

IPhO 2013

Theory problems and solutions · Copenhagen, Denmark · 3 theory problems with complete solutions

eulersolve.org/ipho/2013/

Problem 1: The Maribo Meteorite

Introduction

A meteoroid is a small particle (typically smaller than 1 m) from a comet or an asteroid. A meteoroid that impacts the ground is called a meteorite.

On the night of 17 January 2009 many people near the Baltic Sea saw the glowing trail or fireball of a meteoroid falling through the atmosphere of the Earth. In Sweden a surveillance camera recorded a video of the event, see Fig. 1.1(a). From these pictures and eyewitness accounts it was possible to narrow down the impact area, and six weeks later a meteorite with the mass 0.025 kg was found in the vicinity of the town Maribo in southern Denmark. Measurements on the meteorite, now named Maribo, and its orbit in the sky showed interesting results. Its speed when entering the atmosphere had been exceptionally high. Its age, 4.567×10^9 year, shows that it had been formed shortly after the birth of the solar system. The Maribo meteorite is possibly a part of Comet Encke.

The speed of Maribo

The fireball was moving in westerly direction, heading 285° relative to north, toward the location where the meteorite was subsequently found, as sketched in Fig. 1.1. The meteorite was found at a distance 195 km from the surveillance camera in the direction 230° relative to north.

Part	Question	Points
1.1	Use this and the data in Fig. 1.1 to calculate the average speed of Maribo in the time interval between frames 155 and 161. The curvature of the Earth and the gravitational force on the meteoroid can both be neglected.	1.3

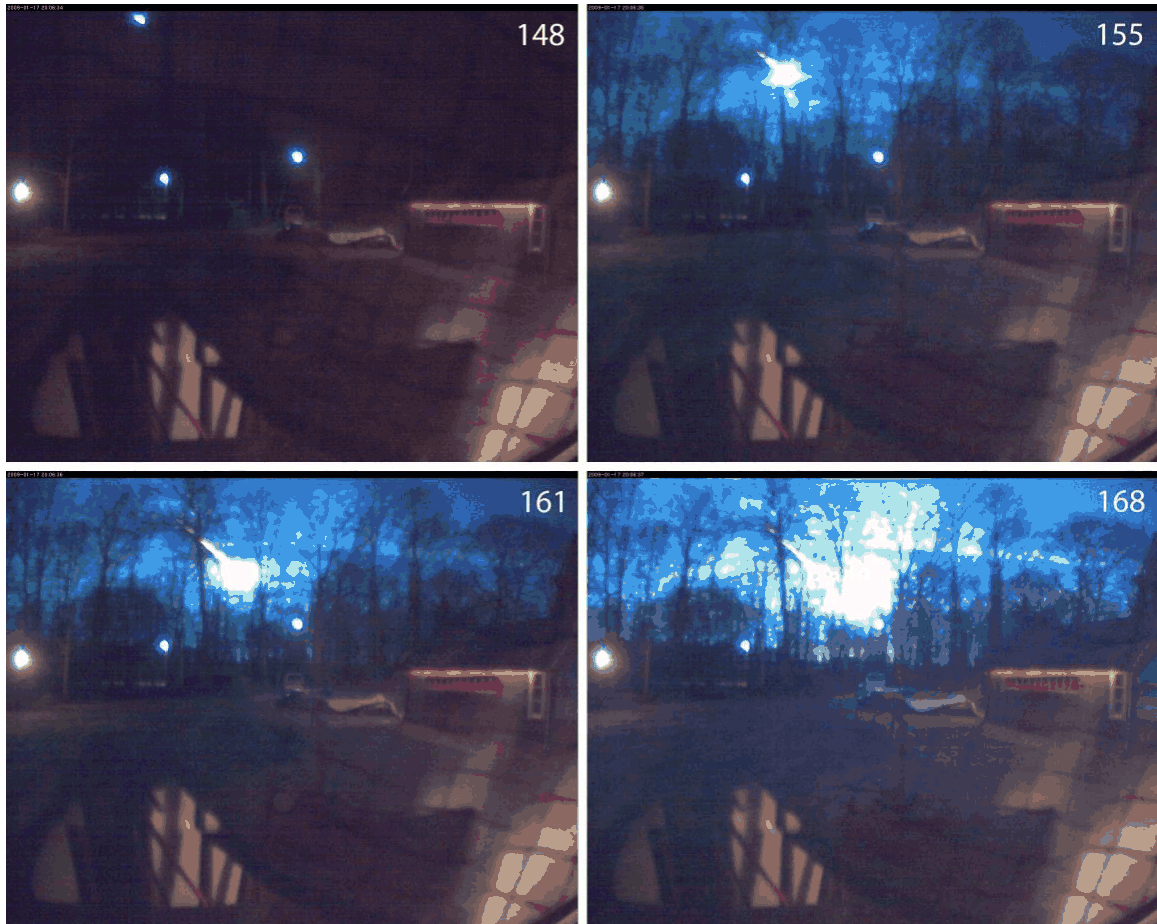
Through the atmosphere and melting?

The friction from the air on a meteoroid moving in the higher atmosphere depends in a complicated way on the shape and velocity of the meteoroid, and on the temperature and density of the atmosphere. As a reasonable approximation the friction force F in the upper atmosphere is given by the expression $F = k\rho_{\text{atm}}Av^2$, where k is a constant, ρ_{atm} the density of the atmosphere, A the projected cross-section area of the meteorite, and v its speed.

The following simplifying assumptions are made to analyze the meteoroid: The object entering the atmosphere was a sphere of mass $m_M = 30$ kg, radius $R_M = 0.13$ m, temperature $T_0 = 200$ K, and speed $v_M = 2.91 \times 10^4$ m/s. The density of the atmosphere is constant (its value 40 km above the surface of the Earth), $\rho_{\text{atm}} = 4.1 \times 10^{-3}$ kg/m³, and the friction coefficient is $k = 0.60$.

Part	Question	Points
1.2a	Estimate how long time after entering the atmosphere it takes the meteoroid to have its speed reduced by 10 % from v_M to $0.90v_M$. You can neglect the gravitational force on the meteoroid and assume, that it maintains its mass and shape.	0.7
1.2b	Calculate how many times larger the kinetic energy E_{kin} of the meteoroid entering the atmosphere is than the energy E_{melt} necessary for melting it completely (see data sheet).	0.3

(a)



(b)

Frame	Time	Azimuth	Altitude
155	1.46 s	215°	19.2°
161	2.28 s	221°	14.7°
Landing at M		230°	0.0°

(c)

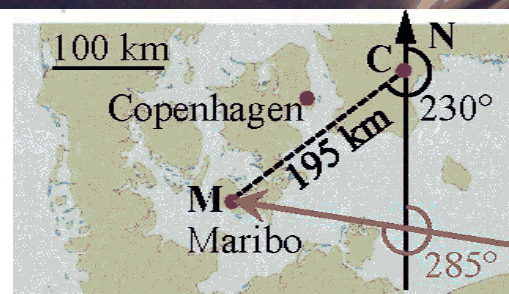


Figure 1.1 (a) Azimuth is the clockwise angular position from north in the horizontal plane, and altitude is the angular position above the horizon. A series of frames recorded by the surveillance camera in Sweden, showing the motion of Maribo as a fireball on its way down through the atmosphere. (b) The data from two frames indicating the time, the direction (azimuth) in degrees, as seen by the camera (C), and the height above the horizon (altitude) in degrees. (c) Sketch of the directions of the path (magenta arrow) of Maribo relative to north (N) and of the landing site (M) in Denmark as seen by the camera (C).

Heating of Maribo during its fall in the atmosphere

When the stony meteoroid Maribo entered the atmosphere at supersonic speed it appeared as a fireball because the surrounding air was glowing. Nevertheless, only the outermost layer of Maribo was heated. Assume that Maribo is a homogenous sphere with density ρ_{sm} , specific heat capacity c_{sm} , and thermal conductivity k_{sm} (for values see the data sheet). Furthermore, when entering the atmosphere, it had the temperature $T_0 = 200$ K. While falling through the atmosphere its surface temperature was constant $T_s = 1000$ K due to the air friction, thus gradually heating the interior.

After falling a time t in the atmosphere, an outer shell of Maribo of thickness x will have been heated to a temperature significantly larger than T_0 . This thickness can be estimated by dimensional analysis as the simple product of powers of the thermodynamic parameters:

$$x \approx t^\alpha \rho_{\text{sm}}^\beta c_{\text{sm}}^\gamma k_{\text{sm}}^\delta.$$

Part	Question	Points
1.3a	Determine by dimensional (unit) analysis the value of the four powers α , β , γ , and δ .	0.6
1.3b	Calculate the thickness x after a fall time $t = 5$ s, and determine the ratio x/R_M .	0.4

The age of a meteorite

The chemical properties of radioactive elements may be different, so during the crystallization of the minerals in a given meteorite, some minerals will have a high content of a specific radioactive element and others a low content. This difference can be used to determine the age of a meteorite by radiometric dating of its radioactive minerals.

