

IPhO 2014

Theory problems and solutions · Astana, Kazakhstan · 3 theory problems with complete solutions

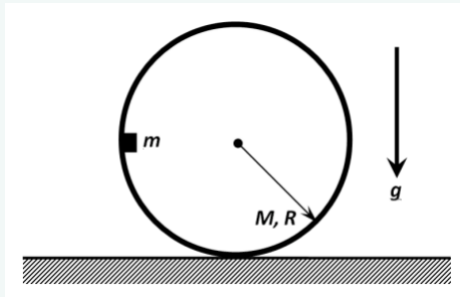
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Problem 1: Three independent parts: puck in a rolling cylinder, soap bubble with gas, LC circuit with a switch

This problem consists of three independent parts.

Part A (3 points)

A small puck of mass m is carefully placed onto the inner surface of the thin hollow thin cylinder of mass M and of radius R . Initially, the cylinder rests on the horizontal plane and the puck is located at the height R above the plane as shown in the figure on the left. Find the interaction force F between the puck and the cylinder at the moment when the puck passes the lowest point of its trajectory. Assume that the friction between the puck and the inner surface of the cylinder is absent, and the cylinder moves on the plane without slipping. The free fall acceleration is g .



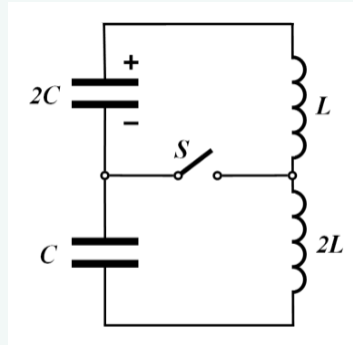
Part B (3 points)

A bubble of radius $r = 5.00 \text{ cm}$, containing a diatomic ideal gas, has the soap film of thickness $h = 10.0 \text{ }\mu\text{m}$ and is placed in vacuum. The soap film has the surface tension $\sigma = 4.00 \cdot 10^{-2} \frac{\text{N}}{\text{m}}$ and the density $\rho = 1.10 \frac{\text{g}}{\text{cm}^3}$. 1) Find formula for the molar heat capacity of the gas in the bubble for such a process when the gas is heated so slowly that the bubble remains in a mechanical equilibrium and evaluate it; 2) Find formula for the frequency ω of the small radial oscillations of the bubble and evaluate it under the assumption that the heat capacity of the soap film is much greater than the heat capacity of the gas in the bubble. Assume that the thermal equilibrium inside the bubble is reached much faster than the period of oscillations.

Hint: Laplace showed that there is pressure difference between inside and outside of a curved surface, caused by surface tension of the interface between liquid and gas, so that $\Delta p = \frac{2\sigma}{r}$.

Part C (3 points)

Initially, a switch S is unshorted in the circuit shown in the figure on the right, a capacitor of capacitance $2C$ carries the electric charge q_0 , a capacitor of capacitance C is uncharged, and there are no electric currents in both coils of inductance L and $2L$, respectively. The capacitor starts to discharge and at the moment when the current in the coils reaches its maximum value, the switch S is instantly shorted. Find the maximum current $I_{\max}$ through the switch S thereafter.



Official English text of the problem.

Answers

- Part A:

$$F = mg \left(3 + \frac{m}{M} \right).$$

- Part B, 1): With $\mathcal{R}$ denoting the universal gas constant,

$$C_{\text{mol}} = 4\mathcal{R} = 33.3 \text{ J mol}^{-1} \text{ K}^{-1}.$$

- Part B, 2):

$$\omega = \sqrt{\frac{8\sigma}{\rho h r^2}} = 1.08 \times 10^2 \text{ rad s}^{-1}.$$

The corresponding ordinary frequency is $f = \omega/(2\pi) = 17.2 \text{ Hz}$.

- Part C:

$$I_{\max} = \frac{q_0}{\sqrt{2LC}}.$$

Solution

Part A

Let V be the horizontal velocity of the cylinder's centre and v_x the horizontal velocity of the puck, both positive to the right. The cylinder's moment of inertia is

$$I_{\text{cyl}} = MR^2.$$

First we obtain a relation between these velocities. Let f be the signed horizontal friction force exerted by the plane on the cylinder. The puck's normal force passes through the cylinder's centre, so only f produces a torque about that centre.

Taking counterclockwise angular velocity as positive, rolling without slipping gives

$$\Omega_{\text{cyl}} = -\frac{V}{R}.$$

Consequently,

$$I_{\text{cyl}}\dot{\Omega}_{\text{cyl}} = Rf \implies f = -M\dot{V}.$$

For the combined puck–cylinder system, the horizontal equation of motion is

$$m\dot{v}_x + M\dot{V} = f.$$

Combining these equations and using the initial state of rest,

$$\boxed{mv_x + 2MV = 0}.$$

At the lowest point, the puck's velocity is horizontal; denote it by v . Its speed relative to the cylinder's centre is

$$u = v - V.$$

Thus

$$V = -\frac{m}{2M}v, \quad u = \frac{2M+m}{2M}v.$$

Mechanical energy is conserved: the puck–cylinder contact is frictionless, and static friction at the stationary point of contact with the plane does no work. The puck descends through height R , giving

$$mgR = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 + \frac{1}{2}MR^2\left(\frac{V}{R}\right)^2 = \frac{1}{2}mv^2 + MV^2.$$

Substituting the velocity relation,

$$v^2 = \frac{4M}{2M+m}gR, \quad u^2 = \frac{2M+m}{M}gR.$$

To find the vertical acceleration, let θ be the puck's angular displacement from the downward vertical. Its height is

$$y = R - R\cos\theta.$$

At the bottom, $\theta = 0$ and $u = R\dot{\theta}$, so

$$\ddot{y} = R\dot{\theta}^2 = \frac{u^2}{R}.$$

The cylinder's force on the puck is upward there. Newton's second law therefore gives

$$F - mg = m\frac{u^2}{R}.$$

Hence

$$\boxed{F = mg\left(3 + \frac{m}{M}\right)}.$$

Part B, 1)

Treat the soap film as thin compared with the bubble radius and take its surface tension as constant. A soap film has **two** liquid–gas interfaces. Their Laplace pressure jumps add, so equilibrium in vacuum requires

$$p = \frac{4\sigma}{r}.$$

Let n be the number of moles and $\mathcal{R}$ the universal gas constant. Using the ideal-gas law,

$$pV = n\mathcal{R}T, \quad V = \frac{4\pi r^3}{3},$$

we obtain

$$n\mathcal{R}T = \frac{16\pi\sigma}{3}r^2.$$

Thus, during the specified slow equilibrium process,

$$T \propto r^2, \quad \frac{dV}{V} = 3\frac{dr}{r} = \frac{3}{2}\frac{dT}{T}.$$

The gas therefore performs work

$$p dV = \frac{3}{2} \frac{pV}{T} dT = \frac{3}{2} n\mathcal{R} dT.$$

For a diatomic ideal gas with rotational degrees of freedom active and vibrational degrees of freedom unexcited,

$$dU = \frac{5}{2} n\mathcal{R} dT.$$

The first law applied to the gas gives

$$\delta Q = dU + p dV = \left(\frac{5}{2} + \frac{3}{2} \right) n\mathcal{R} dT.$$

Therefore its molar heat capacity for this process is

$$\boxed{C_{\text{mol}} = \frac{\delta Q}{n dT} = 4\mathcal{R}}.$$

Using $\mathcal{R} = 8.31446 \text{ J mol}^{-1} \text{ K}^{-1}$,

$$\boxed{C_{\text{mol}} = 33.3 \text{ J mol}^{-1} \text{ K}^{-1}}.$$

Part B, 2)

The film has much greater heat capacity than the gas, and thermal equilibrium is established rapidly. The film therefore acts as a thermal reservoir during an oscillation, making the gas compression and expansion approximately **isothermal**.

