuclei are in random thermal motion due to the temperature T_c at the core of the Sun, and act as moving neutrino sources. As a result, the energy of emitted neutrinos fluctuates with a root mean square (rms) value ΔE_{rms} .

Part	Question	Marks
B4	If $\Delta E_{\text{rms}} = 5.54 \times 10^{-17} \text{ J}$, calculate the rms speed of the Be nuclei, V_{Be} , and hence estimate T_c . (Hint: ΔE_{rms} depends on the rms value of the component of velocity along the line of sight).	2.0

Official English text of the problem.

Answers

- A1

$$T_s = \left(\frac{L_\odot}{4\pi R_\odot^2 \sigma} \right)^{1/4} \simeq 5.76 \times 10^3 \text{ K}.$$

- A2

$$P_{\text{in}} = \frac{12\pi A R_\odot^2 (k_B T_s)^4}{d_\odot^2 c^2 h^3}.$$

- A3

$$n_\gamma(\nu) = \frac{2\pi A R_\odot^2}{d_\odot^2 c^2} \nu^2 e^{-h\nu/(k_B T_s)}.$$

- A4

$$P_{\text{out}} = \frac{2\pi A R_\odot^2 (k_B T_s)^4}{d_\odot^2 c^2 h^3} x_g (x_g^2 + 2x_g + 2) e^{-x_g}.$$

- A5

$$\eta(x_g) = \frac{x_g (x_g^2 + 2x_g + 2) e^{-x_g}}{6}.$$

- A6 The curve starts at the origin with slope $1/3$, reaches one maximum near $(2.27, 0.457)$, and approaches the horizontal axis from above:

$$\eta(0) = 0, \quad \lim_{x_g \rightarrow \infty} \eta = 0, \quad \eta'(0) = \frac{1}{3}, \quad \lim_{x_g \rightarrow \infty} \eta' = 0 \quad \text{from below.}$$

- A7

$$x_0^3 - x_0^2 - 2x_0 - 2 = 0, \quad x_0 \simeq 2.27, \quad \eta(x_0) \simeq 0.457 = 45.7\%.$$

- A8

$$x_{g,\text{Si}} \simeq 2.24, \quad \eta_{\text{Si}} \simeq 0.457 = 45.7\%.$$

- A9

$$\Omega = -\frac{3GM_\odot^2}{5R_\odot} \simeq -2.29 \times 10^{41} \text{ J}.$$

- A10

$$\tau_{\text{KH}} = \frac{3GM_\odot^2}{5R_\odot L_\odot} \simeq 1.88 \times 10^7 \text{ years}.$$

- B1

$$\Phi_\nu = \frac{2L_\odot}{4\pi d_\odot^2 \Delta E} \simeq 6.8 \times 10^{14} \text{ m}^{-2} \text{ s}^{-1}.$$

• B2

$$f = \frac{6}{5} \left(1 - \frac{N_2}{N_1} \right).$$

• B3

$$E_{\text{imparted}} = \alpha \Delta t + m_e c^2 \left(\frac{1}{\sqrt{1 - 1/n^2}} - 1 \right).$$

• B4

$$V_{\text{Be}} = \sqrt{3} c \frac{\Delta E_{\text{rms}}}{E_\nu} \simeq 2.00 \times 10^5 \text{ m s}^{-1},$$

$$T_c = \frac{m_{\text{Be}} c^2}{k_B} \left(\frac{\Delta E_{\text{rms}}}{E_\nu} \right)^2 \simeq 1.12 \times 10^7 \text{ K}.$$

Solution

For numerical evaluation, use the standard constants

$$\begin{aligned}\sigma &= 5.6704 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}, \\ G &= 6.6743 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}, \\ k_B &= 1.380649 \times 10^{-23} \text{ J K}^{-1}, \\ c &= 2.99792458 \times 10^8 \text{ m s}^{-1},\end{aligned}$$

with $1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ J}$ and $1 \text{ year} = 3.15576 \times 10^7 \text{ s}$.

A1

By the Stefan–Boltzmann law, a blackbody emits power σT^4 per unit surface area. The Sun's emitting area is $4\pi R_\odot^2$, so

$$L_\odot = 4\pi R_\odot^2 \sigma T_s^4.$$

Therefore,

$$T_s = \left[\frac{3.85 \times 10^{26}}{4\pi(7.00 \times 10^8)^2(5.6704 \times 10^{-8})} \right]^{1/4} = \boxed{5.76 \times 10^3 \text{ K}}.$$

A2

The total incident power is the integral over all frequencies:

$$P_{\text{in}} = A \frac{R_\odot^2}{d_\odot^2} \frac{2\pi h}{c^2} \int_0^\infty \nu^3 e^{-h\nu/(k_B T_s)} d\nu.$$

Set

$$x = \frac{h\nu}{k_B T_s}, \quad d\nu = \frac{k_B T_s}{h} dx.$$

Then

$$P_{\text{in}} = \frac{2\pi A R_\odot^2 (k_B T_s)^4}{d_\odot^2 c^2 h^3} \int_0^\infty x^3 e^{-x} dx.$$

The supplied integration formula gives

$$\int_0^{\infty} x^3 e^{-x} dx = [-e^{-x}(x^3 + 3x^2 + 6x + 6)]_0^{\infty} = 6.$$

Thus

$$P_{\text{in}} = \frac{12\pi AR_{\odot}^2 (k_B T_s)^4}{d_{\odot}^2 c^2 h^3}.$$

All efficiencies below use this same Wien-approximation input power.

A3

Each photon of frequency ν carries energy $h\nu$. Dividing the spectral power by this energy gives the spectral photon arrival rate:

$$n_{\gamma}(\nu) = \frac{u(\nu)}{h\nu}.$$

Hence

$$n_{\gamma}(\nu) = \frac{2\pi AR_{\odot}^2}{d_{\odot}^2 c^2} \nu^2 e^{-h\nu/(k_B T_s)}.$$

Its units are photons per second per hertz.

A4

Only photons with $\nu \geq \nu_g$ contribute, and each contributes useful energy E_g . Consequently,

$$P_{\text{out}} = E_g \int_{\nu_g}^{\infty} n_{\gamma}(\nu) d\nu.$$

Using $E_g = x_g k_B T_s$ and the substitution from A2,

$$P_{\text{out}} = \frac{2\pi AR_{\odot}^2 (k_B T_s)^4}{d_{\odot}^2 c^2 h^3} x_g \int_{x_g}^{\infty} x^2 e^{-x} dx.$$

Now

$$\int_{x_g}^{\infty} x^2 e^{-x} dx = [-(x^2 + 2x + 2)e^{-x}]_{x_g}^{\infty} = (x_g^2 + 2x_g + 2)e^{-x_g}.$$

Therefore,

$$P_{\text{out}} = \frac{2\pi AR_{\odot}^2 (k_B T_s)^4}{d_{\odot}^2 c^2 h^3} x_g (x_g^2 + 2x_g + 2)e^{-x_g}.$$

A5

Efficiency is useful output power divided by incident power:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}}.$$

The common dimensional and geometrical factors cancel, giving

$$\eta(x_g) = \frac{x_g(x_g^2 + 2x_g + 2)e^{-x_g}}{6}, \quad x_g \geq 0.$$

A6

At zero band gap,

$$\eta(0) = 0.$$

Although photons can then excite electrons, the useful energy per electron is zero.

For large x_g ,

$$\eta(x_g) \sim \frac{x_g^3 e^{-x_g}}{6},$$

so

$$\lim_{x_g \rightarrow \infty} \eta(x_g) = 0.$$

The exponentially small number of photons above the gap makes the useful output vanish.