As a specific example, we study the isotope ^{87}Rb (element no. 37), which decays into the stable isotope ^{87}Sr (element no. 38) with a half-life of $T_{1/2} = 4.9 \times 10^{10}$ year, relative to the stable isotope ^{86}Sr . At the time of crystallization the ratio $^{87}\text{Sr}/^{86}\text{Sr}$ was identical for all minerals, while the ratio $^{87}\text{Rb}/^{86}\text{Sr}$ was different. As time passes on, the amount of ^{87}Rb decreases by decay, while consequently the amount of ^{87}Sr increases. As a result, the ratio $^{87}\text{Sr}/^{86}\text{Sr}$ will be different today. In Fig. 1.2(a), the points on the horizontal line refer to the ratio $^{87}\text{Rb}/^{86}\text{Sr}$ in different minerals at the time, when they are crystallized.

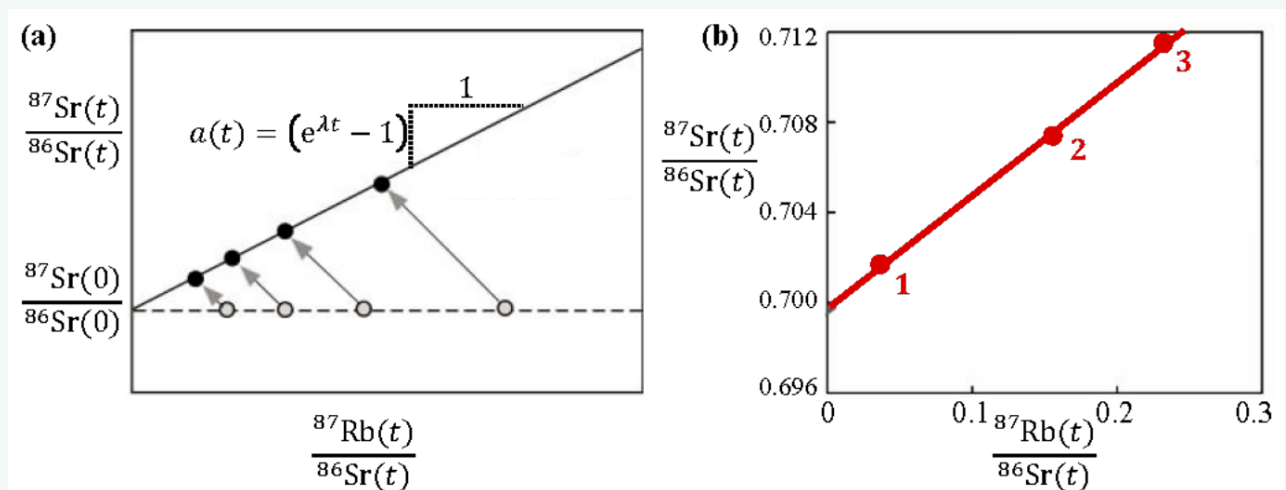


Figure 1.2 (a) The ratio $^{87}\text{Sr}/^{86}\text{Sr}$ in different minerals at the time $t = 0$ of crystallization (open circles) and at present time (filled circles). (b) The isochron-line for three different mineral samples taken from a meteorite at present time.

Part	Question	Points
1.4a	Write down the decay scheme for the transformation of $^{87}_{37}\text{Rb}$ to $^{87}_{38}\text{Sr}$.	0.3
1.4b	Show that the present-time ratio $^{87}\text{Sr}/^{86}\text{Sr}$ plotted versus the present-time ratio $^{87}\text{Rb}/^{86}\text{Sr}$ in different mineral samples from the same meteorite forms a straight line, the so-called isochron-line, with slope $a(t) = (e^{\lambda t} - 1)$. Here t is the time since the formation of the minerals, while λ is the decay constant inversely proportional to half-life $T_{1/2}$.	0.7
1.4c	Determine the age τ_M of the meteorite using the isochron-line of Fig. 1.2(b).	0.4

Comet Encke, from which Maribo may originate

In its orbit around the Sun, the minimum and maximum distances between comet Encke and the Sun are $a_{\min} = 4.95 \times 10^{10} \text{ m}$ and $a_{\max} = 6.16 \times 10^{11} \text{ m}$, respectively.

Part	Question	Points
1.5	Calculate the orbital period t_{Encke} of comet Encke.	0.6

Consequences of an asteroid impact on Earth

65 million years ago Earth was hit by a huge asteroid with density $\rho_{\text{ast}} = 3.0 \times 10^3 \text{ kg m}^{-3}$, radius $R_{\text{ast}} = 5.0 \text{ km}$, and final speed of $v_{\text{ast}} = 2.5 \times 10^4 \text{ m/s}$. This impact resulted in the extermination of most of the life on Earth and the formation of the enormous Chicxulub Crater. Assume that an identical asteroid would hit Earth today in a completely inelastic collision, and use the fact that the moment of inertia of Earth is 0.83 times that for a homogeneous sphere of the same mass and radius. The moment of inertia of a homogeneous sphere with mass M and radius R is $\frac{2}{5}MR^2$. Neglect changes in the orbit of the Earth.

Part	Question	Points
1.6a	Let the asteroid hit the North Pole. Find the maximum change in angular orientation of the axis of Earth after the impact.	0.7
1.6b	Let the asteroid hit the Equator in a radial impact. Find the change $\Delta\tau_{\text{vrt}}$ in the duration of one revolution of Earth after the impact.	0.7
1.6c	Let the asteroid hit the Equator in a tangential impact in the equatorial plane. Find the change $\Delta\tau_{\text{tan}}$ in the duration of one revolution of Earth after the impact.	0.7

Maximum impact speed

Consider a celestial body, gravitationally bound in the solar system, which impacts the surface of Earth with a speed v_{imp} . Initially the effect of the gravitational field of the Earth on the body can be neglected. Disregard the friction in the atmosphere, the effect of other celestial bodies, and the rotation of the Earth.

Part	Question	Points
1.7	Calculate $v_{\text{imp}}^{\max}$, the largest possible value of v_{imp} .	1.6

Data sheet: Table of physical parameters

Physical parameter	Value
Speed of light in vacuum	$c = 2.998 \times 10^8 \text{ m s}^{-1}$

Physical parameter	Value
Planck's constant over 2π	$\hbar = 1.055 \times 10^{-34} \text{ J s}$
Gravitational constant	$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Gravitational acceleration	$g = 9.82 \text{ m s}^{-2}$
Elementary charge	$e = 1.602 \times 10^{-19} \text{ C}$
Electric permittivity of vacuum	$\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{ J}^{-1} \text{ m}^{-1}$
Electron mass	$m_e = 9.109 \times 10^{-31} \text{ kg}$
Avogadro constant	$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	$k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$
Stony meteorite, specific heat	$c_{\text{sm}} = 1.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Stony meteorite, thermal conductivity	$k_{\text{sm}} = 2.0 \text{ W m}^{-1} \text{ K}^{-1}$
Stony meteorite, density	$\rho_{\text{sm}} = 3.3 \times 10^3 \text{ kg m}^{-3}$
Stony meteorite, melting point	$T_{\text{sm}} = 1.7 \times 10^3 \text{ K}$
Stony meteorite, specific melting heat	$L_{\text{sm}} = 2.6 \times 10^5 \text{ J kg}^{-1}$
Silver, molar mass	$M_{\text{Ag}} = 1.079 \times 10^{-1} \text{ kg mol}^{-1}$
Silver, density	$\rho_{\text{Ag}} = 1.049 \times 10^4 \text{ kg m}^{-3}$
Silver, specific heat capacity	$c_{\text{Ag}} = 2.40 \times 10^2 \text{ J kg}^{-1} \text{ K}^{-1}$
Water, molar mass	$M_{\text{wa}} = 1.801 \times 10^{-2} \text{ kg mol}^{-1}$
Water, density	$\rho_{\text{wa}} = 0.998 \times 10^3 \text{ kg m}^{-3}$
Water, specific heat capacity	$c_{\text{wa}} = 4.181 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Water, heat of vaporization	$L_{\text{wa}} = 2.260 \times 10^6 \text{ J kg}^{-1}$
Water, boiling temperature	$T_{100} = 100^\circ \text{C} = 373.15 \text{ K}$
Ice, density of glacier	$\rho_{\text{ice}} = 0.917 \times 10^3 \text{ kg m}^{-3}$
Steam, specific heat capacity	$c_{\text{st}} = 2.080 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Earth, mass of the	$m_E = 5.97 \times 10^{24} \text{ kg}$
Earth, radius of the	$R_E = 6.38 \times 10^6 \text{ m}$
Sun, mass of the	$m_S = 1.99 \times 10^{30} \text{ kg}$
Sun, radius of the	$R_S = 6.96 \times 10^8 \text{ m}$
Average Sun-Earth distance	$a_E = 1.50 \times 10^{11} \text{ m}$

Official English text of the problem.