Let the equilibrium radius be r , and write the instantaneous radius as

$$s = r + \xi, \quad |\xi| \ll r.$$

With equilibrium pressure $p_0 = 4\sigma/r$, the isothermal gas law gives

$$p(s) = p_0 \left(\frac{r}{s} \right)^3.$$

The net outward pressure on the film is

$$p(s) - \frac{4\sigma}{s}.$$

Expanding to first order in ξ ,

$$p(s) \simeq p_0 \left(1 - \frac{3\xi}{r}\right), \quad \frac{4\sigma}{s} \simeq \frac{4\sigma}{r} \left(1 - \frac{\xi}{r}\right),$$

so

$$p(s) - \frac{4\sigma}{s} \simeq -\frac{8\sigma}{r^2}\xi.$$

The film's constant mass, expressed using its equilibrium dimensions, is

$$m_f = 4\pi r^2 \rho h.$$

Neglecting gas inertia and dissipative effects, the equation for radial motion is, to first order,

$$m_f \ddot{\xi} = 4\pi r^2 \left(-\frac{8\sigma}{r^2}\xi\right) = -32\pi\sigma\xi.$$

Thus

$$\ddot{\xi} + \omega^2 \xi = 0, \quad \boxed{\omega^2 = \frac{32\pi\sigma}{m_f} = \frac{8\sigma}{\rho h r^2}}.$$

Converting the given quantities to SI units,

$$r = 5.00 \times 10^{-2} \text{ m}, \quad h = 1.00 \times 10^{-5} \text{ m}, \quad \rho = 1.10 \times 10^3 \text{ kg m}^{-3},$$

we find

$$\omega = \sqrt{\frac{8(4.00 \times 10^{-2})}{(1.10 \times 10^3)(1.00 \times 10^{-5})(5.00 \times 10^{-2})^2}}.$$

Therefore

$$\boxed{\omega = 1.08 \times 10^2 \text{ rad s}^{-1}}, \quad \boxed{f = 17.2 \text{ Hz}}.$$

Part C

Before the switch closes, both inductors carry the same current i . Let q be the charge that has passed downward through the inductors since the discharge began, so $i = \dot{q}$.

The charge on the upper plate of the $2C$ capacitor is then $q_0 - q$, while the charge on the upper plate of the C capacitor is $-q$. Defining both capacitor voltages from top to bottom,

$$U_1 = \frac{q_0 - q}{2C}, \quad U_2 = -\frac{q}{C}.$$

Kirchhoff's voltage law gives

$$3L\dot{i} = U_1 + U_2 = \frac{q_0 - 3q}{2C}.$$

At the maximum current, $\dot{i} = 0$. Hence, at the switching instant,

$$q = \frac{q_0}{3}, \quad \boxed{U_1 = \frac{q_0}{3C}, \quad U_2 = -\frac{q_0}{3C}}.$$

Now set $t = 0$ when the switch closes. The upper capacitor and upper inductor form one LC loop, while the lower pair form another. Their angular frequencies are equal:

$$\Omega_1 = \frac{1}{\sqrt{L(2C)}}, \quad \Omega_2 = \frac{1}{\sqrt{(2L)C}},$$

so

$$\Omega_1 = \Omega_2 = \Omega = \frac{1}{\sqrt{2LC}}.$$

Let i_1 and i_2 be the upper and lower inductor currents, both positive downward. Their equations are

$$\begin{aligned} L\dot{i}_1 &= U_1, & 2C\dot{U}_1 &= -i_1, \\ 2L\dot{i}_2 &= U_2, & C\dot{U}_2 &= -i_2. \end{aligned}$$

Consequently,

$$\ddot{i}_1 + \Omega^2 i_1 = 0, \quad \ddot{i}_2 + \Omega^2 i_2 = 0.$$

Take the switch current as positive from the inductor junction toward the capacitor junction. Kirchhoff's current law gives

$$i_S = i_1 - i_2,$$

and therefore

$$\ddot{i}_S + \Omega^2 i_S = 0.$$

Inductor currents are continuous at switching and were equal beforehand, so

$$i_S(0) = 0.$$

Capacitor voltages are also continuous, giving

$$\dot{i}_S(0) = \frac{U_1(0)}{L} - \frac{U_2(0)}{2L} = \frac{q_0}{3LC} + \frac{q_0}{6LC} = \frac{q_0}{2LC}.$$

The resulting switch current is

$$i_S(t) = \frac{\dot{i}_S(0)}{\Omega} \sin(\Omega t) = \frac{q_0}{\sqrt{2LC}} \sin\left(\frac{t}{\sqrt{2LC}}\right).$$

Its maximum magnitude is thus

$$I_{\max} = \frac{q_0}{\sqrt{2LC}}.$$

The solution was checked against the official solution.

Problem 2: Van der Waals equation of state

In a well-known model of an ideal gas, whose equation of state obeys the Clapeyron-Mendeleev law, the following important physical effects are neglected. First, molecules of a real gas have a finite size and, secondly, they interact with one another. In all parts of this problem *one mole of water* is considered.

Part A. Non-ideal gas equation of state (2 points)

Taking into account the finite size of the molecules, the gaseous equation of state takes the form

$$P(V - b) = RT, \quad (1)$$

where P , V , T stands for the gas pressure, its volume per mole and temperature, respectively, R denotes the universal gas constant, and b is a specific constant extracting some volume.

A1 Estimate b and express it in terms of the diameter of the molecules d . (0.3 points)

With account of intermolecular attraction forces, van der Waals proposed the following equation of state that neatly describes both the gaseous and liquid states of matter

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT. \quad (2)$$

where a is another specific constant.

At temperatures T below a certain critical value T_c the isotherm of equation (2) is well represented by a non-monotonic curve 1 shown in Figure 1 which is then called van der Waals isotherm. In the same figure curve 2 shows the isotherm of an ideal gas at the same temperature. A real isotherm differs from the van der Waals isotherm by a straight segment AB drawn at some constant pressure P_{LG} . This straight segment is located between the volumes V_L and V_G , and corresponds to the equilibrium of the liquid phase (indicated by L) and the gaseous phase (referred to by G). From the second law of thermodynamics J. Maxwell showed that the pressure P_{LG} must be chosen such that the areas I and II shown in Figure 1 must be equal.