Differentiating,

$$\eta'(x_g) = \frac{e^{-x_g}}{6} (-x_g^3 + x_g^2 + 2x_g + 2).$$

Thus

$$\eta'(0) = \frac{1}{3}, \quad \lim_{x_g \rightarrow \infty} \eta'(x_g) = 0 \text{ from below}.$$

The sketch starts at the origin with tangent $\eta = x_g/3$, rises to a single maximum near $x_g = 2.27$, and then decreases toward the horizontal asymptote $\eta = 0$, remaining positive. For additional shape information,

$$\eta''(x_g) = \frac{e^{-x_g}}{6} x_g^2 (x_g - 4),$$

so the curve is concave downward for $0 < x_g < 4$ and concave upward for $x_g > 4$.

Representative points are:

x_g	$\eta(x_g)$
0	0
1	0.307
2	0.451
2.27	0.457
4	0.317
6	0.124
10	0.00923
$\rightarrow \infty$	$\rightarrow 0$

A7

At the maximum, $\eta'(x_0) = 0$. Since the exponential factor never vanishes,

$$-x_0^3 + x_0^2 + 2x_0 + 2 = 0,$$

or

$$\boxed{x_0^3 - x_0^2 - 2x_0 - 2 = 0}.$$

For $x_0 > 0$, this is equivalent to

$$x_0 = 1 + \frac{2}{x_0} + \frac{2}{x_0^2}.$$

The left-hand side increases strictly with x_0 , while the right-hand side decreases strictly. Therefore, there is only one positive stationary point. Together with the endpoint behaviour in A6, this establishes that it is the global maximum.

Let $F(x) = x^3 - x^2 - 2x - 2$. Then

$$F(2) = -2, \quad F(2.5) = 2.375.$$

Thus $2 < x_0 < 2.5$, already giving $x_0 = 2.25 \pm 0.25$. Refinement gives

$$\boxed{x_0 \simeq 2.27}.$$

Substituting into the efficiency,

$$\eta(x_0) = \frac{x_0(x_0^2 + 2x_0 + 2)e^{-x_0}}{6} \simeq 0.45703.$$

Hence

$$\boxed{\eta_{\max} \simeq 45.7\%}.$$

A8

At the surface temperature found in A1,

$$k_B T_s \simeq 0.4966 \text{ eV}.$$

For silicon,

$$x_{g,\text{Si}} = \frac{E_g}{k_B T_s} = \frac{1.11}{0.4966} \simeq 2.235.$$

Therefore,

$$\eta_{\text{Si}} = \frac{2.235 (2.235^2 + 2(2.235) + 2)e^{-2.235}}{6} \simeq 0.45694.$$

Thus

$$\boxed{\eta_{\text{Si}} \simeq 45.7\%}.$$

This is very close to the maximum because the silicon gap is close to the optimal gap $x_0 k_B T_s \simeq 1.13 \text{ eV}$.

A9

Build the uniform sphere by bringing successive thin spherical shells from infinity. At radius r , the mass already assembled is

$$M(r) = M_{\odot} \frac{r^3}{R_{\odot}^3},$$

and the shell mass is

$$dM = \frac{3M_{\odot} r^2}{R_{\odot}^3} dr.$$

By the shell theorem, the potential energy added is

$$d\Omega = -\frac{GM(r) dM}{r}.$$

Therefore,

$$\Omega = -\frac{3GM_{\odot}^2}{R_{\odot}^6} \int_0^{R_{\odot}} r^4 dr = -\frac{3GM_{\odot}^2}{R_{\odot}^6} \frac{R_{\odot}^5}{5}.$$

Hence

$$\boxed{\Omega = -\frac{3GM_{\odot}^2}{5R_{\odot}}} \simeq -2.29 \times 10^{41} \text{ J}.$$

The negative sign corresponds to taking zero potential energy at infinite separation.

A10

The initial very extended cloud has approximately zero gravitational potential energy. Its contraction therefore releases at most

$$E_{\text{available}} = -\Omega = \frac{3GM_{\odot}^2}{5R_{\odot}}.$$

The maximum possible shining time follows by taking all this released energy to have been radiated. At constant luminosity,

$$L_{\odot} \tau_{\text{KH}} = \frac{3GM_{\odot}^2}{5R_{\odot}},$$

so

$$\boxed{\tau_{\text{KH}} = \frac{3GM_{\odot}^2}{5R_{\odot} L_{\odot}}}.$$

Numerically,

$$\tau_{\text{KH}} = \frac{2.288 \times 10^{41}}{3.85 \times 10^{26}} \simeq 5.94 \times 10^{14} \text{ s} = \boxed{1.88 \times 10^7 \text{ years}}.$$

Any released energy retained as internal energy would reduce the duration supported by gravitational contraction, so this is an upper bound.

B1

Neglecting energy carried away by neutrinos, the number of fusion reactions per second is

$$\dot{N}_{\text{reaction}} = \frac{L_{\odot}}{\Delta E}.$$

Each reaction produces two neutrinos:

–

$$\dot{N}_\nu = \frac{2L_\odot}{\Delta E}.$$

For spherical emission, these neutrinos cross an area $4\pi d_\odot^2$ at Earth's distance. Thus

$$\Phi_\nu = \frac{2L_\odot}{4\pi d_\odot^2 \Delta E}.$$

Substitution gives

$$\Phi_\nu = \frac{2(3.85 \times 10^{26})}{4\pi(1.50 \times 10^{11})^2(4.0 \times 10^{-12})} = \boxed{6.8 \times 10^{14} \text{ m}^{-2} \text{ s}^{-1}}.$$

B2

After conversion, the fraction $1 - f$ remains as electron neutrinos and contributes $(1 - f)N_1$ detections per year. The converted fraction contributes $fN_1/6$, because its detection efficiency is six times smaller.

Consequently,

$$N_2 = (1 - f)N_1 + \frac{fN_1}{6} = N_1 \left(1 - \frac{5f}{6}\right).$$

Solving,

$$f = \frac{6}{5} \left(1 - \frac{N_2}{N_1}\right).$$

The physical range $0 \leq f \leq 1$ corresponds to $N_1/6 \leq N_2 \leq N_1$.

B3

Cherenkov radiation ceases when the electron slows to the speed of light in water:

$$v_{\text{th}} = \frac{c}{n}.$$

The electron's kinetic energy at that instant is relativistic:

$$K_{\text{th}} = (\gamma_{\text{th}} - 1)m_e c^2, \quad \gamma_{\text{th}} = \frac{1}{\sqrt{1 - v_{\text{th}}^2/c^2}} = \frac{1}{\sqrt{1 - 1/n^2}}.$$

During the emission time Δt , the electron loses energy $\alpha \Delta t$. Its initial kinetic energy immediately after the neutrino interaction is therefore

$$K_i = K_{\text{th}} + \alpha \Delta t.$$

Since the electron was initially at rest before the interaction, the energy imparted by the neutrino equals K_i . Hence

$$E_{\text{imparted}} = \alpha \Delta t + m_e c^2 \left(\frac{1}{\sqrt{1 - 1/n^2}} - 1 \right).$$

The second term is essential: the electron still has kinetic energy when Cherenkov emission stops.

B4

Use the stated model in which the energy spread comes from thermal motion of the Be sources. Let $v_{\parallel}$ be the source velocity component toward Earth.