Answers

- **1.1:** $\bar{v} = 2.86 \times 10^4 \text{ m s}^{-1} = 28.6 \text{ km s}^{-1}$.
- **1.2a:** $t = 0.88 \text{ s}$.
- **1.2b:** $E_{\text{kin}}/E_{\text{melt}} \approx 2.1 \times 10^2$.
- **1.3a:** $\alpha = \delta = \frac{1}{2}, \quad \beta = \gamma = -\frac{1}{2}$, so $x \approx \sqrt{k_{\text{sm}} t / (\rho_{\text{sm}} c_{\text{sm}})}$.

- **1.3b:** $x \approx 1.6 \text{ mm}, \quad x/R_M \approx 0.012 = 1.2\%$.
- **1.4a:** ${}^{87}_{37}\text{Rb} \rightarrow {}^{87}_{38}\text{Sr} + {}^0_{-1}e + \bar{\nu}_e$.
- **1.4b:** $Y(t) = Y_0 + (e^{\lambda t} - 1)X(t)$, where $Y = {}^{87}\text{Sr}/{}^{86}\text{Sr}$, $X = {}^{87}\text{Rb}/{}^{86}\text{Sr}$, and $\lambda = \ln 2/T_{\frac{1}{2}}$.
- **1.4c:** $\tau_M \approx 3.5 \times 10^9 \text{ yr}$, from a graph slope of approximately 0.050.
- **1.5:** $t_{\text{Encke}} = 1.05 \times 10^8 \text{ s} \approx 3.32 \text{ yr}$.
- **1.6a:** $\theta_{\text{max}} \approx 4.3 \times 10^{-8} \text{ rad} \approx 2.4 \times 10^{-6} \text{ degree}$.
- **1.6b:** $\Delta\tau_{\text{vrt}} \approx +6.8 \times 10^{-5} \text{ s}$.
- **1.6c:** $\Delta\tau_{\text{tan}} \approx -3.6 \times 10^{-3} \text{ s}$ for a prograde impact; $\Delta\tau_{\text{tan}} \approx +3.8 \times 10^{-3} \text{ s}$ for a retrograde impact.
- **1.7:** $v_{\text{imp}}^{\text{max}} \approx 7.27 \times 10^4 \text{ m s}^{-1} = 72.7 \text{ km s}^{-1}$, understood as a limiting upper bound for a gravitationally bound body.

Solution

1.1

Let C denote the camera, M the landing site, and P_i the horizontal projection of the meteoroid's position in frame i . Write

$$CM = D = 195 \text{ km}, \quad CP_i = r_i, \quad MP_i = s_i.$$

The observed quantities are:

Frame	Time t_i	Azimuth ϕ_i	Altitude angle h_i
155	1.46 s	215°	19.2°
161	2.28 s	221°	14.7°

The horizontal trajectory has heading 285°. Viewed from M , the camera has bearing 50°, while the earlier meteoroid positions have bearing 105°. Hence the angle at M in triangle $CM P_i$ is 55°. The angle at C is $230^\circ - \phi_i$.

The sine rule therefore gives

$$r_i = D \frac{\sin 55^\circ}{\sin(285^\circ - \phi_i)}, \quad s_i = D \frac{\sin(230^\circ - \phi_i)}{\sin(285^\circ - \phi_i)}.$$

Since Earth's curvature is neglected, the meteoroid's height is

$$z_i = r_i \tan h_i.$$

Numerically,

Frame	r_i (km)	s_i (km)	z_i (km)
155	169.986	53.709	59.195
161	177.721	33.940	46.624

Thus the horizontal and vertical distances travelled are

$$\Delta s = 19.769 \text{ km}, \quad |\Delta z| = 12.571 \text{ km}.$$

Taking the trajectory over this short interval to be straight,

$$\Delta \ell = \sqrt{(\Delta s)^2 + (\Delta z)^2} = 23.428 \text{ km}.$$

The elapsed time is $2.28 - 1.46 = 0.82 \text{ s}$, so

$$\bar{v} = \frac{23.428 \text{ km}}{0.82 \text{ s}} = 28.6 \text{ km s}^{-1}.$$

For the subsequent parts, we use the separately specified initial speed $v_M = 2.91 \times 10^4 \text{ m s}^{-1}$.

1.2a

The projected area of the sphere is $A = \pi R_M^2$. Newton's second law, with gravity neglected, gives

$$m_M \frac{dv}{dt} = -k\rho_{\text{atm}}\pi R_M^2 v^2.$$

Separating variables and integrating from v_M to v ,

$$\frac{1}{v} - \frac{1}{v_M} = \frac{k\rho_{\text{atm}}\pi R_M^2}{m_M} t.$$

For $v = 0.90v_M$,

$$t = \frac{m_M}{k\rho_{\text{atm}}\pi R_M^2 v_M} \left(\frac{1}{0.90} - 1 \right).$$

Substitution yields

$$t = \frac{30}{(0.60)(4.1 \times 10^{-3})\pi(0.13)^2(2.91 \times 10^4)} \left(\frac{1}{0.90} - 1 \right) = 0.877 \text{ s}.$$

Therefore,

$$t \approx 0.88 \text{ s}.$$

1.2b

Complete melting requires both heating the material from T_0 to its melting point and supplying its latent heat of fusion. Using the constant specific heat given in the data sheet,

$$E_{\text{melt}} = m_M [c_{\text{sm}}(T_{\text{sm}} - T_0) + L_{\text{sm}}].$$

Thus

$$E_{\text{melt}} = 30 [(1.2 \times 10^3)(1700 - 200) + 2.6 \times 10^5] = 6.18 \times 10^7 \text{ J}.$$

The initial kinetic energy is

$$E_{\text{kin}} = \frac{1}{2} m_M v_M^2 = \frac{1}{2} (30)(2.91 \times 10^4)^2 = 1.2702 \times 10^{10} \text{ J}.$$

Consequently,

$$\frac{E_{\text{kin}}}{E_{\text{melt}}} = \frac{v_M^2}{2[c_{\text{sm}}(T_{\text{sm}} - T_0) + L_{\text{sm}}]} = 205.5 \approx 2.1 \times 10^2.$$

This energy comparison does not imply complete melting: the kinetic energy need not be deposited as heat throughout the meteoroid.

1.3a

Use M , L , T , Θ for the dimensions of mass, length, time, and temperature. The relevant dimensions are

$$[\rho_{\text{sm}}] = ML^{-3}, \quad [c_{\text{sm}}] = L^2T^{-2}\Theta^{-1}, \quad [k_{\text{sm}}] = MLT^{-3}\Theta^{-1}.$$

Therefore,

$$[t^\alpha \rho_{\text{sm}}^\beta c_{\text{sm}}^\gamma k_{\text{sm}}^\delta] = M^{\beta+\delta} L^{-3\beta+2\gamma+\delta} T^{\alpha-2\gamma-3\delta} \Theta^{-\gamma-\delta}.$$

Equating this to the dimension L of x gives

$$\begin{aligned} \beta + \delta &= 0, \\ -3\beta + 2\gamma + \delta &= 1, \\ \alpha - 2\gamma - 3\delta &= 0, \\ -\gamma - \delta &= 0. \end{aligned}$$

The first and fourth equations give $\beta = \gamma = -\delta$. The length equation then gives $2\delta = 1$, and the time equation gives $\alpha = \delta$. Hence

$$\alpha = \delta = \frac{1}{2}, \quad \beta = \gamma = -\frac{1}{2}.$$

The resulting estimate is

$$x \approx \sqrt{\frac{k_{\text{sm}} t}{\rho_{\text{sm}} c_{\text{sm}}}}.$$

Dimensional analysis determines the exponents but leaves an undetermined dimensionless coefficient. The requested estimate takes this coefficient to be unity.