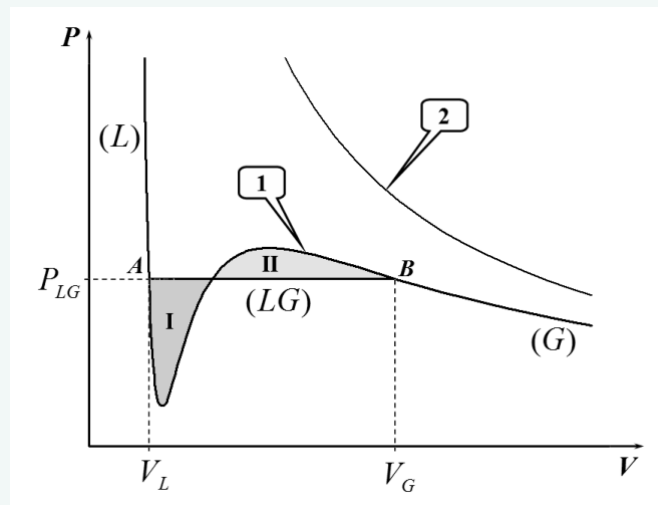


Figure 1. Van der Waals isotherm of gas/liquid (curve 1) and the isotherm of an ideal gas (curve 2).

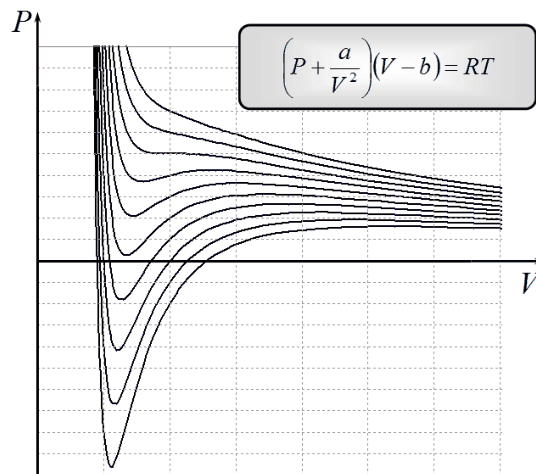


Figure 2. Several isotherms for van der Waals equation of state.

With increasing temperature the straight segment AB on the isotherm shrinks to a single point when the temperature and the pressure reaches some values T_c and $P_{LG} = P_c$, respectively. The parameters P_c and T_c are called critical and can be measured experimentally with high degree of accuracy.

A2	Express the van der Waals constants a and b in terms of T_c and P_c . (1.3 points)
A3	For water $T_c = 647$ K and $P_c = 2.2 \cdot 10^7$ Pa. Calculate a_w and b_w for water. (0.2 points)
A4	Estimate the diameter of water molecules d_w . (0.2 points)

Part B. Properties of gas and liquid (6 points)

This part of the problem deals with the properties of water in the gaseous and liquid states at temperature $T = 100$ °C. The saturated vapor pressure at this temperature is known to be $p_{LG} = p_0 = 1.0 \cdot 10^5$ Pa, and the molar mass of water is $\mu = 1.8 \cdot 10^{-2} \frac{\text{kg}}{\text{mole}}$.

Gaseous state

It is reasonable to assume that the inequality $V_G \gg b$ is valid for the description of water properties in a gaseous state.

B1	Derive the formula for the volume V_G and express it in terms of R, T, p_0 , and a . (0.8 points)
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Almost the same volume V_{G0} can be approximately evaluated using the ideal gas law.

B2	Evaluate in percentage the relative decrease in the gas volume due to intermolecular forces, $\frac{\Delta V_G}{V_{G0}} = \frac{V_{G0} - V_G}{V_{G0}}$. (0.3 points)
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If the system volume is reduced below V_G , the gas starts to condense. However, thoroughly purified gas can remain in a mechanically metastable state (called supercooled vapor) until its volume reaches a certain value $V_{G\min}$.

The condition of mechanical stability of supercooled gas at constant temperature is written as: $\frac{dP}{dV} < 0$.

B3	Find and evaluate how many times the volume of water vapor can be reduced and still remains in a metastable state. In other words, what is $V_G/V_{G\min}$? (0.7 points)
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Liquid state

For the van der Waals' description of water in a liquid state it is reasonable to assume that the following inequality holds $P \ll a/V^2$.

B4 Express the volume of liquid water V_L in terms of a, b, R , and T . **(1.0 points)**

Assuming that $bRT \ll a$, find the following characteristics of water. *Do not be surprised if some of the data evaluated do not coincide with the well-known tabulated values!*

B5 Express the liquid water density ρ_L in some of the terms of μ, a, b, R and evaluate it. **(0.5 points)**

B6 Express the volume thermal expansion coefficient $\alpha = \frac{1}{V_L} \frac{\Delta V_L}{\Delta T}$ in terms of a, b, R , and evaluate it. **(0.6 points)**

B7 Express the specific heat of water vaporization L in terms of μ, a, b, R and evaluate it. **(1.1 points)**

B8 Considering the monomolecular layer of water, estimate the surface tension σ of water. **(1.2 points)**

Part C. Liquid-gas system (3 points)

From Maxwell's rule (equalities of areas, by applying trivial integration) and the van der Waals' equation of state together with the approximations made in Part B, it can be shown that the saturated vapor pressure p_{LG} depends on temperature T as follows

$$\ln p_{LG} = A + \frac{B}{T}, \quad (3)$$

where A and B are some constants, that can be expressed in terms of a and b as

$$A = \ln \left(\frac{a}{b^2} \right) - 1; \quad B = -\frac{a}{bR}$$

W. Thomson showed that the pressure of saturated vapor depends on the curvature of the liquid surface. Consider a liquid that does not wet the material of a capillary (contact angle 180°). When the capillary is immersed into the liquid, the liquid in the capillary drops to a certain level because of the surface tension (see Figure 3).

C1 Find a small change in pressure Δp_T of the saturated vapor over the curved surface of liquid and express it in terms of the vapor density ρ_s , the liquid density ρ_L , the surface tension σ and the radius of surface curvature r . **(1.3 points)**

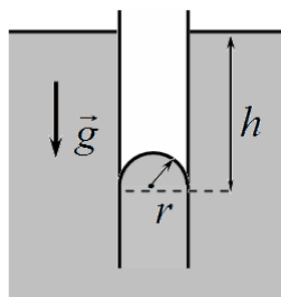


Figure 3. Capillary immersed in a liquid that does not wet its material

Metastable states, considered in part B3, are widely used in real experimental setups, such as the cloud chamber designed for registration of elementary particles. They also occur in natural phenomena such as the formation of morning dew. Supercooled vapor is subject to condensation

by forming liquid droplets. Very small droplets evaporate quickly but large enough ones can still grow.

- C2 Suppose that at the evening temperature of $t_e = 20^\circ\text{C}$ the water vapor in the air was saturated, but in the morning the ambient temperature has fallen by a small amount of $\Delta t = 5.0^\circ\text{C}$. Assuming that the vapor pressure has remained unchanged, estimate the minimum radius of droplets that can grow. Use the tabulated value of water surface tension $\sigma = 7.3 \cdot 10^{-2} \text{ N/m}$. **(1.7 points)**

Official English text of the problem.