For a massless neutrino, momentum in the source rest frame is $p_{\nu} = E_{\nu}/c$. To first order in the nonrelativistic source velocity, the Lorentz transformation of energy gives

$$E_{\text{obs}} \simeq E_{\nu} + v_{\parallel} \frac{E_{\nu}}{c}.$$

Thus the Doppler energy shift is

$$\delta E \simeq E_{\nu} \frac{v_{\parallel}}{c}.$$

The thermal motion is isotropic, so

$$\langle v_{\parallel} \rangle = 0, \quad \langle v_{\parallel}^2 \rangle = \frac{\langle v^2 \rangle}{3} = \frac{V_{\text{Be}}^2}{3}.$$

Taking the rms energy shift,

$$\Delta E_{\text{rms}} = \frac{E_{\nu}}{c} \sqrt{\langle v_{\parallel}^2 \rangle} = \frac{E_{\nu} V_{\text{Be}}}{\sqrt{3} c}.$$

Therefore,

$$V_{\text{Be}} = \sqrt{3} c \frac{\Delta E_{\text{rms}}}{E_{\nu}}.$$

Numerically,

$$\frac{\Delta E_{\text{rms}}}{E_{\nu}} = \frac{5.54 \times 10^{-17}}{1.44 \times 10^{-13}} = 3.847 \times 10^{-4},$$

and hence

$$V_{\text{Be}} \simeq 2.00 \times 10^5 \text{ m s}^{-1}.$$

The ratio $V_{\text{Be}}/c \simeq 6.66 \times 10^{-4}$ validates the nonrelativistic approximation.

By equipartition for the three translational degrees of freedom,

$$\frac{1}{2} m_{\text{Be}} V_{\text{Be}}^2 = \frac{3}{2} k_B T_c.$$

Thus

$$T_c = \frac{m_{\text{Be}} V_{\text{Be}}^2}{3 k_B} = \frac{m_{\text{Be}} c^2}{k_B} \left(\frac{\Delta E_{\text{rms}}}{E_{\nu}} \right)^2.$$

Substituting the data,

$$T_c = \frac{(11.65 \times 10^{-27})(2.9979 \times 10^8)^2}{1.380649 \times 10^{-23}} \left(\frac{5.54 \times 10^{-17}}{1.44 \times 10^{-13}} \right)^2,$$

which gives

$$T_c \simeq 1.12 \times 10^7 \text{ K}.$$

The solution was checked against the official solution.

Problem 2: The Extremum Principle

A The Extremum Principle in Mechanics

Consider a horizontal frictionless $x - y$ plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation $x = x_1$. The potential energy of a point particle of mass m in region I is $V = 0$ while it is $V = V_0$ in region II. The particle is sent from the origin O with speed v_1 along a line making an angle θ_1 with the x -axis. It reaches point P in region II traveling with speed v_2 along a line that makes an angle θ_2 with the x -axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

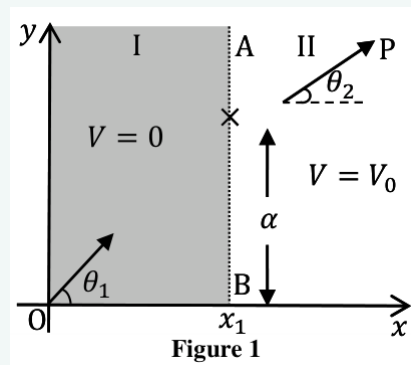


Figure 1

A1	Obtain an expression for v_2 in terms of m , v_1 and V_0 .	0.2
A2	Express v_2 in terms of v_1 , θ_1 and θ_2 .	0.3

We define a quantity called action $A = m \int v(s) ds$, where ds is the infinitesimal length along the trajectory of a particle of mass m moving with speed $v(s)$. The integral is taken over the path. As an example, for a particle moving with constant speed v on a circular path of radius R , the action A for one revolution will be $2\pi m R v$. For a particle with constant energy E , it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which A defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

A3	PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates $(0, 0)$ and (x_0, y_0) respectively and the boundary point where the particle transits from region I to region II have coordinates (x_1, α) . Note that x_1 is fixed and the action depends on the coordinate α only. State the expression for the action $A(\alpha)$. Use PLA to obtain the relationship between v_1/v_2 and these coordinates.	1.0
----	---	-----

B The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices n_1 and n_2 respectively. The two media are separated by a line parallel to the x -axis. The light ray makes an angle i_1 with the y -axis in medium I and i_2 in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

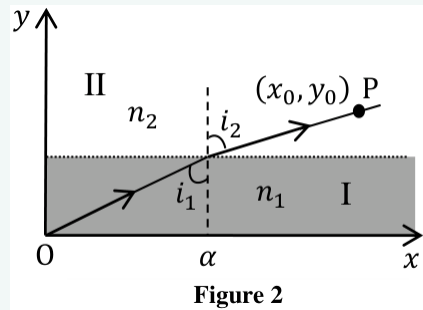


Figure 2

B1	The principle states that between two fixed points, a light ray moves along a path such that time taken between the two points is an extremum. Derive the relation between $\sin i_1$ and $\sin i_2$ on the basis of Fermat's principle.	0.5
----	--	-----

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

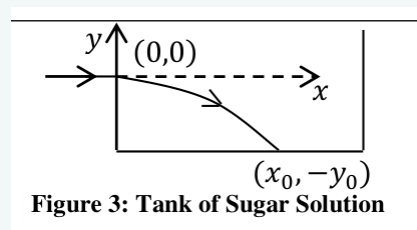


Figure 3: Tank of Sugar Solution

B2	Assume that the refractive index $n(y)$ depends only on y . Use the equation obtained in B1 to obtain the expression for the slope dy/dx of the beam's path in terms of refractive index n_0 at $y = 0$ and $n(y)$.	1.5
B3	The laser beam is directed horizontally from the origin $(0, 0)$ into the sugar solution at a height y_0 from the bottom of the tank as shown in figure 3. Take $n(y) = n_0 - ky$ where n_0 and k are positive constants. Obtain an expression for x in terms of y and related quantities for the actual trajectory of the laser beam.	1.2

¹ Manoj Harbola (IIT-Kanpur) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.

	You may use: $\int \sec \theta d\theta = \ln(\sec \theta + \tan \theta) + \text{constant}$, where $\sec \theta = 1/\cos \theta$ or $\int \frac{dx}{\sqrt{x^2 - 1}} = \ln(x + \sqrt{x^2 - 1}) + \text{constant}$	
B4	Obtain the value of x_0 , the point where the beam meets the bottom of the tank. Take $y_0 = 10.0 \text{ cm}$, $n_0 = 1.50$, $k = 0.050 \text{ cm}^{-1}$ ($1 \text{ cm} = 10^{-2} \text{ m}$).	0.8

C The Extremum Principle and the Wave Nature of Matter

We now explore the connection between the PLA and the wave nature of a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

C1	As the particle moves along its trajectory by an infinitesimal distance Δs , relate the change $\Delta\varphi$ in the phase of its de Broglie wave to the change ΔA in the action and the Planck constant.	0.6
C2	Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width d in AB such that $d \ll (x_0 - x_1)$ and $d \ll x_1$. Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference $\Delta\varphi_{CD}$ between the two paths to first order.	1.2

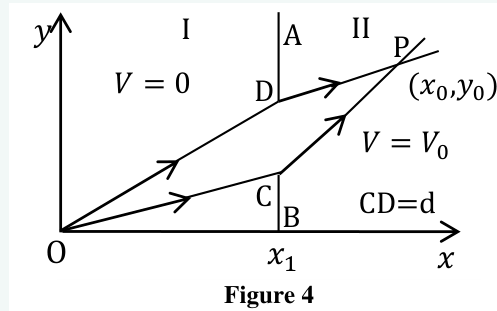


Figure 4

D Matter Wave Interference

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition A_1B_1 at $x = x_1$ such that OFP is a straight line. P is a point on the screen at $x = x_0$ (see Fig. 5). The speed in I is $v_1 = 2.0000 \times 10^7 \text{ m s}^{-1}$ and $\theta = 10.0000^\circ$. The potential in II is such that speed $v_2 = 1.9900 \times 10^7 \text{ m s}^{-1}$. The distance $x_0 - x_1$ is 250.00 mm ($1\text{mm} = 10^{-3}\text{m}$). Ignore electron-electron interaction.