1.3b

The thermal diffusivity is

$$\chi = \frac{k_{\text{sm}}}{\rho_{\text{sm}} c_{\text{sm}}} = \frac{2.0}{(3.3 \times 10^3)(1.2 \times 10^3)} = 5.05 \times 10^{-7} \text{ m}^2 \text{ s}^{-1}.$$

At $t = 5 \text{ s}$,

$$x \approx \sqrt{\chi t} = \sqrt{(5.05 \times 10^{-7})(5)} = 1.589 \times 10^{-3} \text{ m}.$$

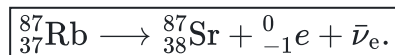
Thus

$$x \approx 1.6 \text{ mm}, \quad \frac{x}{R_{\text{M}}} = \frac{1.589 \times 10^{-3}}{0.13} \approx 0.012 = 1.2\%.$$

The heated depth is small compared with the radius, consistent with only an outer layer warming appreciably during the fall.

1.4a

The atomic number increases by one while the mass number remains unchanged. This is beta-minus decay:



An electron and an electron antineutrino are emitted.

1.4b

Consider one mineral sample as a closed system after crystallization. Let $N_{\text{Rb}}(t)$, $N_{87}(t)$, and N_{86} be the numbers of ^{87}Rb , ^{87}Sr , and ^{86}Sr nuclei. The stable reference population N_{86} is constant.

Radioactive decay gives

$$N_{\text{Rb}}(t) = N_{\text{Rb}}(0)e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{T_{\frac{1}{2}}}.$$

Each decayed rubidium nucleus produces one strontium-87 nucleus, so

$$\begin{aligned} N_{87}(t) &= N_{87}(0) + N_{\text{Rb}}(0) - N_{\text{Rb}}(t) \\ &= N_{87}(0) + N_{\text{Rb}}(t)(e^{\lambda t} - 1). \end{aligned}$$

Dividing by N_{86} , define

$$Y(t) = \frac{N_{87}(t)}{N_{86}}, \quad X(t) = \frac{N_{\text{Rb}}(t)}{N_{86}}, \quad Y_0 = \frac{N_{87}(0)}{N_{86}}.$$

Then

$$Y(t) = Y_0 + (e^{\lambda t} - 1)X(t).$$

All samples have the same initial strontium isotope ratio Y_0 and the same age t . Their present values therefore lie on a straight line with vertical intercept Y_0 and slope

$$a(t) = e^{\lambda t} - 1.$$

At crystallization the slope is zero; it subsequently increases with age.

1.4c

The line in Fig. 1.2(b) passes approximately through

$$(X, Y) \approx (0.05, 0.702) \quad \text{and} \quad (0.25, 0.712).$$

Its slope is therefore approximately

$$a \approx \frac{0.712 - 0.702}{0.25 - 0.05} = 0.050.$$

Inverting the result of part 1.4b,

$$\tau_{\text{M}} = \frac{\ln(1 + a)}{\lambda} = \frac{T_{\frac{1}{2}}}{\ln 2} \ln(1 + a).$$

Hence

$$\tau_{\text{M}} \approx \frac{4.9 \times 10^{10} \text{ yr}}{\ln 2} \ln(1.050) = 3.45 \times 10^9 \text{ yr}.$$

Allowing for the graphical reading,

$$\tau_{\text{M}} \approx 3.5 \times 10^9 \text{ yr}.$$

Figure 1.2(b) refers to samples from “a meteorite”; its inferred age need not equal Maribo’s age quoted in the introduction.

1.5

For an elliptical orbit, the perihelion and aphelion distances sum to twice the semimajor axis:

$$a = \frac{a_{\min} + a_{\max}}{2} = \frac{4.95 \times 10^{10} + 6.16 \times 10^{11}}{2} = 3.3275 \times 10^{11} \text{ m.}$$

Neglecting the comet's mass compared with the Sun's mass, Kepler's third law gives

$$t_{\text{Encke}}^2 = \frac{4\pi^2 a^3}{Gm_S}.$$

Thus

$$t_{\text{Encke}} = 2\pi \sqrt{\frac{(3.3275 \times 10^{11})^3}{(6.67 \times 10^{-11})(1.99 \times 10^{30})}} = 1.0468 \times 10^8 \text{ s.}$$

Using $1 \text{ yr} \approx 3.156 \times 10^7 \text{ s}$,

$$t_{\text{Encke}} \approx 1.05 \times 10^8 \text{ s} \approx 3.32 \text{ yr.}$$

1.6a

First calculate the quantities used in all three impact questions. The asteroid's mass is

$$m_{\text{ast}} = \frac{4}{3}\pi R_{\text{ast}}^3 \rho_{\text{ast}} = \frac{4}{3}\pi (5.0 \times 10^3)^3 (3.0 \times 10^3) = 1.571 \times 10^{15} \text{ kg.}$$

Earth's moment of inertia is

$$I_E = 0.83 \frac{2}{5} m_E R_E^2 = 8.068 \times 10^{37} \text{ kg m}^2.$$

Taking its initial rotation period to be $\tau_E = 24 \text{ h} = 86400 \text{ s}$,

$$\omega_E = \frac{2\pi}{\tau_E} = 7.272 \times 10^{-5} \text{ s}^{-1},$$

and its spin angular momentum is

$$L_E = I_E \omega_E = 5.867 \times 10^{33} \text{ kg m}^2 \text{ s}^{-1}.$$

We treat the accreted asteroid as a point mass at Earth's surface. Its size and the displacement of the common center of mass give negligible corrections.

At the North Pole, the asteroid's position vector is parallel to Earth's initial rotation axis. Consequently, its angular momentum

$$\mathbf{L}_{\text{ast}} = \mathbf{r} \times m_{\text{ast}} \mathbf{v}_{\text{ast}}$$

is perpendicular to Earth's initial spin angular momentum. Its magnitude is largest for a tangential impact:

$$L_{\text{ast}}^{\max} = m_{\text{ast}} R_E v_{\text{ast}} = 2.505 \times 10^{26} \text{ kg m}^2 \text{ s}^{-1}.$$

Conservation of angular momentum then gives, to leading order in the small asteroid-to-Earth mass ratio,

$$\tan \theta_{\max} \simeq \frac{L_{\text{ast}}^{\max}}{L_E}.$$

The angle is extremely small, so

$$\theta_{\max} \simeq \frac{m_{\text{ast}} R_E v_{\text{ast}}}{I_E \omega_E} = 4.27 \times 10^{-8} \text{ rad} \approx 2.4 \times 10^{-6} \text{ degree.}$$

The change in the inertia tensor contributes only a higher-order correction to this small tilt.

1.6b

For a radial impact, the incoming asteroid has zero angular momentum about Earth's center. After it sticks at the Equator, the moment of inertia increases to

$$I' = I_E + m_{\text{ast}} R_E^2.$$

Angular momentum conservation requires

$$I_E \omega_E = (I_E + m_{\text{ast}} R_E^2) \omega'.$$

Since the period is inversely proportional to angular speed,

$$\tau' = \tau_E \left(1 + \frac{m_{\text{ast}} R_E^2}{I_E} \right).$$

Therefore,

$$\Delta \tau_{\text{vrt}} = \tau_E \frac{m_{\text{ast}} R_E^2}{I_E} = \tau_E \frac{m_{\text{ast}}}{0.332 m_E}.$$

Numerically,

$$\Delta \tau_{\text{vrt}} = 6.85 \times 10^{-5} \text{ s} \approx +6.8 \times 10^{-5} \text{ s.}$$

The day becomes approximately $68 \mu\text{s}$ longer.

1.6c

The direction of the tangential impact matters. Let $\sigma = +1$ for a prograde impact, which adds spin angular momentum, and $\sigma = -1$ for a retrograde impact.

Conservation of angular momentum gives

$$I_E \omega_E + \sigma m_{\text{ast}} R_E v_{\text{ast}} = (I_E + m_{\text{ast}} R_E^2) \omega'.$$

Define the small dimensionless quantities

$$\epsilon = \frac{m_{\text{ast}} R_E^2}{I_E} = 7.925 \times 10^{-10}, \quad \eta = \frac{m_{\text{ast}} R_E v_{\text{ast}}}{I_E \omega_E} = 4.270 \times 10^{-8}.$$

The new period and its change are

$$\tau' = \tau_E \frac{1 + \epsilon}{1 + \sigma \eta},$$

$$\Delta \tau_{\text{tan}} = \tau_E \frac{\epsilon - \sigma \eta}{1 + \sigma \eta}.$$

Since both ϵ and η are very small, this is also accurately

$$\Delta \tau_{\text{tan}} \approx \tau_E (\epsilon - \sigma \eta).$$

Thus:

- **Prograde impact:**

$$\Delta\tau_{\text{tan}} = -3.621 \times 10^{-3} \text{ s} \approx -3.6 \text{ ms.}$$

- **Retrograde impact:**

$$\Delta\tau_{\text{tan}} = +3.758 \times 10^{-3} \text{ s} \approx +3.8 \text{ ms.}$$

The added moment of inertia lengthens the day in both cases. The incoming angular momentum produces the larger effect and determines the sign.