Answers

Numerical calculations use $R = 8.31 \text{ J mol}^{-1}\text{K}^{-1}$, $N_A = 6.02 \times 10^{23} \text{ mol}^{-1}$, and $T = 373 \text{ K}$ in Part B. Volumes are molar volumes. The molecular estimates below use the standard hard-sphere excluded-volume convention; their model dependence is discussed in the solutions of A1, A4 and B8.

- **A1:** $b \sim N_A d^3$; the hard-sphere estimate is

$$b = \frac{2\pi}{3} N_A d^3.$$

- **A2:**

$$a = \frac{27R^2T_c^2}{64P_c}, \quad b = \frac{RT_c}{8P_c}.$$

- **A3:**

$$a_w \simeq 0.55 \text{ Pa m}^6 \text{ mol}^{-2}, \quad b_w \simeq 3.1 \times 10^{-5} \text{ m}^3 \text{ mol}^{-1}.$$

- **A4:**

$$d_w = \left(\frac{3b_w}{2\pi N_A} \right)^{1/3} \simeq 2.9 \times 10^{-10} \text{ m}.$$

- **B1:**

$$V_G = \frac{RT + \sqrt{R^2T^2 - 4ap_0}}{2p_0} \simeq 3.08 \times 10^{-2} \text{ m}^3 \text{ mol}^{-1}.$$

- **B2:**

$$\frac{\Delta V_G}{V_{G0}} = \frac{1 - \sqrt{1 - \frac{4ap_0}{R^2T^2}}}{2} \simeq 0.58\%.$$

- **B3:**

$$V_{G\min} = \frac{2a}{RT} \simeq 3.6 \times 10^{-4} \text{ m}^3 \text{ mol}^{-1}, \quad \frac{V_G}{V_{G\min}} \simeq 86.$$

- **B4:**

$$V_L = \frac{a - \sqrt{a^2 - 4abRT}}{2RT} = \frac{2b}{1 + \sqrt{1 - \frac{4bRT}{a}}}.$$

- **B5:**

$$\rho_L \simeq \frac{\mu}{b} \simeq 5.9 \times 10^2 \text{ kg m}^{-3}.$$

- **B6:**

$$\alpha \simeq \frac{bR}{a} \simeq 4.6 \times 10^{-4} \text{ K}^{-1}.$$

- **B7:**

$$L \simeq \frac{a}{\mu b} \simeq 1.0 \times 10^6 \text{ J kg}^{-1}.$$

- **B8:** A monolayer estimate with half the bulk cohesive contribution missing gives

$$\sigma \sim \frac{1}{2} \rho_L L d_w = \frac{a d_w}{2b^2} \simeq 8.6 \times 10^{-2} \text{ N m}^{-1}.$$

Its reliable precision is an order-of-magnitude estimate, $\sigma \sim 10^{-1} \text{ N m}^{-1}$.

- **C1:** For the convex liquid surface shown,

$$\Delta p_T = \frac{2\sigma\rho_s}{r(\rho_L - \rho_s)} \simeq \frac{2\sigma\rho_s}{r\rho_L} > 0.$$

- **C2:** With $T_e = 293 \text{ K}$ and $\Delta T = 5.0 \text{ K}$,

$$r_{\min} \simeq \frac{2\sigma T_e}{\rho_L L \Delta T} = \frac{2\sigma b^2 T_e}{a \Delta T} \simeq 1.4 \times 10^{-8} \text{ m} = 14 \text{ nm}.$$

Droplets larger than this critical radius can grow.

Solution

A1. Molecular size and excluded volume

Model the molecules as hard spheres of diameter d . The center of one molecule cannot enter a sphere of radius d around the center of another. The excluded volume associated with a pair is therefore

$$v_{\text{ex}} = \frac{4\pi d^3}{3}.$$

In the standard dilute hard-sphere excluded-volume calculation, each pair is counted once, giving an excluded-volume parameter per molecule of $v_{\text{ex}}/2$. For one mole,

$$b = \frac{1}{2} N_A v_{\text{ex}} = \frac{2\pi}{3} N_A d^3.$$

Thus the usual van der Waals b is four times the total geometrical volume of the spherical molecules. Independently of this geometric prefactor, the requested scale is $b \sim N_A d^3$.

A2. Critical-point conditions

Write the equation of state as

$$P(V, T) = \frac{RT}{V - b} - \frac{a}{V^2}.$$

At the critical point, the two extrema of the subcritical isotherm merge into a horizontal inflection point. Hence

$$\left(\frac{\partial P}{\partial V} \right)_{T_c, V_c} = 0, \quad \left(\frac{\partial^2 P}{\partial V^2} \right)_{T_c, V_c} = 0.$$

These conditions give

$$\frac{RT_c}{(V_c - b)^2} = \frac{2a}{V_c^3}, \quad \frac{RT_c}{(V_c - b)^3} = \frac{3a}{V_c^4}.$$

Dividing the second equation by the first,

$$\frac{1}{V_c - b} = \frac{3}{2V_c},$$

so

$$V_c = 3b.$$

Substitution into the first derivative condition and the equation of state yields

$$RT_c = \frac{8a}{27b}, \quad P_c = \frac{a}{27b^2}.$$

Solving for the constants,

$$\boxed{b = \frac{RT_c}{8P_c}, \quad a = \frac{27R^2T_c^2}{64P_c}.$$

A3. Constants for water

Using the given critical data,

$$b_w = \frac{8.31(647)}{8(2.2 \times 10^7)} = 3.055 \times 10^{-5} \text{ m}^3 \text{ mol}^{-1},$$

and

$$a_w = \frac{27(8.31)^2(647)^2}{64(2.2 \times 10^7)} = 0.5543 \text{ Pa m}^6 \text{ mol}^{-2}.$$

To two significant figures,

$$\boxed{a_w \simeq 0.55 \text{ Pa m}^6 \text{ mol}^{-2}, \quad b_w \simeq 3.1 \times 10^{-5} \text{ m}^3 \text{ mol}^{-1}.$$

The unrounded values are retained in subsequent calculations.

A4. Molecular diameter

From A1,

$$d_w = \left(\frac{3b_w}{2\pi N_A} \right)^{1/3}.$$

Consequently,

$$d_w \simeq 2.9 \times 10^{-10} \text{ m} = 0.29 \text{ nm}.$$

This is an effective diameter within the hard-sphere model.