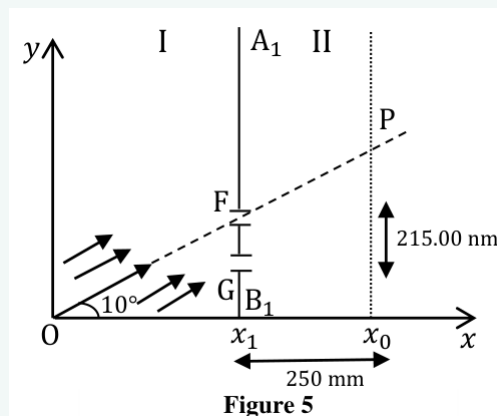


Figure 5

D1	If the electrons at O have been accelerated from rest, calculate the accelerating potential U_1 .	0.3
D2	Another identical slit G is made in the partition A_1B_1 at a distance of 215.00 nm ($1\text{nm} = 10^{-9}\text{m}$) below slit F (Fig. 5). If the phase difference between de Broglie waves	0.8

	arriving at P through the slits F and G is $2\pi\beta$, calculate β .	
D3	What is the smallest distance Δy from P at which null (zero) electron detection maybe expected on the screen? [Note: you may find the approximation $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta \cos\theta$ useful]	1.2
D4	The beam has a square cross section of $500\text{nm} \times 500\text{nm}$ and the setup is 2 m long. What should be the minimum flux density $I_{\min}$ (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time?	0.4

Official English text of the problem.

Answers

- **A1:** $v_2 = \sqrt{v_1^2 - \frac{2V_0}{m}}.$

- **A2:** $v_2 = v_1 \frac{\sin\theta_1}{\sin\theta_2}.$

- **A3:**

$$A(\alpha) = mv_1 \sqrt{x_1^2 + \alpha^2} + mv_2 \sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2},$$

$$\frac{v_1}{v_2} = \frac{(y_0 - \alpha) \sqrt{x_1^2 + \alpha^2}}{\alpha \sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2}}.$$

- **B1:** $n_1 \sin i_1 = n_2 \sin i_2.$

- **B2:** $\frac{dy}{dx} = -\sqrt{\frac{n(y)^2}{n_0^2} - 1}.$

- **B3:**

$$x(y) = \frac{n_0}{k} \ln \left[\frac{n_0 - ky + \sqrt{(n_0 - ky)^2 - n_0^2}}{n_0} \right].$$

Equivalently, $y(x) = \frac{n_0}{k} \left[1 - \cosh\left(\frac{kx}{n_0}\right) \right].$

- **B4:** $x_0 = 23.86 \dots \text{ cm} \approx 24 \text{ cm}.$

- **C1:** $\Delta\varphi = \frac{2\pi}{h} \Delta A = \frac{\Delta A}{\hbar}.$

- **C2:** $\Delta\varphi_{CD} = 0$ to first order in d ; the leading correction is quadratic.

- **D1:** $U_1 \approx 1.14 \times 10^3 \text{ V}.$

- **D2:** $\beta = \frac{m_e d (v_1 - v_2) \sin\theta}{h} \approx 5.13.$

- **D3:** $\Delta y \approx 1.6 \times 10^{-5} \text{ m} = 16 \text{ }\mu\text{m}, \text{ below P}.$

- **D4:** $I_{\min} \approx 4 \times 10^{19} \text{ m}^{-2} \text{ s}^{-1}.$

Solution

For the numerical electron calculations, use

$$m_e = 9.1093837 \times 10^{-31} \text{ kg}, \quad e = 1.602176634 \times 10^{-19} \text{ C}, \quad h = 6.62607015 \times 10^{-34} \text{ J s},$$

where e is the magnitude of the electron charge.

A1

The potential is time independent, so mechanical energy is conserved:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 + V_0.$$

Therefore,

$$v_2 = \sqrt{v_1^2 - \frac{2V_0}{m}}.$$

The problem assumes that the particle is transmitted into region II.

A2

The potential depends only on x , so the force has no y -component. Consequently, the momentum parallel to the boundary is conserved:

$$mv_1 \sin \theta_1 = mv_2 \sin \theta_2.$$

Thus,

$$v_2 = v_1 \frac{\sin \theta_1}{\sin \theta_2}.$$

A3

Let the boundary crossing point be $Q = (x_1, \alpha)$. The two straight segments have lengths

$$L_1 = \sqrt{x_1^2 + \alpha^2}, \quad L_2 = \sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2}.$$

At fixed total energy, the speeds in the two regions are fixed. Hence

$$A(\alpha) = m(v_1 L_1 + v_2 L_2).$$

The actual trajectory satisfies

$$\frac{dA}{d\alpha} = m \left[v_1 \frac{\alpha}{L_1} - v_2 \frac{y_0 - \alpha}{L_2} \right] = 0.$$

Therefore,

$$\frac{v_1}{v_2} = \frac{(y_0 - \alpha)L_1}{\alpha L_2} = \frac{(y_0 - \alpha)\sqrt{x_1^2 + \alpha^2}}{\alpha\sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2}}.$$

Since $\alpha/L_1 = \sin \theta_1$ and $(y_0 - \alpha)/L_2 = \sin \theta_2$, this reproduces A2. Moreover,

$$\frac{d^2 A}{d\alpha^2} = m \left[\frac{v_1 x_1^2}{L_1^3} + \frac{v_2 (x_0 - x_1)^2}{L_2^3} \right] > 0,$$

so this stationary action is a minimum.

B1

Write the horizontal interface as $y = b$, and let the ray cross it at (α, b) . The speed of light in medium j is c/n_j . Its travel time from O to P is therefore

$$T(\alpha) = \frac{n_1}{c} \sqrt{\alpha^2 + b^2} + \frac{n_2}{c} \sqrt{(x_0 - \alpha)^2 + (y_0 - b)^2}.$$

Fermat's principle requires $dT/d\alpha = 0$:

$$n_1 \frac{\alpha}{\sqrt{\alpha^2 + b^2}} = n_2 \frac{x_0 - \alpha}{\sqrt{(x_0 - \alpha)^2 + (y_0 - b)^2}}.$$

Because the angles are measured from the interface normal, the two geometric ratios are $\sin i_1$ and $\sin i_2$. Thus

$$\boxed{n_1 \sin i_1 = n_2 \sin i_2}, \quad \boxed{\frac{\sin i_1}{\sin i_2} = \frac{n_2}{n_1}}.$$

B2

Regard the liquid as a succession of thin horizontal layers. Applying B1 between adjacent layers shows that

$$n(y) \sin i(y) = \text{constant}.$$

The initial ray is horizontal, so $i(0) = 90^\circ$, giving

$$n(y) \sin i(y) = n_0.$$

For a ray traveling towards increasing x ,

$$\sin i = \frac{dx}{ds} = \frac{1}{\sqrt{1 + (dy/dx)^2}}.$$

Consequently,

$$\frac{n(y)}{\sqrt{1 + (dy/dx)^2}} = n_0,$$

and hence

$$\left(\frac{dy}{dx}\right)^2 = \frac{n(y)^2}{n_0^2} - 1.$$

The ray bends towards the larger refractive index, which is downward. Choosing the negative slope,

$$\boxed{\frac{dy}{dx} = -\sqrt{\frac{n(y)^2}{n_0^2} - 1}}.$$

B3

Substitute $n(y) = n_0 - ky$:

$$\frac{dx}{dy} = -\frac{n_0}{\sqrt{(n_0 - ky)^2 - n_0^2}}.$$

Integrating from the entry point $(0, 0)$,

$$x = -n_0 \int_0^y \frac{d\eta}{\sqrt{(n_0 - k\eta)^2 - n_0^2}}.$$

With $u = (n_0 - k\eta)/n_0$, this becomes

$$x = \frac{n_0}{k} \int_1^{(n_0 - ky)/n_0} \frac{du}{\sqrt{u^2 - 1}}.$$

Using the supplied integral,

$$x(y) = \frac{n_0}{k} \ln \left[\frac{n_0 - ky + \sqrt{(n_0 - ky)^2 - n_0^2}}{n_0} \right].$$

This is the branch with $x \geq 0$ and $-y_0 \leq y \leq 0$.