1.7

First consider the body's motion relative to the Sun, before Earth's gravity becomes significant. At Earth's orbital distance, a gravitationally bound body has negative specific orbital energy:

$$\mathcal{E} = \frac{u^2}{2} - \frac{Gm_S}{a_E} < 0.$$

Its heliocentric speed is therefore bounded by the local solar escape speed:

$$u < u_{\text{max}} = \sqrt{\frac{2Gm_S}{a_E}}.$$

The limiting case is an orbit whose binding energy tends to zero.

Approximating Earth's orbit as circular at the given mean distance, its orbital speed is

$$v_E = \sqrt{\frac{Gm_S}{a_E}}.$$

The largest relative approach speed occurs when the body moves directly opposite to Earth's orbital motion. Hence

$$v_{\infty}^{\text{max}} = u_{\text{max}} + v_E = (1 + \sqrt{2})\sqrt{\frac{Gm_S}{a_E}}.$$

Numerically,

$$v_E = 29.75 \text{ km s}^{-1}, \quad u_{\text{max}} = 42.07 \text{ km s}^{-1},$$

so

$$v_{\infty}^{\text{max}} = 71.82 \text{ km s}^{-1}.$$

Earth's gravity further accelerates the body. In the terrestrial encounter, conservation of mechanical energy gives

$$\frac{1}{2}v_{\text{imp}}^2 - \frac{Gm_E}{R_E} = \frac{1}{2}v_{\infty}^2.$$

Thus

$$v_{\text{imp}}^{\text{max}} = \sqrt{(1 + \sqrt{2})^2 \frac{Gm_S}{a_E} + \frac{2Gm_E}{R_E}}.$$

Earth's surface escape speed is

$$\sqrt{\frac{2Gm_E}{R_E}} = 11.17 \text{ km s}^{-1}.$$

Therefore,

$$v_{\text{imp}}^{\text{max}} = \sqrt{(71.82)^2 + (11.17)^2} \text{ km s}^{-1} \approx 72.7 \text{ km s}^{-1}.$$

For a strictly bound orbit, the speed is strictly below this value. It can approach the value arbitrarily closely as the initial solar binding energy tends to zero.

The solution was checked against the official solution.

Problem 2: Plasmonic Steam Generator

Introduction

In this problem we study an efficient process of steam production that has been demonstrated to work experimentally. An aqueous solution of spherical nanometer-sized silver spheres (nanoparticles) with only about 10^{13} particles per liter is illuminated by a focused light beam. A fraction of the light is absorbed by the nanoparticles, which are heated up and generate steam locally around them without heating up the entire water solution. The steam is released from the system in the form of escaping steam bubbles. Not all details of the process are well understood at present, but the core process is known to be absorption of light through the so-called collective electron oscillations of the metallic nanoparticles. The device is known as a plasmonic steam generator.

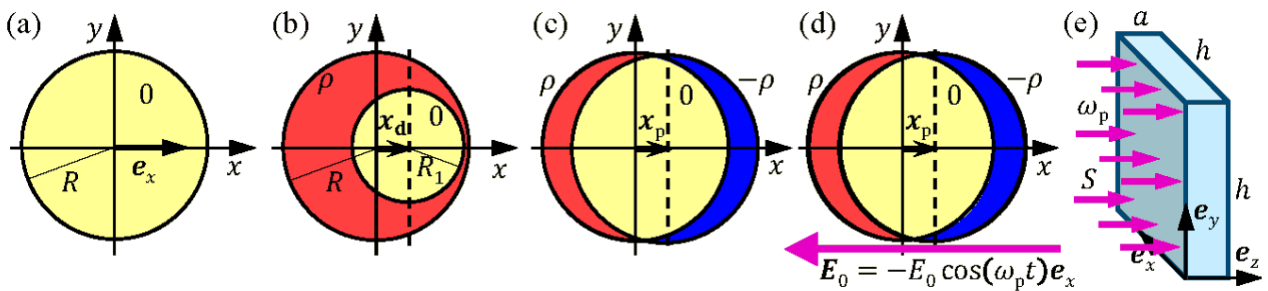


Figure 2.1 (a) A spherical charge-neutral nanoparticle of radius R placed at the center of the coordinate system. **(b)** A sphere with a positive homogeneous charge density ρ (red), and containing a smaller spherical charge-neutral region (0, yellow) of radius R_1 , with its center displaced by $\mathbf{x}_d = x_d \mathbf{e}_x$. **(c)** The sphere with positive charge density ρ of the nanoparticle silver ions is fixed in the center of the coordinate system. The center of the spherical region with negative spherical charge density $-\rho$ (blue) of the electron cloud is displaced by $\mathbf{x}_p$, where $x_p \ll R$. **(d)** An external homogeneous electric field $\mathbf{E}_0 = -E_0 \mathbf{e}_x$. For time-dependent $\mathbf{E}_0$, the electron cloud moves with velocity $\mathbf{v} = d\mathbf{x}_p/dt$. **(e)** The rectangular vessel ($h \times h \times a$) containing the aqueous solution of nanoparticles illuminated by monochromatic light propagating along the z -axis with angular frequency ω_p and intensity S .

A single spherical silver nanoparticle

Throughout this problem we consider a spherical silver nanoparticle of radius $R = 10.0 \text{ nm}$ and with its center fixed at the origin of the coordinate system, see Fig. 2.1(a). All motions, forces and driving fields are parallel to the horizontal x -axis (with unit vector $\mathbf{e}_x$). The nanoparticle contains

free (conduction) electrons moving within the whole nanoparticle volume without being bound to any silver atom. Each silver atom is a positive ion that has donated one such free electron.

Part	Question	Points
2.1	Find the following quantities: The volume V and mass M of the nanoparticle, the number N and charge density ρ of silver ions in the particle, and for the free electrons their concentration n , their total charge Q , and their total mass m_0 .	0.7

The electric field in a charge-neutral region inside a charged sphere

For the rest of the problem assume that the relative dielectric permittivity of all materials is $\varepsilon = 1$. Inside a charged sphere of homogeneous charge density ρ and radius R is created a small spherical charge-neutral region of radius R_1 by adding the opposite charge density $-\rho$, with its center displaced by $\mathbf{x}_d = x_d \mathbf{e}_x$ from the center of the R -sphere, see Fig. 2.1(b).

Part	Question	Points
2.2	Show that the electric field inside the charge-neutral region is homogenous of the form $\mathbf{E} = A(\rho/\varepsilon_0) \mathbf{x}_d$, and determine the pre-factor A .	1.2

The restoring force on the displaced electron cloud

In the following, we study the collective motion of the free electrons, and therefore model them as a single negatively charged sphere of homogeneous charge density $-\rho$ with a center position $\mathbf{x}_p$, which can move along the x -axis relative to the center of the positively charged sphere (silver ions) fixed at the origin of the coordinate system, see Fig. 2.1(c). Assume that an external force $\mathbf{F}_{\text{ext}}$ displaces the electron cloud to a new equilibrium position $\mathbf{x}_p = x_p \mathbf{e}_x$ with $|x_p| \ll R$. Except for tiny net charges at opposite ends of the nanoparticle, most of its interior remains charge-neutral.

Part	Question	Points
2.3	Express in terms of $\mathbf{x}_p$ and n the following two quantities: The restoring force $\mathbf{F}$ exerted on the electron cloud and the work W_{el} done on the electron cloud during displacement.	1.0

The spherical silver nanoparticle in an external constant electric field

A nanoparticle is placed in vacuum and influenced by an external force $\mathbf{F}_{\text{ext}}$ due to an applied static homogeneous electric field $\mathbf{E}_0 = -E_0 \mathbf{e}_x$, which displaces the electron cloud the small distance $|x_p|$, where $|x_p| \ll R$.

Part	Question	Points
2.4	Find the displacement x_p of the electron cloud in terms of E_0 and n , and determine the amount $-\Delta Q$ of electron charge displaced through the yz -plane at the center of the nanoparticle in terms of n , R and x_p .	0.6

The equivalent capacitance and inductance of the silver nanoparticle

For both a constant and a time-dependent field $\mathbf{E}_0$, the nanoparticle can be modeled as an equivalent electric circuit. The equivalent capacitance can be found by relating the work W_{el} done on the separation of charges ΔQ , to the energy of a capacitor, carrying charge $\pm \Delta Q$. The charge separation will cause a certain equivalent voltage V_0 across the equivalent capacitor.