B1. Saturated vapor volume

Apply the prescribed approximation $V_G \gg b$:

$$\left(p_0 + \frac{a}{V_G^2} \right) V_G = RT.$$

Multiplication by V_G gives

$$p_0 V_G^2 - RT V_G + a = 0.$$

The gaseous state corresponds to the larger root, which approaches RT/p_0 as $a \rightarrow 0$:

$$V_G = \frac{RT + \sqrt{R^2 T^2 - 4ap_0}}{2p_0}.$$

At $T = 373 \text{ K}$,

$$V_G = 0.0308164 \text{ m}^3 \text{ mol}^{-1} \simeq 3.08 \times 10^{-2} \text{ m}^3 \text{ mol}^{-1}.$$

For weak attraction, expansion of the square root also gives

$$V_G \simeq \frac{RT}{p_0} - \frac{a}{RT}.$$

B2. Relative volume decrease

The ideal-gas volume is

$$V_{G0} = \frac{RT}{p_0} = 0.0309963 \text{ m}^3 \text{ mol}^{-1}.$$

Using B1,

$$\frac{V_{G0} - V_G}{V_{G0}} = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4ap_0}{R^2 T^2}} \right).$$

Numerically,

$$\frac{\Delta V_G}{V_{G0}} = 5.803 \times 10^{-3},$$

so

$$\frac{\Delta V_G}{V_{G0}} \simeq 0.58\%.$$

To first order, this fraction is $ap_0/(R^2 T^2)$.

B3. Limit of metastable compression

Within the same gas approximation,

$$P(V) = \frac{RT}{V} - \frac{a}{V^2}.$$

Mechanical stability requires

$$\frac{dP}{dV} = -\frac{RT}{V^2} + \frac{2a}{V^3} = \frac{2a - RTV}{V^3} < 0.$$

Thus the gas remains mechanically stable against small volume perturbations when

$$V > \frac{2a}{RT}.$$

The limiting, marginally stable volume is therefore

$$V_{G \min} = \frac{2a}{RT} = 3.58 \times 10^{-4} \text{ m}^3 \text{ mol}^{-1}.$$

The compression factor is

$$\frac{V_G}{V_{G \min}} = \frac{RT}{2a} \frac{RT + \sqrt{R^2 T^2 - 4ap_0}}{2p_0} \simeq 86.2.$$

Hence

$$\frac{V_G}{V_{G \min}} \simeq 86.$$

At the limiting volume, $V_{G \min}/b \simeq 12$, so the result is an estimate within the prescribed approximation that neglects b .

B4. Liquid volume

For the liquid, neglect P relative to a/V_L^2 :

$$\frac{a}{V_L^2}(V_L - b) = RT.$$

Therefore

$$RTV_L^2 - aV_L + ab = 0.$$

The liquid is the smaller-volume root, which approaches b as $T \rightarrow 0$:

$$V_L = \frac{a - \sqrt{a^2 - 4abRT}}{2RT} = \frac{2b}{1 + \sqrt{1 - 4bRT/a}}.$$

At 373 K, this expression gives approximately

$$V_L = 3.91 \times 10^{-5} \text{ m}^3 \text{ mol}^{-1}.$$

For the remaining questions, introduce the stipulated small parameter

$$\varepsilon = \frac{bRT}{a} \ll 1.$$

Expanding the liquid root,

$$V_L = b \left(1 + \frac{bRT}{a} + O(\varepsilon^2) \right).$$

For the supplied water parameters, $\varepsilon \simeq 0.17$, so the following leading-order results are approximate.

B5. Liquid density

To leading order, $V_L \simeq b$. Since one mole has mass μ ,

$$\rho_L = \frac{\mu}{V_L} \simeq \frac{\mu}{b}.$$

Numerically,

$$\rho_L \simeq \frac{1.8 \times 10^{-2}}{3.055 \times 10^{-5}} = 5.9 \times 10^2 \text{ kg m}^{-3}.$$

This is the density predicted by the specified approximation.

B6. Thermal expansion coefficient

Retaining the first temperature-dependent term from B4,

$$V_L \simeq b + \frac{b^2 RT}{a}.$$

Thus, at fixed external pressure within the approximation used,

$$\frac{\partial V_L}{\partial T} \simeq \frac{b^2 R}{a}.$$

Using $V_L \simeq b$ in the denominator,

$$\alpha = \frac{1}{V_L} \left(\frac{\partial V_L}{\partial T} \right)_P \simeq \frac{bR}{a}.$$

Numerically,

$$\alpha \simeq 4.6 \times 10^{-4} \text{ K}^{-1}.$$

B7. Specific latent heat of vaporization

The thermodynamic identity

$$\left(\frac{\partial U}{\partial V} \right)_T = T \left(\frac{\partial P}{\partial T} \right)_V - P$$

relates the internal energy to the equation of state. For the van der Waals fluid,

$$T \left(\frac{\partial P}{\partial T} \right)_V - P = \frac{RT}{V-b} - \left(\frac{RT}{V-b} - \frac{a}{V^2} \right) = \frac{a}{V^2}.$$

Hence the molar internal energy has the form

$$U(T, V) = U_0(T) - \frac{a}{V}.$$

At the same temperature, the internal-energy increase upon vaporization is

$$U_G - U_L = a \left(\frac{1}{V_L} - \frac{1}{V_G} \right).$$

Vaporization at constant pressure also requires expansion work, so the heat supplied per mole is

$$\mu L = a \left(\frac{1}{V_L} - \frac{1}{V_G} \right) + p_0(V_G - V_L).$$

Since $V_G \gg V_L \simeq b$, and $RT \ll a/b$, the leading contribution is a/b . Therefore

$$L \simeq \frac{a}{\mu b}.$$

Numerically,

$$L \simeq 1.0 \times 10^6 \text{ J kg}^{-1}.$$

The omission of thermal corrections is consistent: retaining $V_L \simeq b(1 + bRT/a)$ gives $a/V_L \simeq a/b - RT$, whose $-RT$ correction cancels the leading expansion-work term $p_0 V_G \simeq RT$.

B8. Surface tension from a monomolecular layer

Surface tension is the surface free energy per unit area. For an energetic estimate, neglect the surface entropy contribution and use the cohesive energy found in B7.

The bulk cohesive energy per molecule is approximately

$$E_{\text{coh}} \simeq \frac{\mu L}{N_A} \simeq \frac{a}{b N_A}.$$

A molecule near a free surface lacks neighbors on the vapor side. In a simple estimate, its assigned cohesive binding is reduced by about one half, giving a surface excess energy of order $E_{\text{coh}}/2$.

A surface layer of area S and thickness d_w contains approximately

$$N_{\text{layer}} \simeq \frac{\rho_L S d_w}{\mu} N_A$$

molecules. Its excess energy is therefore

$$E_{\text{surface}} \sim \frac{1}{2} E_{\text{coh}} N_{\text{layer}} = \frac{1}{2} \rho_L L d_w S.$$

Identifying this with σS ,

$$\sigma \sim \frac{1}{2} \rho_L L d_w = \frac{a d_w}{2 b^2}.$$

Using the estimates above,

$$\sigma \sim 8.6 \times 10^{-2} \text{ N m}^{-1}.$$

The monolayer construction fixes the scale, $\sigma \sim 10^{-1} \text{ N m}^{-1}$; its numerical prefactor depends on the assumed molecular arrangement and missing interactions.

C1. Saturation pressure above a curved surface

Let p_s be the saturated vapor pressure above the flat liquid surface. The nonwetting meniscus is convex toward the vapor and lies a depth h below that surface.