Equivalently,

$$\frac{kx}{n_0} = \operatorname{arcosh} \left(1 - \frac{ky}{n_0} \right),$$

so

$$y(x) = \frac{n_0}{k} \left[1 - \cosh \left(\frac{kx}{n_0} \right) \right].$$

It passes through the origin with a horizontal tangent and curves increasingly downward.

B4

At the bottom, $y = -y_0$. Therefore,

$$x_0 = \frac{n_0}{k} \operatorname{arcosh} \left(1 + \frac{ky_0}{n_0} \right).$$

Using centimetres consistently,

$$\frac{n_0}{k} = 30.0 \text{ cm}, \quad 1 + \frac{ky_0}{n_0} = 1 + \frac{0.050(10.0)}{1.50} = \frac{4}{3}.$$

Thus

$$x_0 = 30.0 \ln \left(\frac{4 + \sqrt{7}}{3} \right) \text{ cm} = 23.86096 \dots \text{ cm}.$$

Hence

$$x_0 \approx 24 \text{ cm} = 0.24 \text{ m}.$$

C1

The de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{mv}.$$

The spatial phase accumulated over an infinitesimal distance is

$$\Delta\varphi = \frac{2\pi}{\lambda} \Delta s = \frac{2\pi}{h} mv \Delta s.$$

Since $\Delta A = mv \Delta s$,

$$\boxed{\Delta\varphi = \frac{2\pi}{h} \Delta A = \frac{\Delta A}{\hbar}}.$$

For stationary waves of the same energy, the common temporal phase cancels when comparing paths at the same observation time.

C2

Let C have coordinate (x_1, α) , so D has coordinate $(x_1, \alpha + d)$. To first order,

$$OD - OC = d \sin \theta_1 + O(d^2),$$

while

$$DP - CP = -d \sin \theta_2 + O(d^2).$$

Therefore,

$$A_{ODP} - A_{OCP} = md(v_1 \sin \theta_1 - v_2 \sin \theta_2) + O(d^2).$$

Because OCP is the classical trajectory, A2 gives

$$v_1 \sin \theta_1 = v_2 \sin \theta_2.$$

Using C1,

$$\Delta\varphi_{CD} = \frac{2\pi md}{h}(v_1 \sin \theta_1 - v_2 \sin \theta_2) + O(d^2).$$

Thus

$$\boxed{\Delta\varphi_{CD} = 0 \quad \text{to first order in } d}.$$

The phase difference is generally nonzero at second order. Stationary action makes neighboring paths have the same phase to first order.

D1

Acceleration from rest through a potential difference of magnitude U_1 gives

$$eU_1 = \frac{1}{2} m_e v_1^2.$$

Hence

$$U_1 = \frac{m_e v_1^2}{2e} = 1137.126 \dots \text{ V},$$

or

$$\boxed{U_1 \approx 1.14 \times 10^3 \text{ V}}.$$

D2

Let

$$d = 215.00 \times 10^{-9} \text{ m}, \quad L = x_0 - x_1 = 0.25000 \text{ m}.$$

We treat the incident collimated beam as a coherent plane wave and use the far-field path-length differences to first order in the slit separation.

Since F is above G, the incident phase difference is

$$\varphi_F^{\text{in}} - \varphi_G^{\text{in}} = \frac{m_e v_1}{\hbar} d \sin \theta.$$

At P, which lies on the straight continuation of OF,

$$FP - GP \simeq -d \sin \theta.$$

Propagation in region II therefore contributes

$$-\frac{m_e v_2}{\hbar} d \sin \theta.$$

Taking the total phase difference as the phase through F minus that through G,

$$2\pi\beta = \frac{m_e d}{\hbar} (v_1 - v_2) \sin \theta.$$

Thus

$$\boxed{\beta = \frac{m_e d (v_1 - v_2) \sin \theta}{h}}.$$

Numerically,

$$\beta = 5.132650 \dots, \quad \boxed{\beta \approx 5.13}.$$

This is the unreduced phase difference in cycles; modulo one cycle it is 0.132650...

D3

For identical slits with equal amplitudes, destructive interference occurs when the phase difference is an odd multiple of π , or a half-integer number of cycles.

Let the outgoing direction towards a nearby screen point be $\theta + \delta\theta$. The incident phase difference remains fixed, while the propagation phase changes. The phase difference in cycles becomes

$$b(\delta\theta) = \frac{m_e d}{h} [v_1 \sin \theta - v_2 \sin(\theta + \delta\theta)].$$

Using the given small-angle expansion,

$$b(\delta\theta) \simeq \beta - \frac{m_e v_2 d}{h} \cos \theta \delta\theta.$$

The neighboring half-integers are 4.5 and 5.5. Their distances from β are

$$\beta - 4.5 = 0.632650 \dots, \quad 5.5 - \beta = 0.367350 \dots$$

The nearest minimum therefore has $b = 5.5$, giving

$$\delta\theta = -\frac{h(5.5 - \beta)}{m_e v_2 d \cos \theta}.$$

The negative sign means the minimum lies below P.

The signed displacement on the vertical screen is

$$\delta y = L [\tan(\theta + \delta\theta) - \tan \theta] \simeq \frac{L}{\cos^2 \theta} \delta\theta.$$

Thus its magnitude is

$$\Delta y = \frac{hL(5.5 - \beta)}{m_e v_2 d \cos^3 \theta}.$$

Numerically,

$$\frac{hL}{m_e v_2 d \cos^3 \theta} = 44.5001 \dots \mu\text{m},$$

so the leading-order result is

$$\Delta y = (44.5001 \dots)(0.3673499 \dots) \mu\text{m} = 16.3471 \dots \mu\text{m}.$$

Therefore,

$$\Delta y \approx 16 \mu\text{m}, \quad \text{below P}.$$

The small-angle and far-field approximations are appropriate here: $|\delta\theta| \simeq 6.34 \times 10^{-5} \text{ rad}$, and $d/L = 8.60 \times 10^{-7}$. The quoted result neglects higher-order path-length corrections.