Part	Question	Points
2.5a	Express the systems equivalent capacitance C in terms of ε_0 and R , and find its value.	0.7
2.5b	For this capacitance, determine in terms of E_0 and R the equivalent voltage V_0 that should be connected to the equivalent capacitor in order to accumulate the charge ΔQ .	0.4

For a time-dependent field $\mathbf{E}_0$, the electron cloud moves with velocity $\mathbf{v} = v\mathbf{e}_x$, Fig. 2.1(d). It has the kinetic energy W_{kin} and forms an electric current I flowing through the fixed yz -plane. The kinetic energy of the electron cloud can be attributed to the energy of an equivalent inductor of inductance L carrying the current I .

Part	Question	Points
2.6a	Express both W_{kin} and I in terms of the velocity v .	0.7
2.6b	Express the equivalent inductance L in terms of the particle radius R , the electron charge e and mass m_e , the electron concentration n , and calculate its value.	0.5

The plasmon resonance of the silver nanoparticle

From the above analysis it follows that the motion, arising from displacing the electron cloud from its equilibrium position and then releasing it, can be modeled by an ideal LC -circuit oscillating at resonance. This dynamical mode of the electron cloud is known as the plasmon resonance, which oscillates at the so-called angular plasmon frequency ω_p .

Part	Question	Points
2.7a	Find an expression for the angular plasmon frequency ω_p of the electron cloud in terms of the electron charge e and mass m_e , the electron density n , and the permittivity ε_0 .	0.5
2.7b	Calculate ω_p in rad/s and the wavelength λ_p in nm of light in vacuum having angular frequency $\omega = \omega_p$.	0.4

The silver nanoparticle illuminated with light at the plasmon frequency

In the rest of the problem, the nanoparticle is illuminated by monochromatic light at the angular plasmon frequency ω_p with the incident intensity $S = \frac{1}{2}c\varepsilon_0 E_0^2 = 1.00 \text{ MW m}^{-2}$. As the wavelength is large, $\lambda_p \gg R$, the nanoparticle can be considered as being placed in a homogeneous harmonically oscillating field $\mathbf{E}_0 = -E_0 \cos(\omega_p t)\mathbf{e}_x$. Driven by $\mathbf{E}_0$, the center $\mathbf{x}_p(t)$ of the electron cloud oscillates at the same frequency with velocity $\mathbf{v} = d\mathbf{x}_p/dt$ and constant amplitude x_0 . This oscillating electron motion leads to absorption of light. The energy captured by the particle is either converted into Joule heating inside the particle or re-emitted by the particle as scattered light.

Joule heating is caused by random inelastic collisions, where any given free electron once in a while hits a silver ion and loses its total kinetic energy, which is converted into vibrations of the silver ions (heat). The average time between the collisions is $\tau \gg 1/\omega_p$, where for silver nanoparticle we use $\tau = 5.24 \times 10^{-15} \text{ s}$.

Part	Question	Points
2.8a	Find an expression for the time-averaged Joule heating power P_{heat} in the nanoparticle as well as the time-averaged current squared $\langle I^2 \rangle$, which includes explicitly the time-averaged velocity squared $\langle v^2 \rangle$ of the electron cloud.	1.0

Part	Question	Points
2.8b	Find an expression for the equivalent ohmic resistance R_{heat} in an equivalent resistor-model of the nanoparticle having the Joule heating power P_{heat} due to the electron cloud current I . Calculate the numerical value of R_{heat} .	1.0

The incident light beam loses some time-averaged power P_{scat} by scattering on the oscillating electron cloud (re-emission). P_{scat} depends on the scattering source amplitude x_0 , charge Q , angular frequency ω_p and properties of the light (the speed of light c and permittivity ϵ_0 in vacuum). In terms of these four variables, P_{scat} is given by

$$P_{\text{scat}} = \frac{Q^2 x_0^2 \omega_p^4}{12\pi\epsilon_0 c^3}.$$

Part	Question	Points
2.9	By use of P_{scat} , find an expression for the equivalent scattering resistance R_{scat} (in analogy with R_{heat}) in an equivalent resistor-model, and calculate its value.	1.0

The above equivalent circuit elements are combined into an LCR series circuit model of the silver nanoparticle, which is driven by a harmonically oscillating equivalent voltage $V = V_0 \cos(\omega_p t)$ determined by the electric field E_0 of the incident light.

Part	Question	Points
2.10a	Derive expressions for the time-averaged power losses P_{heat} and P_{scat} involving the amplitude E_0 of the electric field in the incident light at the plasmon resonance $\omega = \omega_p$.	1.2
2.10b	Calculate the numerical value of E_0 , P_{heat} , and P_{scat} .	0.3

Steam generation by light

An aqueous solution of silver nanoparticles is prepared with a concentration $n_{\text{np}} = 7.3 \times 10^{15} \text{ m}^{-3}$. It is placed inside a rectangular transparent vessel of size $h \times h \times a = 10 \times 10 \times 1.0 \text{ cm}^3$ and illuminated by light at the plasmon frequency with the same intensity $S = 1.00 \text{ MW m}^{-2}$ at normal incidence as above, see Fig. 2.1(e). The temperature of the water is $T_{\text{wa}} = 20^\circ\text{C}$ and we assume, in fair agreement with observations, that in steady state all Joule heating of the nanoparticle goes to the production of steam of temperature $T_{\text{st}} = 110^\circ\text{C}$, without raising the temperature of the water.

The thermodynamic efficiency η of the plasmonic steam generator is defined by the power ratio $\eta = P_{\text{st}}/P_{\text{tot}}$, where P_{st} is the power going into the production of steam in the entire vessel, while P_{tot} is the total power of the incoming light that enters the vessel.

Most of the time any given nanoparticle is surrounded by steam instead of water, and it can thus be described as being in vacuum.

Part	Question	Points
2.11a	Calculate the total mass per second μ_{st} of steam produced by the plasmonic steam generator during illumination by light at the plasmon frequency and intensity S .	0.6
2.11b	Calculate the numerical value of the thermodynamic efficiency η of the plasmonic steam generator.	0.2

Data sheet: Table of physical parameters

Physical parameter	Value
Speed of light in vacuum	$c = 2.998 \times 10^8 \text{ m s}^{-1}$
Planck's constant over 2π	$\hbar = 1.055 \times 10^{-34} \text{ J s}$
Gravitational constant	$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Gravitational acceleration	$g = 9.82 \text{ m s}^{-2}$
Elementary charge	$e = 1.602 \times 10^{-19} \text{ C}$
Electric permittivity of vacuum	$\varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{ J}^{-1} \text{ m}^{-1}$
Electron mass	$m_e = 9.109 \times 10^{-31} \text{ kg}$
Avogadro constant	$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	$k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$
Stony meteorite, specific heat	$c_{\text{sm}} = 1.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Stony meteorite, thermal conductivity	$k_{\text{sm}} = 2.0 \text{ W m}^{-1} \text{ K}^{-1}$
Stony meteorite, density	$\rho_{\text{sm}} = 3.3 \times 10^3 \text{ kg m}^{-3}$
Stony meteorite, melting point	$T_{\text{sm}} = 1.7 \times 10^3 \text{ K}$
Stony meteorite, specific melting heat	$L_{\text{sm}} = 2.6 \times 10^5 \text{ J kg}^{-1}$
Silver, molar mass	$M_{\text{Ag}} = 1.079 \times 10^{-1} \text{ kg mol}^{-1}$
Silver, density	$\rho_{\text{Ag}} = 1.049 \times 10^4 \text{ kg m}^{-3}$
Silver, specific heat capacity	$c_{\text{Ag}} = 2.40 \times 10^2 \text{ J kg}^{-1} \text{ K}^{-1}$
Water, molar mass	$M_{\text{wa}} = 1.801 \times 10^{-2} \text{ kg mol}^{-1}$
Water, density	$\rho_{\text{wa}} = 0.998 \times 10^3 \text{ kg m}^{-3}$
Water, specific heat capacity	$c_{\text{wa}} = 4.181 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Water, heat of vaporization	$L_{\text{wa}} = 2.260 \times 10^6 \text{ J kg}^{-1}$
Water, boiling temperature	$T_{100} = 100^\circ \text{C} = 373.15 \text{ K}$
Ice, density of glacier	$\rho_{\text{ice}} = 0.917 \times 10^3 \text{ kg m}^{-3}$
Steam, specific heat capacity	$c_{\text{st}} = 2.080 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Earth, mass of the	$m_E = 5.97 \times 10^{24} \text{ kg}$
Earth, radius of the	$R_E = 6.38 \times 10^6 \text{ m}$
Sun, mass of the	$m_S = 1.99 \times 10^{30} \text{ kg}$
Sun, radius of the	$R_S = 6.96 \times 10^8 \text{ m}$
Average Sun-Earth distance	$a_E = 1.50 \times 10^{11} \text{ m}$

Official English text of the problem.