For a small height difference, hydrostatic equilibrium gives the pressures at the meniscus level:

$$p_\ell = p_s + \rho_L g h, \quad p_v = p_s + \rho_s g h.$$

The spherical meniscus obeys the Laplace pressure relation

$$p_\ell - p_v = \frac{2\sigma}{r}.$$

Consequently,

$$(\rho_L - \rho_s)gh = \frac{2\sigma}{r}.$$

The change in the equilibrium vapor pressure is

$$\Delta p_T = p_v - p_s = \rho_s gh.$$

Eliminating h ,

$$\Delta p_T = \frac{2\sigma\rho_s}{r(\rho_L - \rho_s)} \simeq \frac{2\sigma\rho_s}{r\rho_L}.$$

The last step uses $\rho_s \ll \rho_L$. The change is positive: a convex liquid surface requires a higher equilibrium vapor pressure than a flat surface.

C2. Minimum radius of a growing droplet

Let $p_s(T)$ denote the saturation pressure over a flat surface. Initially the actual vapor pressure is

$$p = p_s(T_e), \quad T_e = 293 \text{ K}.$$

After cooling by $\Delta T = 5.0 \text{ K}$, the actual pressure stays fixed while the flat-surface saturation pressure falls. To first order,

$$\Delta p_{\text{cool}} = p_s(T_e) - p_s(T_e - \Delta T) \simeq \left. \frac{dp_s}{dT} \right|_{T_e} \Delta T.$$

From equation (3),

$$\frac{1}{p_s} \frac{dp_s}{dT} = -\frac{B}{T^2} = \frac{a}{bRT^2}.$$

Thus

$$\Delta p_{\text{cool}} \simeq p_s \frac{a\Delta T}{bRT_e^2}.$$

A droplet can grow if the ambient vapor pressure exceeds its curvature-dependent equilibrium pressure. At the critical radius, the two pressure changes are equal:

$$p_s \frac{a\Delta T}{bRT_e^2} = \frac{2\sigma\rho_s}{\rho_L r_{\min}}.$$

Treating the dilute vapor as ideal,

$$\rho_s \simeq \frac{\mu p_s}{RT_e}.$$

Substituting and cancelling p_s ,

$$r_{\min} \simeq \frac{2\sigma\mu b T_e}{\rho_L a \Delta T}.$$

Using the Part B results $\rho_L \simeq \mu/b$ and $L \simeq a/(\mu b)$,

$$r_{\min} \simeq \frac{2\sigma b^2 T_e}{a \Delta T} = \frac{2\sigma T_e}{\rho_L L \Delta T}.$$

With the supplied tabulated surface tension,

$$r_{\min} \simeq \frac{2(7.3 \times 10^{-2})(3.055 \times 10^{-5})^2(293)}{(0.5543)(5.0)} = \boxed{1.4 \times 10^{-8} \text{ m} \approx 14 \text{ nm.}}$$

At this radius a droplet is in unstable equilibrium. Larger droplets can grow; smaller droplets tend to evaporate.

The solution was checked against the official solution.

Problem 3: Simplest model of gas discharge

An electric current flowing through a gas is called a gas discharge. There are many types of gas discharges including glow discharge in lighting lamps, arc discharge in welding and the well known spark discharge that occurs between the clouds and the earth in the form of lightning.

Part A. Non-self-sustained gas discharge (4.8points)

In this part of the problem the so-called non-self-sustained gas discharge is studied. To maintain it a permanent operationan external ionizer is needed, which creates Z_{ext} pairs of singly ionized ions and free electrons per unit volume and per unit time uniformly in the volume.

When an external ionizer is switched on, the number of electrons and ions starts to grow. Unlimited increase in the number densities of electrons and ions in the gas is prevented by the recombination process in which a free electron recombineswith an ion to form a neutral atom. The number of recombining events Z_{rec} that occurs in the gas per unit volume and per unit time is given by

$$Z_{\text{rec}} = r n_e n_i,$$

where r is a constant called the recombination coefficient, and n_e, n_i denotethe electron and ion number densities, respectively.

Suppose that at time $t = 0$ the external ionizer is switched on and the initial number densities of electrons and ions in the gas are both equal to zero. Then, the electron number density $n_e(t)$ depends on time t as follows:

$$n_e(t) = n_0 + a \tanh bt,$$

where n_0, a and b are some constants, and $\tanh x$ stands for the hyperbolic tangent.

A1 Find n_0, a, b and express them in terms of Z_{ext} and r . (1.8points)

Assume that there are two external ionizers available. When the first one is switched on, the electron number density in the gas reaches its equilibrium value of be $n_{e1} = 12 \cdot 10^{10} \text{ cm}^{-3}$. When the second external ionizer is switched on, the electron number density reaches its equilibrium value of $n_{e2} = 16 \cdot 10^{10} \text{ cm}^{-3}$.

A2 Find the electron number density n_e at equilibriumwhen both external ionizers are switched on simultaneously. (0.6 points)

Attention! In what follows it is assumed that the external ionizer is switched on for quite long period of time such that all processes have become stationary and do not depend on time. Completely neglect the electric field due the charge carriers.

Assume that the gas fills in the tube between the two parallel conductive plates of area S separated by the distance $L \ll \sqrt{S}$ from each other. The voltage U is applied across the plates to create an electric field between them. Assume that the number densities of both kinds of

charge carriers remain almost constant along the tube.

Assume that both the electrons (denoted by the subscript e) and the ions (denoted by the subscript i) acquire the same ordered speed v due to the electric field strength E found as

$$v = \beta E,$$

where β is a constant called charge mobility.

A3	Express the electric current I in the tube in terms of $U, \beta, L, S, Z_{\text{ext}}, r$ and e which is the elementary charge. (1.7 points)
A4	Find the resistivity ρ_{gas} of the gas at sufficiently small values of the voltage applied and express it in terms of $\beta, L, Z_{\text{ext}}, r$ and e . (0.7 points)

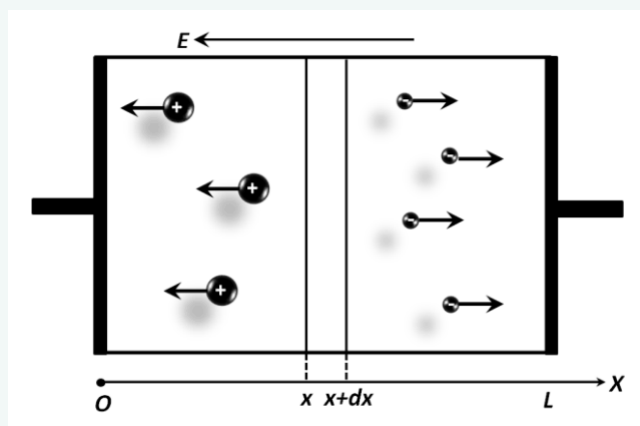
Part B. Self-sustained gas discharge (5.2 points)

In this part of the problem the ignition of the self-sustained gas discharge is considered to show how the electric current in the tube becomes self-maintaining.

Attention! In the sequel assume that the external ionizer continues to operate with the same Z_{ext} rate, neglect the electric field due to the charge carriers such that the electric field is uniform along the tube, and the recombination can be completely ignored.