D4

The beam's normal cross-sectional area is

$$S = (500 \times 10^{-9})^2 = 2.50 \times 10^{-13} \text{ m}^2.$$

For a 2 m flight distance, using $v \simeq 2.0 \times 10^7 \text{ m s}^{-1}$, the transit time is

$$\tau \simeq \frac{2}{2.0 \times 10^7} = 1.0 \times 10^{-7} \text{ s}.$$

If the flux density is I , the number entering per unit time is IS . In steady operation, the average number in the apparatus is

$$\overline{N} = IS\tau.$$

The minimum flux corresponds to $\overline{N} = 1$:

$$I_{\min} = \frac{1}{S\tau}.$$

Therefore,

$$I_{\min} \simeq \frac{1}{(2.50 \times 10^{-13})(1.0 \times 10^{-7})} = 4 \times 10^{19} \text{ m}^{-2} \text{ s}^{-1}.$$

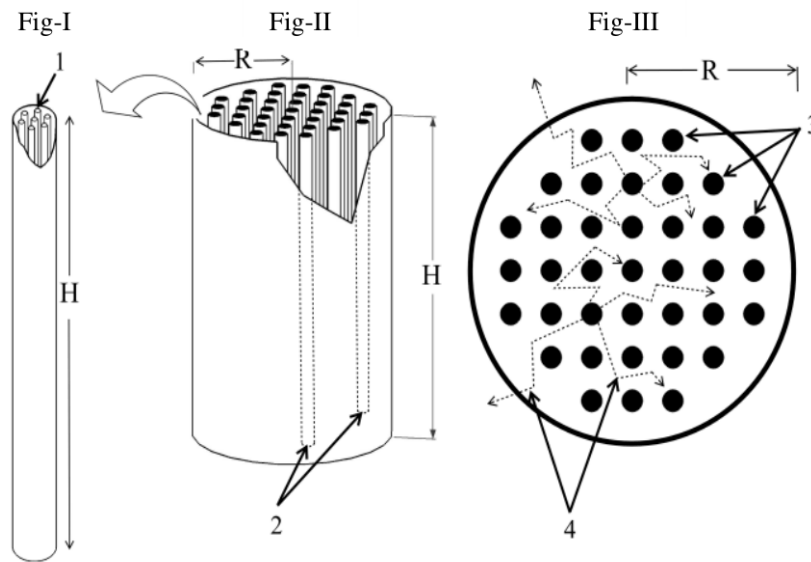
More precisely, $\tau = \ell_1/v_1 + \ell_2/v_2$, where $\ell_1 + \ell_2$ is the flight distance. The two speeds differ by only 0.5%, so replacing them by the common approximate speed does not affect the quoted precision.

The solution was checked against the official solution.

Problem 3: The Design of a Nuclear Reactor

Uranium occurs in nature as UO_2 with only 0.720% of the uranium atoms being ^{235}U . Neutron induced fission occurs readily in ^{235}U with the emission of 2-3 fission neutrons having high kinetic energy. This fission probability will increase if the neutrons inducing fission have low kinetic energy. So by reducing the kinetic energy of the fission neutrons, one can induce a chain of fissions in other ^{235}U nuclei. This forms the basis of the power generating nuclear reactor (NR).

A typical NR consists of a cylindrical tank of height H and radius R filled with a material called moderator. Cylindrical tubes, called fuel channels, each containing a cluster of cylindrical fuel pins of natural UO_2 in solid form of height H , are kept axially in a square array. Fission neutrons, coming outward from a fuel channel, collide with the moderator, losing energy and reach the surrounding fuel channels with low enough energy to cause fission (Figs I-III). Heat generated from fission in the pin is transmitted to a coolant fluid flowing along its length. In the current problem we shall study some of the physics behind the (A) Fuel Pin, (B) Moderator and (C) NR of cylindrical geometry.



Schematic sketch of the Nuclear Reactor (NR)

Fig-I: Enlarged view of a fuel channel (1-Fuel Pins)

Fig-II: A view of the NR (2-Fuel Channels)

Fig-III: Top view of NR (3-Square Arrangement of Fuel Channels and 4-Typical Neutron Paths).

Only components relevant to the problem are shown (e.g. control rods and coolant are not shown).

A Fuel Pin

Data for UO_2	1. Molecular weight $M_w = 0.270 \text{ kg mol}^{-1}$	2. Density $\rho = 1.060 \times 10^4 \text{ kg m}^{-3}$
	3. Melting point $T_m = 3.138 \times 10^3 \text{ K}$	4. Thermal conductivity $\lambda = 3.280 \text{ W m}^{-1} \text{ K}^{-1}$

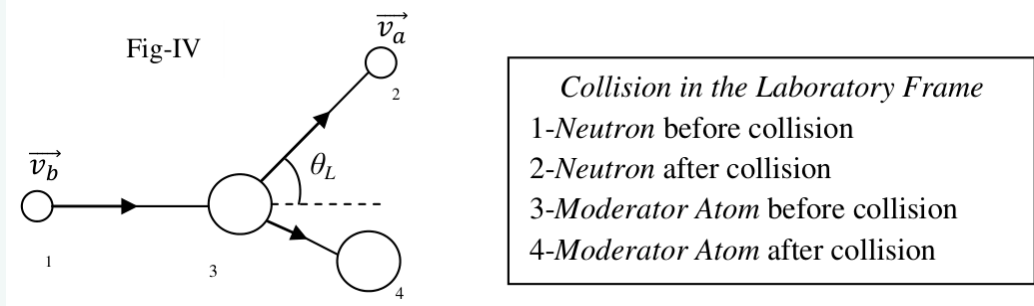
A1	Consider the following fission reaction of a stationary ^{235}U after it absorbs a neutron of negligible kinetic energy. $^{235}\text{U} + {}^1_0\text{n} \longrightarrow {}^{94}\text{Zr} + {}^{140}\text{Ce} + 2 {}^1_0\text{n} + \Delta E$ Estimate ΔE (in MeV) the total fission energy released. The nuclear masses are: $m(^{235}\text{U}) = 235.044 \text{ u}$; $m(^{94}\text{Zr}) = 93.9063 \text{ u}$; $m(^{140}\text{Ce}) = 139.905 \text{ u}$; $m({}^1_0\text{n}) = 1.00867 \text{ u}$ and $1 \text{ u} = 931.502 \text{ MeV } c^{-2}$. Ignore charge imbalance.	0.8
A2	Estimate N the number of ^{235}U atoms per unit volume in natural UO_2 .	0.5
A3	Assume that the neutron flux density, $\varphi = 2.000 \times 10^{18} \text{ m}^{-2} \text{ s}^{-1}$ on the fuel is uniform. The fission cross-section (effective area of the target nucleus) of a ^{235}U nucleus is $\sigma_f = 5.400 \times 10^{-26} \text{ m}^2$. If 80.00% of the fission energy is available as heat, estimate Q (in W m^{-3}), the rate of heat production in the pin per unit volume. $1 \text{ MeV} = 1.602 \times 10^{-13} \text{ J}$	1.2

A4	The steady-state temperature difference between the center (T_c) and the surface (T_s) of the pin can be expressed as $T_c - T_s = k F(Q, a, \lambda)$, where $k = 1/4$ is a dimensionless constant and a is the radius of the pin. Obtain $F(Q, a, \lambda)$ by dimensional analysis. Note that λ is the thermal conductivity of UO_2 .	0.5
A5	The desired temperature of the coolant is 5.770×10^2 K. Estimate the upper limit a_u on the radius a of the pin.	1.0

B The Moderator

Consider the two dimensional elastic collision between a neutron of mass 1 u and a moderator atom of mass A u. Before collision all the moderator atoms are considered at rest in the laboratory frame (LF). Let $\vec{v}_b$ and $\vec{v}_a$ be the velocities of the neutron before and after collision respectively in the LF. Let $\vec{v}_m$ be the velocity of the center of mass (CM) frame relative to LF and θ be the neutron scattering angle in the CM frame. All the particles involved in collisions are moving at nonrelativistic speeds.