Answers

Here $e > 0$, $Q < 0$, and W_{el} denotes the positive work supplied to separate the charges. Choose the circuit-current direction so that $I = d(\Delta Q)/dt$, positive when the electrons move in $+x$.

• 2.1

$$\begin{aligned} V &= 4.19 \times 10^{-24} \text{ m}^3, & M &= 4.39 \times 10^{-20} \text{ kg}, \\ N &= 2.45 \times 10^5, & \rho &= +9.38 \times 10^9 \text{ C m}^{-3}, \\ n &= 5.85 \times 10^{28} \text{ m}^{-3}, & Q &= -3.93 \times 10^{-14} \text{ C}, & m_0 &= 2.23 \times 10^{-25} \text{ kg}. \end{aligned}$$

• 2.2

$$\mathbf{E} = \frac{\rho}{3\varepsilon_0} \mathbf{x}_d, \quad \boxed{A = \frac{1}{3}}.$$

• 2.3

$$\mathbf{F} = -\frac{4\pi n^2 e^2 R^3}{9\varepsilon_0} \mathbf{x}_p, \quad W_{\text{el}} = \frac{2\pi n^2 e^2 R^3}{9\varepsilon_0} x_p^2.$$

• 2.4

$$x_p = \frac{3\varepsilon_0 E_0}{ne}, \quad -\Delta Q = -ne\pi R^2 x_p.$$

• 2.5a

$$C = \frac{9}{4} \pi \varepsilon_0 R = 6.26 \times 10^{-19} \text{ F}.$$

• 2.5b

$$V_0 = \frac{4}{3} R E_0.$$

• 2.6a

$$W_{\text{kin}} = \frac{1}{2} m_0 v^2 = \frac{2\pi}{3} n m_e R^3 v^2, \quad I = ne\pi R^2 v.$$

• 2.6b

$$L = \frac{4m_e}{3\pi ne^2 R} = 2.57 \times 10^{-14} \text{ H}.$$

• 2.7a

$$\omega_p = \sqrt{\frac{ne^2}{3\varepsilon_0 m_e}}.$$

• 2.7b

$$\omega_p = 7.88 \times 10^{15} \text{ rad s}^{-1}, \quad \lambda_p = 239 \text{ nm}.$$

• 2.8a

$$P_{\text{heat}} = \frac{m_0}{2\tau} \langle v^2 \rangle, \quad \langle I^2 \rangle = (ne\pi R^2)^2 \langle v^2 \rangle.$$

• 2.8b

$$R_{\text{heat}} = \frac{L}{2\tau} = \frac{2m_e}{3\pi ne^2 R \tau} = 2.46 \text{ } \Omega.$$

• 2.9

$$R_{\text{scat}} = \frac{8R^2 \omega_p^2}{27\pi \varepsilon_0 c^3} = 2.45 \text{ } \Omega.$$

- **2.10a**

$$P_{\text{heat}} = \frac{8R^2 E_0^2}{9} \frac{R_{\text{heat}}}{(R_{\text{heat}} + R_{\text{scat}})^2}, \quad P_{\text{scat}} = \frac{8R^2 E_0^2}{9} \frac{R_{\text{scat}}}{(R_{\text{heat}} + R_{\text{scat}})^2}.$$

- **2.10b**

$$E_0 = 2.74 \times 10^4 \text{ V m}^{-1}, \quad P_{\text{heat}} = 6.82 \times 10^{-9} \text{ W}, \quad P_{\text{scat}} = 6.82 \times 10^{-9} \text{ W}.$$

- **2.11a**, using uniform illumination throughout the vessel:

$$\mu_{\text{st}} \simeq 1.9 \times 10^{-3} \text{ kg s}^{-1} = 1.9 \text{ g s}^{-1}.$$

- **2.11b**, with the same approximation:

$$\eta \simeq 0.50 = 50\%.$$

Solution

All electrostatic and circuit results below use the prescribed rigid, uniformly charged electron-cloud model. Displacement-dependent results retain the leading terms for $|x_p|/R \ll 1$.

2.1

The particle is a sphere, so

$$V = \frac{4\pi R^3}{3} = \frac{4\pi}{3} (1.00 \times 10^{-8})^3 = 4.19 \times 10^{-24} \text{ m}^3.$$

Its mass is

$$M = \rho_{\text{Ag}} V = 4.39 \times 10^{-20} \text{ kg}.$$

The number of silver atoms, and hence of singly charged silver ions, is

$$N = \frac{M}{M_{\text{Ag}}} N_A = 2.45 \times 10^5.$$

There is one conduction electron per ion. Thus

$$n = \frac{N}{V} = \frac{\rho_{\text{Ag}} N_A}{M_{\text{Ag}}} = 5.85 \times 10^{28} \text{ m}^{-3}.$$

The positive ionic charge density is

$$\rho = ne = +9.38 \times 10^9 \text{ C m}^{-3}.$$

For the electrons,

$$Q = -Ne = -3.93 \times 10^{-14} \text{ C}, \quad m_0 = Nm_e = 2.23 \times 10^{-25} \text{ kg}.$$

2.2

Apply Gauss's law to a concentric Gaussian sphere of radius r inside a uniformly charged sphere:

$$E(r) 4\pi r^2 = \frac{\rho(4\pi r^3/3)}{\epsilon_0}.$$

Consequently, the interior field is

$$\mathbf{E}_+(\mathbf{r}) = \frac{\rho}{3\epsilon_0} \mathbf{r}.$$

The neutral region is formed by superposing a sphere of charge density $-\rho$, centered at $\mathbf{x}_d$. At a point inside this smaller sphere its field is

$$\mathbf{E}_-(\mathbf{r}) = -\frac{\rho}{3\epsilon_0}(\mathbf{r} - \mathbf{x}_d).$$

Adding the two fields gives

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\epsilon_0}\mathbf{x}_d.$$

The position $\mathbf{r}$ cancels, so the field is homogeneous and

$$\boxed{A = \frac{1}{3}}.$$

2.3

The displaced electron cloud has charge density $-ne$, while the fixed ions have charge density $+ne$. By 2.2, the field in their overlapping, neutral interior is

$$\mathbf{E}_{\text{int}} = \frac{ne}{3\epsilon_0}\mathbf{x}_p.$$

To first order in displacement this field gives the restoring force on the cloud:

$$\mathbf{F} = Q\mathbf{E}_{\text{int}} = -\frac{Nne^2}{3\epsilon_0}\mathbf{x}_p.$$

The small nonoverlapping regions only affect higher-order terms. With $N = 4\pi nR^3/3$,

$$\boxed{\mathbf{F} = -\frac{4\pi n^2 e^2 R^3}{9\epsilon_0}\mathbf{x}_p}.$$

Define the effective spring constant

$$k = \frac{Nne^2}{3\epsilon_0}.$$

The work supplied in a quasistatic displacement from zero to x_p is

$$W_{\text{el}} = -\int_0^{x_p} F_x(x) dx = \int_0^{x_p} kx dx = \frac{1}{2}kx_p^2.$$

Therefore

$$\boxed{W_{\text{el}} = \frac{Nne^2}{6\epsilon_0}x_p^2 = \frac{2\pi n^2 e^2 R^3}{9\epsilon_0}x_p^2}.$$

This is the stored electrostatic energy used in 2.5. The work done by the restoring force itself is $-W_{\text{el}}$.

2.4

The external field points in $-x$, so the force on the negative electron cloud points in $+x$:

$$\mathbf{F}_{\text{ext}} = Q\mathbf{E}_0 = NeE_0\mathbf{e}_x.$$

At equilibrium,

$$NeE_0 - \frac{Nne^2}{3\epsilon_0}x_p = 0.$$

Hence

$$x_p = \frac{3\varepsilon_0 E_0}{ne}.$$

For a small translation x_p , the volume of electron cloud transported through the central plane is, to leading order,

$$\Delta V = \pi R^2 x_p.$$

Multiplying by its charge density gives

$$-\Delta Q = -ne\pi R^2 x_p.$$

Thus the positive charge-separation variable is

$$\Delta Q = ne\pi R^2 x_p = 3\pi\varepsilon_0 R^2 E_0.$$

2.5a

Equate the electrostatic energy to the energy of the equivalent capacitor:

$$W_{\text{el}} = \frac{(\Delta Q)^2}{2C}.$$

Using the results above,

$$C = \frac{(ne\pi R^2 x_p)^2}{2(2\pi n^2 e^2 R^3 x_p^2 / (9\varepsilon_0))} = \frac{9}{4}\pi\varepsilon_0 R.$$

Therefore

$$C = 6.26 \times 10^{-19} \text{ F}.$$

2.5b

The equivalent voltage is

$$V_0 = \frac{\Delta Q}{C}.$$

Substitution yields

$$V_0 = \frac{3\pi\varepsilon_0 R^2 E_0}{(9/4)\pi\varepsilon_0 R},$$

so

$$V_0 = \frac{4}{3} R E_0.$$

2.6a

All N electrons participate in the collective motion. Their kinetic energy is

$$W_{\text{kin}} = \frac{1}{2} N m_e v^2 = \frac{1}{2} m_0 v^2 = \frac{2\pi}{3} n m_e R^3 v^2.$$

Differentiating the charge-separation relation gives

$$I = \frac{d(\Delta Q)}{dt} = ne\pi R^2 v.$$

Here positive circuit current corresponds to conventional current in the $-x$ direction. If conventional current through the plane were instead defined positive in $+x$, it would be $I_x = -ne\pi R^2 v$. The energy and resistance results depend on I^2 , so this orientation choice does not affect them.