For the self-sustained gas discharge there are two important processes not considered above. The first process is a secondary electron emission, and the second one is a formation of electron avalanche. The secondary electron emission occurs when ions hit on the negative electrode, called a cathode, and the electrons are knocked out of it to move towards the positive electrode, called an anode. The ratio of the number of the knocked electrons $\dot{N}_e$ per unit time to the number of ions $\dot{N}_i$ hitting the cathode per unit time is called the coefficient of the secondary electron emission, $\gamma = \dot{N}_e / \dot{N}_i$. The formation of the electron avalanche is explained as follows. The electric field accelerates free electrons which acquire enough kinetic energy to ionize the atoms in the gas by hitting them. As a result the number of free electrons moving towards the anode significantly increases. This process is described by the Townsend coefficient α , which characterizes an increase in the number of electrons dN_e due to moving N_e electrons that have passed the distance dl , i.e.

$$\frac{dN_e}{dl} = \alpha N_e.$$



The total current I in any cross section of the gas tube consists of the ion $I_i(x)$ and the electron $I_e(x)$ currents which, in the steady state, depend on the coordinate x , shown in the figure above. The electron current $I_e(x)$ varies along the x -axis according to the formula

$$I_e(x) = C_1 e^{A_1 x} + A_2,$$

where A_1, A_2, C_1 are some constants.

B1 Find A_1, A_2 and express them in terms of $Z_{\text{ext}}, \alpha, e, L, S$. **(2 points)**

The ion current $I_i(x)$ varies along the x -axis according to the formula

$$I_i(x) = C_2 + B_1 e^{B_2 x},$$

where B_1, B_2, C_2 are some constants.

B2 Find B_1, B_2 and express them in terms of $Z_{\text{ext}}, \alpha, e, L, S, C_1$. **(0.6 points)**

B3 Write down the condition for $I_i(x)$ at $x = L$. **(0.3 points)**

B4 Write down the condition for $I_i(x)$ and $I_e(x)$ at $x = 0$. **(0.6 points)**

B5 Find the total current I and express it in terms of $Z_{\text{ext}}, \alpha, \gamma, e, L, S$. Assume that it remains finite. **(1.2 points)**

Let the Townsend coefficient α be constant. When the length of the tube turns out greater than some critical value, i.e. $L > L_{\text{cr}}$, the external ionizer can be turned off and the discharge becomes self-sustained.

B6 Find L_{cr} and express it in terms of $Z_{\text{ext}}, \alpha, \gamma, e, L, S$. **(0.5 points)**

Official English text of the problem.

Answers

Currents below denote magnitudes, and $U \geq 0$.

• **A1**

$$n_0 = 0, \quad a = \sqrt{\frac{Z_{\text{ext}}}{r}}, \quad b = \sqrt{r Z_{\text{ext}}}.$$

• **A2**

$$n_e = 2.0 \times 10^{11} \text{ cm}^{-3} = 2.0 \times 10^{17} \text{ m}^{-3}.$$

• **A3**, using the stated uniform-density collection approximation:

$$I = \frac{e\beta S U}{rL} \left[\sqrt{\frac{\beta^2 U^2}{L^4} + 4r Z_{\text{ext}}} - \frac{\beta U}{L^2} \right].$$

• **A4**

$$\rho_{\text{gas}} = \frac{1}{2e\beta} \sqrt{\frac{r}{Z_{\text{ext}}}}.$$

- B1

$$A_1 = \alpha, \quad A_2 = -\frac{eSZ_{\text{ext}}}{\alpha}.$$

- B2

$$B_1 = -C_1, \quad B_2 = \alpha.$$

- B3

$$I_i(L) = 0.$$

- B4

$$I_e(0) = \gamma I_i(0).$$

- B5

$$I = \frac{eSZ_{\text{ext}}}{\alpha} \frac{(1 + \gamma)(e^{\alpha L} - 1)}{1 - \gamma(e^{\alpha L} - 1)}.$$

A physical finite stationary solution with $Z_{\text{ext}} > 0$ requires

$$\gamma(e^{\alpha L} - 1) < 1.$$

- B6, for $\alpha > 0, \gamma > 0$:

$$L_{\text{cr}} = \frac{1}{\alpha} \ln \left(1 + \frac{1}{\gamma} \right).$$

Solution

A1. Growth of the carrier density

Ionization creates one electron and one ion per event, while recombination removes one of each. Since their initial densities are equal, they remain equal:

$$n_e(t) = n_i(t) = n(t).$$

The particle balance per unit volume is therefore

$$\frac{dn}{dt} = Z_{\text{ext}} - rn^2, \quad n(0) = 0.$$

The prescribed expression gives $n(0) = n_0$, so

$$n_0 = 0.$$

Substituting $n = a \tanh(bt)$ into the balance equation,

$$ab [1 - \tanh^2(bt)] = Z_{\text{ext}} - ra^2 \tanh^2(bt).$$

Matching the constant terms and the coefficients of $\tanh^2(bt)$ yields

$$ab = Z_{\text{ext}}, \quad ab = ra^2.$$

Taking a, b positive,

$$a = \sqrt{\frac{Z_{\text{ext}}}{r}}, \quad b = \sqrt{rZ_{\text{ext}}}.$$

Thus

$$n_e(t) = \sqrt{\frac{Z_{\text{ext}}}{r}} \tanh\left(\sqrt{rZ_{\text{ext}}} t\right).$$

Its initial slope is Z_{ext} , and its limiting density is $\sqrt{Z_{\text{ext}}/r}$, as required by generation–recombination balance. Simultaneously reversing the signs of a and b describes the same function.

A2. Two ionizers

Let the individual production rates be Z_1 and Z_2 . At equilibrium,

$$Z_1 = rn_{e1}^2, \quad Z_2 = rn_{e2}^2.$$

The independent production rates add when both ionizers operate:

$$rn_e^2 = Z_1 + Z_2 = r(n_{e1}^2 + n_{e2}^2).$$

Consequently,

$$n_e = \sqrt{n_{e1}^2 + n_{e2}^2} = \sqrt{12^2 + 16^2} \times 10^{10} \text{ cm}^{-3}.$$

Therefore,

$$n_e = 2.0 \times 10^{11} \text{ cm}^{-3} = 2.0 \times 10^{17} \text{ m}^{-3}.$$

The densities themselves do not add because the recombination rate is quadratic in density.