B1	The collision in LF is shown schematically, where θ_L is the scattering angle (Fig-IV). Sketch the collision schematically in CM frame. Label the particle velocities for 1, 2 and 3 in terms of $\vec{v}_b$, $\vec{v}_a$ and $\vec{v}_m$. Indicate the scattering angle θ .	1.0
----	--	-----



B2	Obtain v and V , the speeds of the neutron and moderator atom in the CM frame after collision, in terms of A and v_b .	1.0
B3	Derive an expression for $G(\alpha, \theta) = E_a/E_b$, where E_b and E_a are the kinetic energies of the neutron, in the LF, before and after the collision respectively and $\alpha \equiv [(A - 1)/(A + 1)]^2$.	1.0
B4	Assume that the above expression holds for D_2O molecule. Calculate the maximum possible fractional energy loss $f_l \equiv (E_b - E_a)/E_b$ of the neutron for the D_2O (20 u) moderator.	0.5

C The Nuclear Reactor

To operate the NR at any constant neutron flux ψ (steady state), the leakage of neutrons has to be compensated by an excess production of neutrons in the reactor. For a reactor in cylindrical geometry the leakage rate is $k_1[(2.405/R)^2 + (\pi/H)^2]\psi$ and the excess production rate is $k_2\psi$. The constants k_1 and k_2 depend on the material properties of the NR.

C1	Consider a NR with $k_1 = 1.021 \times 10^{-2}$ m and $k_2 = 8.787 \times 10^{-3} \text{ m}^{-1}$. Noting that for a fixed volume the leakage rate is to be minimized for efficient fuel utilization, obtain the dimensions of the NR in the steady state.	1.5
C2	The fuel channels are in a square arrangement (Fig-III) with the nearest neighbour distance 0.286 m. The effective radius of a fuel channel (if it were solid) is 3.617×10^{-2} m. Estimate the number of fuel channels F_n in the reactor and the mass M of UO_2 required to operate the NR in steady state.	1.0

¹ Joseph Amal Nathan (BARC) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.

Official English text of the problem.

Answers

- **A1:** $\Delta E \simeq 208.7 \text{ MeV}$.
- **A2:** $N \simeq 1.70 \times 10^{26} \text{ m}^{-3}$.
- **A3:** $Q \simeq 4.92 \times 10^8 \text{ W m}^{-3}$.
- **A4:** $F(Q, a, \lambda) = \frac{Qa^2}{\lambda}$, so $T_c - T_s = \frac{Qa^2}{4\lambda}$.
- **A5:** $a_u \simeq 8.27 \times 10^{-3} \text{ m} = 8.27 \text{ mm}$.
- **B1:** CM velocity labels: 1 : $\vec{v}_b - \vec{v}_m$, 2 : $\vec{v}_a - \vec{v}_m$, 3 : $-\vec{v}_m$. The angle between the initial and final neutron velocities is θ ; the two outgoing particles move in opposite directions.
- **B2:** $v = \frac{A}{A+1} v_b$, $V = \frac{1}{A+1} v_b$.
- **B3:** $G(\alpha, \theta) = \frac{1 + \alpha + (1 - \alpha) \cos \theta}{2}$.
- **B4:** $f_{l,\max} = \frac{80}{441} \simeq 0.1814 = 18.14\%$.
- **C1:** $R = 2.405 \sqrt{\frac{3k_1}{2k_2}} \simeq 3.175 \text{ m}$, $H = \pi \sqrt{\frac{3k_1}{k_2}} \simeq 5.865 \text{ m}$.
- **C2:** $F_n \simeq 387$ fuel channels; $M \simeq 9.89 \times 10^4 \text{ kg}$, or 98.9 tonnes of UO_2 .

Solution

A1

Since the initial kinetic energy is negligible, conservation of energy gives the released energy as the decrease in total rest energy:

$$\Delta E = [m(^{235}\text{U}) + m(^1\text{n}) - m(^{94}\text{Zr}) - m(^{140}\text{Ce}) - 2m(^1\text{n})]c^2.$$

Using the supplied masses without any charge correction,

$$\begin{aligned} \Delta m &= [235.044 + 1.00867 - (93.9063 + 139.905 + 2(1.00867))] \text{u} \\ &= 0.22403 \text{ u}. \end{aligned}$$

Therefore,

$$\Delta E = 0.22403(931.502) \text{ MeV} = 208.684 \dots \text{ MeV},$$

or

$$\boxed{\Delta E \simeq 208.7 \text{ MeV}}.$$

A2

Each formula unit of UO_2 contains one uranium atom. The number of formula units per unit volume is

$$n_{\text{UO}_2} = \frac{\rho}{M_w} N_A,$$

where the standard Avogadro constant is $N_A \simeq 6.022 \times 10^{23} \text{ mol}^{-1}$.

The fraction of uranium atoms that are ^{235}U is 0.00720. Hence

$$\begin{aligned} N &= 0.00720 \frac{\rho N_A}{M_w} \\ &= 0.00720 \frac{1.060 \times 10^4}{0.270} (6.022 \times 10^{23}) \text{ m}^{-3} \\ &= 1.7022 \dots \times 10^{26} \text{ m}^{-3}. \end{aligned}$$

Thus

$$\boxed{N \simeq 1.70 \times 10^{26} \text{ m}^{-3}}.$$

A3

The fission rate of one target nucleus in the prescribed uniform neutron flux is $\varphi\sigma_f$. Therefore the fission rate per unit volume is

$$r_f = N\varphi\sigma_f.$$

Numerically,

$$\varphi\sigma_f = (2.000 \times 10^{18})(5.400 \times 10^{-26}) = 1.080 \times 10^{-7} \text{ s}^{-1},$$

and

$$r_f = 1.8384 \dots \times 10^{19} \text{ m}^{-3} \text{ s}^{-1}.$$

The heat deposited per fission is

$$E_h = 0.8000 \Delta E = 0.8000(208.684 \dots)(1.602 \times 10^{-13}) \text{ J}.$$

Consequently,

$$\begin{aligned} Q &= N\varphi\sigma_f E_h \\ &= 4.9168 \dots \times 10^8 \text{ W m}^{-3}, \end{aligned}$$

giving

$$\boxed{Q \simeq 4.92 \times 10^8 \text{ W m}^{-3}}.$$

A4

Let F have the dimensional form $Q^x a^y \lambda^z$. Denoting dimensions of power, length and temperature by P, L, Θ ,

$$[Q] = PL^{-3}, \quad [a] = L, \quad [\lambda] = PL^{-1}\Theta^{-1}.$$

Thus

$$[F] = P^{x+z} L^{-3x+y-z} \Theta^{-z}.$$

For F to have the dimensions of temperature,

$$x + z = 0, \quad -3x + y - z = 0, \quad -z = 1.$$

These equations give

$$x = 1, \quad y = 2, \quad z = -1.$$

The dimensionless numerical factor is already supplied as k , so

$$\boxed{F(Q, a, \lambda) = \frac{Qa^2}{\lambda}}, \quad \boxed{T_c - T_s = \frac{Qa^2}{4\lambda}}.$$

A5

With uniform internal heat production, the center is the hottest part of the pin. Avoiding melting requires

$$T_c \leq T_m.$$

Heat must flow from the pin surface to the coolant, so the surface temperature cannot be below the coolant temperature:

$$T_s \geq T_{\text{cool}}.$$

It follows that

$$\frac{Qa^2}{4\lambda} = T_c - T_s \leq T_m - T_{\text{cool}}.$$

The limiting radius is therefore

$$a_u = \sqrt{\frac{4\lambda(T_m - T_{\text{cool}})}{Q}}.$$

Using

$$T_m - T_{\text{cool}} = 3138 - 577.0 = 2561 \text{ K},$$

we obtain

$$a_u = \sqrt{\frac{4(3.280)(2561)}{4.9168 \times 10^8}} = 8.2667 \times 10^{-3} \text{ m}.$$

Thus

$$\boxed{a_u \simeq 8.27 \text{ mm}}.$$

This upper limit corresponds to the ideal limiting case $T_s = T_{\text{cool}}$. A finite temperature drop between the surface and coolant would reduce the permissible radius.