A3. Stationary current with recombination and collection

Neglecting edge effects and the field of the carriers,

$$E = \frac{U}{L}, \quad v = \frac{\beta U}{L}.$$

Apply the stipulated uniform-density approximation to the volume SL . In a time dt , a collecting electrode removes $n_e Sv dt$ electrons; the opposite electrode removes $n_i Sv dt$ ions. The stationary particle balances are

$$\begin{aligned} Z_{\text{ext}}SL - rn_en_iSL - n_eSv &= 0, \\ Z_{\text{ext}}SL - rn_en_iSL - n_iSv &= 0. \end{aligned}$$

For $v > 0$, subtraction gives $n_e = n_i = n$. Hence

$$rn^2 + \frac{v}{L}n - Z_{\text{ext}} = 0.$$

Only the nonnegative root is physical:

$$n = \frac{1}{2r} \left[\sqrt{\left(\frac{v}{L}\right)^2 + 4rZ_{\text{ext}}} - \frac{v}{L} \right].$$

Although the electrons and ions move in opposite directions, their charges also have opposite signs. Their conventional currents therefore add:

$$I = eSn_ev + eSn_iv = 2eSnv.$$

Substitution gives

$$I = \frac{eSv}{r} \left[\sqrt{\left(\frac{v}{L}\right)^2 + 4rZ_{\text{ext}}} - \frac{v}{L} \right].$$

Using $v = \beta U/L$,

$$I = \frac{e\beta SU}{rL} \left[\sqrt{\frac{\beta^2 U^2}{L^4} + 4rZ_{\text{ext}}} - \frac{\beta U}{L^2} \right].$$

This is the result of applying the spatially uniform collection model literally. Strictly, the densities cannot stay uniform up to the plates: the electron density must vanish at the cathode and the ion density at the anode. The formula is therefore reliable while these depleted layers are thin compared with L , as in the small-voltage regime of A4; at high voltage it tends to $2eZ_{\text{ext}}SL$, twice the true saturation current $eZ_{\text{ext}}SL$.

A4. Small-voltage resistivity

For sufficiently small voltage,

$$\frac{\beta U}{L^2} \ll \sqrt{rZ_{\text{ext}}},$$

collection perturbs the generation–recombination equilibrium only slightly. To leading order,

$$n \simeq \sqrt{\frac{Z_{\text{ext}}}{r}}.$$

Thus the current becomes proportional to voltage:

$$I \simeq 2e\beta \sqrt{\frac{Z_{\text{ext}}}{r}} \frac{S}{L} U.$$

Comparing this with

$$I = \frac{S}{\rho_{\text{gas}} L} U$$

gives

$$\rho_{\text{gas}} = \frac{1}{2e\beta} \sqrt{\frac{r}{Z_{\text{ext}}}}.$$

The length L cancels. With SI quantities, the result is in $\Omega \text{ m}$.

B1. Electron-current equation

In Part B, recombination is neglected. Take I_e and I_i as nonnegative current magnitudes, so that

$$I = I_e(x) + I_i(x).$$

Consider a thin slice of thickness dx . The external ionizer creates

$$Z_{\text{ext}} S dx$$

electrons per unit time in the slice. Avalanche multiplication creates an additional

$$\alpha \frac{I_e(x)}{e} dx$$

electrons per unit time. Each new electron adds e times its production rate to the electron current leaving the slice. Therefore,

$$I_e(x + dx) - I_e(x) = [eSZ_{\text{ext}} + \alpha I_e(x)]dx.$$

In differential form,

$$\frac{dI_e}{dx} = \alpha I_e + eSZ_{\text{ext}}.$$

For constant $\alpha > 0$, its general solution is

$$I_e(x) = C_1 e^{\alpha x} - \frac{eSZ_{\text{ext}}}{\alpha}.$$

Comparison with the prescribed expression gives

$$\boxed{A_1 = \alpha, \quad A_2 = -\frac{eSZ_{\text{ext}}}{\alpha}.}$$

B2. Ion-current equation

Every ionization event also produces one ion. Ions move toward decreasing x , so the ion current leaving the slice through its left face exceeds that entering through its right face:

$$I_i(x) - I_i(x + dx) = [eSZ_{\text{ext}} + \alpha I_e(x)]dx.$$

Thus

$$\frac{dI_i}{dx} = -eSZ_{\text{ext}} - \alpha I_e(x).$$

Using the electron-current solution,

$$\frac{dI_i}{dx} = -\alpha C_1 e^{\alpha x},$$

and integration gives

$$I_i(x) = C_2 - C_1 e^{\alpha x}.$$

Consequently,

$$\boxed{B_1 = -C_1, \quad B_2 = \alpha.}$$

The two differential equations also show that

$$\frac{d}{dx}(I_e + I_i) = 0,$$

which expresses stationary charge conservation.

B3. Boundary condition at the anode

The anode is at $x = L$. No ions enter the gas from this electrode, and ions created inside the gas drift away from it. Therefore,

$$\boxed{I_i(L) = 0.}$$

Equivalently,

$$C_2 = C_1 e^{\alpha L}.$$

B4. Boundary condition at the cathode

At $x = 0$, the ion arrival rate is $I_i(0)/e$. Each arriving ion releases, on average, γ electrons. Hence

$$\frac{I_e(0)}{e} = \gamma \frac{I_i(0)}{e},$$

or

$$I_e(0) = \gamma I_i(0).$$

B5. Total stationary current

For brevity, define

$$K = \frac{eSZ_{\text{ext}}}{\alpha}, \quad T = e^{\alpha L}.$$

Then

$$I_e(x) = C_1 e^{\alpha x} - K, \quad I_i(x) = C_2 - C_1 e^{\alpha x}.$$

The anode condition gives

$$C_2 = C_1 T.$$

The cathode condition becomes

$$C_1 - K = \gamma(C_2 - C_1) = \gamma C_1(T - 1).$$

Therefore,

$$C_1 = \frac{K}{1 - \gamma(T - 1)}.$$

Adding the electron and ion currents,

$$I = C_2 - K = C_1 T - K.$$

Substituting C_1 and simplifying,

$$I = K \frac{T - 1 + \gamma(T - 1)}{1 - \gamma(T - 1)}.$$

Thus

$$I = \frac{eSZ_{\text{ext}}}{\alpha} \frac{(1 + \gamma)(e^{\alpha L} - 1)}{1 - \gamma(e^{\alpha L} - 1)}.$$

With positive external ionization, a physical finite stationary current requires

$$1 - \gamma(e^{\alpha L} - 1) > 0.$$

At zero denominator, the linear stationary model predicts divergence; a negative denominator would give an unphysical negative current magnitude.

B6. Critical length for self-sustained discharge

One electron emitted from the cathode grows into

$$e^{\alpha L}$$

electrons by the time the avalanche reaches the anode. Since the original electron was already present, the avalanche creates

$$e^{\alpha L} - 1$$

new electron–ion pairs.

With recombination neglected, all these ions reach the cathode. They release, on average,

$$\mathcal{R} = \gamma(e^{\alpha L} - 1)$$

secondary electrons. This is the number of cathode electrons produced by one preceding cathode electron.

If $\mathcal{R} < 1$, successive generations decrease and external ionization is necessary. The threshold occurs when each generation exactly replaces itself:

$$\gamma(e^{\alpha L_{\text{cr}}} - 1) = 1.$$

Solving,

$$L_{\text{cr}} = \frac{1}{\alpha} \ln \left(1 + \frac{1}{\gamma} \right).$$

For $L > L_{\text{cr}}$, the feedback amplifies a seed discharge even after the external ionizer is removed. The present linear model determines the ignition threshold; additional physical effects are needed to determine a finite operating current above threshold.

Neither Z_{ext} , e , nor S enters the threshold. If $\alpha = 0$ or $\gamma = 0$, this mechanism cannot sustain a discharge at any finite length.

The solution was checked against the official solution.