B1

Let the neutron mass be m , so the moderator mass is Am . Before collision the moderator is stationary in the LF. The CM velocity is therefore

$$\vec{v}_m = \frac{m\vec{v}_b + Am\vec{0}}{(A+1)m} = \frac{\vec{v}_b}{A+1}.$$

A Galilean transformation to the CM frame subtracts $\vec{v}_m$ from every LF velocity. Thus the required labels are

$$\boxed{\vec{u}_1 = \vec{v}_b - \vec{v}_m, \quad \vec{u}_2 = \vec{v}_a - \vec{v}_m, \quad \vec{u}_3 = -\vec{v}_m}.$$

The sketch should have these features:

- Before collision, neutron **1** approaches from the left, moving horizontally right with velocity $\vec{v}_b - \vec{v}_m$.

- Moderator **3** approaches from the right, moving horizontally left with velocity $-\vec{v}_m$.
- After collision, neutron **2** leaves the collision point with velocity $\vec{v}_a - \vec{v}_m$. Mark θ between this arrow and the original rightward neutron direction.
- Moderator **4** leaves in the direction exactly opposite to neutron **2**, because the total CM momentum is zero. Its velocity is

$$\vec{u}_4 = -\frac{\vec{u}_2}{A}.$$

For example, if the outgoing neutron arrow points upward and right, the outgoing moderator arrow points downward and left.

B2

The initial CM speeds are

$$u_{n,b} = v_b - v_m = \frac{A}{A+1}v_b, \quad u_{M,b} = v_m = \frac{1}{A+1}v_b.$$

Conservation of momentum in the CM frame requires the final momenta to be equal and opposite. Their magnitudes therefore satisfy

$$mv = AmV, \quad \text{so} \quad v = AV.$$

Because the collision is elastic, kinetic energy is conserved:

$$\frac{1}{2}mu_{n,b}^2 + \frac{1}{2}Amu_{M,b}^2 = \frac{1}{2}mv^2 + \frac{1}{2}AmV^2.$$

After cancelling $m/2$,

$$\frac{A}{A+1}v_b^2 = v^2 + AV^2.$$

Substituting $v = AV$ gives

$$A(A+1)V^2 = \frac{A}{A+1}v_b^2.$$

Hence

$$\boxed{V = \frac{v_b}{A+1}}, \quad \boxed{v = \frac{A}{A+1}v_b}.$$

The collision changes the velocity directions, but each particle's speed in the CM frame remains unchanged.

B3

The neutron's final LF velocity is

$$\vec{v}_a = \vec{v}_m + \vec{u}_2.$$

The vector $\vec{v}_m$ points along the initial neutron direction, while $\vec{u}_2$ makes angle θ with that direction. Consequently,

$$v_a^2 = v_m^2 + v^2 + 2v_mv \cos \theta.$$

Using the results above,

$$\frac{v_a^2}{v_b^2} = \frac{1 + A^2 + 2A \cos \theta}{(A + 1)^2}.$$

Since both energies refer to the same neutron in the LF,

$$G(\alpha, \theta) = \frac{E_a}{E_b} = \frac{v_a^2}{v_b^2}.$$

Now

$$\alpha = \frac{(A - 1)^2}{(A + 1)^2}$$

implies

$$\frac{1 + \alpha}{2} = \frac{A^2 + 1}{(A + 1)^2}, \quad \frac{1 - \alpha}{2} = \frac{2A}{(A + 1)^2}.$$

Therefore,

$$G(\alpha, \theta) = \frac{1 + \alpha + (1 - \alpha) \cos \theta}{2}.$$

An equivalent form is

$$G(\alpha, \theta) = 1 - (1 - \alpha) \sin^2 \frac{\theta}{2}.$$

In particular, forward scattering gives $G = 1$, while backward scattering in the CM frame gives $G = \alpha$.

B4

The fractional energy loss is

$$f_l = 1 - G = (1 - \alpha) \sin^2 \frac{\theta}{2}.$$

It is largest for CM backscattering, $\theta = \pi$. Thus

$$f_{l,\max} = 1 - \alpha = 1 - \left(\frac{A - 1}{A + 1} \right)^2 = \frac{4A}{(A + 1)^2}.$$

Following the problem's approximation that the whole D_2O molecule is a target of mass 20 u, set $A = 20$:

$$f_{l,\max} = \frac{4(20)}{21^2} = \frac{80}{441} \simeq 0.1814 = 18.14\%.$$

C1

For a nonzero steady neutron flux, production and leakage must balance:

$$k_1 \left[\frac{2.405^2}{R^2} + \frac{\pi^2}{H^2} \right] \psi = k_2 \psi.$$

Cancelling ψ ,

$$\frac{2.405^2}{R^2} + \frac{\pi^2}{H^2} = \frac{k_2}{k_1}. \quad (1)$$

We first determine the shape that minimizes leakage at fixed volume. Write

$$\mathcal{V} = \pi R^2 H, \quad b = 2.405,$$

so that

$$H = \frac{\mathcal{V}}{\pi R^2}.$$

The geometric factor in the leakage rate becomes

$$S(R) = \frac{b^2}{R^2} + \frac{\pi^4 R^4}{\mathcal{V}^2}.$$

Its derivative vanishes when

$$-\frac{2b^2}{R^3} + \frac{4\pi^4 R^3}{\mathcal{V}^2} = 0.$$

Rearranging and using $\mathcal{V} = \pi R^2 H$,

$$\boxed{\frac{b^2}{R^2} = 2 \frac{\pi^2}{H^2}}. \quad (2)$$

This stationary point is a minimum, since

$$S''(R) = \frac{6b^2}{R^4} + \frac{12\pi^4 R^2}{\mathcal{V}^2} > 0.$$

The optimum aspect ratio is therefore

$$\frac{H}{R} = \frac{\sqrt{2}\pi}{b} = 1.84735 \dots$$

Combining (1) and (2),

$$3 \frac{\pi^2}{H^2} = \frac{k_2}{k_1},$$

which gives

$$\boxed{H = \pi \sqrt{\frac{3k_1}{k_2}}}, \quad \boxed{R = b \sqrt{\frac{3k_1}{2k_2}}}.$$

Numerically,

$$H = \pi \sqrt{\frac{3(1.021 \times 10^{-2})}{8.787 \times 10^{-3}}} \simeq \boxed{5.865 \text{ m}},$$

$$R = 2.405 \sqrt{\frac{3(1.021 \times 10^{-2})}{2(8.787 \times 10^{-3})}} \simeq \boxed{3.175 \text{ m}}.$$

C2

Let the square-lattice spacing be $d = 0.286$ m. Neglecting boundary corrections, each fuel channel occupies one square cell of cross-sectional area d^2 . Therefore,

$$F_n \simeq \frac{\pi R^2}{d^2}.$$

Substitution gives

$$F_n \simeq \frac{\pi(3.17507\dots)^2}{(0.286)^2} = 387.19\dots$$

Hence the estimated number is

$$\boxed{F_n \simeq 387 \text{ fuel channels}}.$$

With effective solid-fuel radius $r_f = 3.617 \times 10^{-2} \text{ m}$, the volume of UO_2 in one channel is $\pi r_f^2 H$. Its mass is therefore $\rho \pi r_f^2 H$, and the total fuel mass is

$$M = F_n \rho \pi r_f^2 H.$$

Using the unrounded area estimate,

$$\begin{aligned} M &\simeq (387.19\dots)(1.060 \times 10^4) \pi (3.617 \times 10^{-2})^2 (5.86547\dots) \text{ kg} \\ &= 9.8942\dots \times 10^4 \text{ kg}. \end{aligned}$$

Thus

$$\boxed{M \simeq 9.89 \times 10^4 \text{ kg} = 98.9 \text{ tonnes of } \text{UO}_2}.$$

The solution was checked against the official solution.