2.6b

Identify the kinetic energy with the energy of the equivalent inductor:

$$\frac{1}{2}LI^2 = \frac{1}{2}m_0v^2.$$

It follows that

$$L = \frac{m_0}{(ne\pi R^2)^2}.$$

Using $m_0 = (4\pi R^3/3)nm_e$,

$$L = \frac{4m_e}{3\pi ne^2 R} = 2.57 \times 10^{-14} \text{ H}.$$

2.7a

For a freely oscillating ideal LC circuit,

$$L \frac{d^2(\Delta Q)}{dt^2} + \frac{\Delta Q}{C} = 0,$$

so

$$\omega_p^2 = \frac{1}{LC}.$$

Since

$$LC = \frac{4m_e}{3\pi ne^2 R} \frac{9\pi\epsilon_0 R}{4} = \frac{3\epsilon_0 m_e}{ne^2},$$

we obtain

$$\omega_p = \sqrt{\frac{ne^2}{3\epsilon_0 m_e}}.$$

The same result follows directly from $\omega_p^2 = k/m_0$.

2.7b

Substitution of the given constants gives

$$\omega_p = 7.88 \times 10^{15} \text{ rad s}^{-1}.$$

Because $\omega = 2\pi c/\lambda$,

$$\lambda_p = \frac{2\pi c}{\omega_p} = 2.39 \times 10^{-7} \text{ m}.$$

Thus

$$\lambda_p = 239 \text{ nm}.$$

2.8a

In the stated collision model, each electron loses its entire collective kinetic energy in a collision. Its time-averaged energy loss per collision is

$$\frac{1}{2}m_e\langle v^2\rangle.$$

There are, on average, N/τ collisions per second among all electrons. Random collision times allow us to use the cycle-averaged kinetic energy, giving

$$P_{\text{heat}} = \frac{Nm_e}{2\tau}\langle v^2\rangle = \frac{m_0}{2\tau}\langle v^2\rangle.$$

From the current-velocity relation,

$$\langle I^2\rangle = (ne\pi R^2)^2\langle v^2\rangle.$$

2.8b

The equivalent resistance is defined by

$$P_{\text{heat}} = R_{\text{heat}}\langle I^2\rangle.$$

Consequently,

$$R_{\text{heat}} = \frac{m_0}{2\tau(ne\pi R^2)^2} = \frac{L}{2\tau}.$$

Thus

$$R_{\text{heat}} = \frac{2m_e}{3\pi ne^2 R\tau} = 2.46\,\Omega.$$

The factor $1/2$ follows from the kinetic energy $\frac{1}{2}m_e v^2$ lost per collision in the specific model given in the problem.

2.9

For harmonic motion of amplitude x_0 ,

$$x_p(t) = x_0 \cos(\omega_p t + \phi),$$

and hence

$$\langle v^2\rangle = \frac{1}{2}\omega_p^2 x_0^2.$$

Therefore

$$\langle I^2\rangle = \frac{1}{2}(ne\pi R^2)^2\omega_p^2 x_0^2.$$

Define the scattering resistance by

$$P_{\text{scat}} = R_{\text{scat}}\langle I^2\rangle.$$

Using the supplied radiation formula,

$$R_{\text{scat}} = \frac{Q^2 x_0^2 \omega_p^4}{12\pi\epsilon_0 c^3} \frac{2}{(ne\pi R^2)^2 \omega_p^2 x_0^2}.$$

Since

$$Q = -\frac{4}{3}\pi R^3 ne, \quad \frac{Q^2}{(ne\pi R^2)^2} = \frac{16R^2}{9},$$

we obtain

$$R_{\text{scat}} = \frac{8R^2\omega_p^2}{27\pi\epsilon_0 c^3} = 2.45 \, \Omega.$$

Equivalently,

$$R_{\text{scat}} = \frac{8ne^2 R^2}{81\pi\epsilon_0^2 m_e c^3}.$$

2.10a

Both heating and scattering remove energy from the same oscillating current. Their equivalent resistances therefore add in the series circuit:

$$R_{\Sigma} = R_{\text{heat}} + R_{\text{scat}}.$$

At $\omega = \omega_p = 1/\sqrt{LC}$, the inductive and capacitive reactances cancel:

$$\omega_p L - \frac{1}{\omega_p C} = 0.$$

Thus the current is in phase with the driving voltage and has amplitude

$$I_0 = \frac{V_0}{R_{\Sigma}} = \frac{4RE_0}{3R_{\Sigma}}.$$

For a sinusoidal current,

$$\langle I^2 \rangle = \frac{I_0^2}{2} = \frac{8R^2 E_0^2}{9R_{\Sigma}^2}.$$

The two average powers are therefore

$$P_{\text{heat}} = \frac{8R^2 E_0^2}{9} \frac{R_{\text{heat}}}{(R_{\text{heat}} + R_{\text{scat}})^2}$$

and

$$P_{\text{scat}} = \frac{8R^2 E_0^2}{9} \frac{R_{\text{scat}}}{(R_{\text{heat}} + R_{\text{scat}})^2}.$$

2.10b

From the specified incident intensity,

$$S = \frac{1}{2} c \epsilon_0 E_0^2,$$

so

$$E_0 = \sqrt{\frac{2S}{c\epsilon_0}} = \sqrt{\frac{2(1.00 \times 10^6)}{(2.998 \times 10^8)(8.854 \times 10^{-12})}}.$$

Thus

$$E_0 = 2.74 \times 10^4 \, \text{V m}^{-1}.$$

Using unrounded intermediate values,

$$V_0 = 3.66 \times 10^{-4} \text{ V}, \quad R_\Sigma = 4.910 \, \Omega, \quad I_0 = 7.45 \times 10^{-5} \text{ A}.$$

Consequently,

$$\boxed{P_{\text{heat}} = 6.82 \times 10^{-9} \text{ W}}, \quad \boxed{P_{\text{scat}} = 6.82 \times 10^{-9} \text{ W}}.$$

Their near equality follows from the near equality of the two resistances.

The displacement amplitude can also be checked:

$$x_0 = \frac{I_0}{ne\pi R^2 \omega_p} = 3.21 \times 10^{-15} \text{ m},$$

so $x_0/R = 3.21 \times 10^{-7} \ll 1$. Also, $\omega_p \tau = 41.3$, consistent with the assumed separation of oscillation and collision timescales.

2.11a

Apply the per-particle heating power from 2.10 throughout the illuminated volume. This uses uniform illumination and does not introduce an additional model for attenuation or multiple scattering.

The vessel volume is

$$V_{\text{vessel}} = h^2 a = (0.10)^2 (0.010) = 1.0 \times 10^{-4} \text{ m}^3.$$

It contains

$$N_{\text{np}} = n_{\text{np}} h^2 a = 7.3 \times 10^{11}$$

nanoparticles. All their Joule heating is assumed to produce steam, so

$$P_{\text{st}} = N_{\text{np}} P_{\text{heat}} = 4.98 \times 10^3 \text{ W}.$$

Each kilogram that becomes steam must first be heated from 20°C to 100°C , then vaporized, and finally heated to 110°C . The required energy per unit mass is

$$q_{\text{st}} = c_{\text{wa}}(T_{100} - T_{\text{wa}}) + L_{\text{wa}} + c_{\text{st}}(T_{\text{st}} - T_{100}).$$

Numerically,

$$\begin{aligned} q_{\text{st}} &= (4.181 \times 10^3)(80) + 2.260 \times 10^6 + (2.080 \times 10^3)(10) \\ &= 2.61528 \times 10^6 \text{ J kg}^{-1}. \end{aligned}$$

The bulk water remaining at 20°C does not eliminate the heating required for the portion that evaporates.

Energy conservation gives

$$\mu_{\text{st}} q_{\text{st}} = P_{\text{st}}.$$

Thus

$$\mu_{\text{st}} = \frac{n_{\text{np}} h^2 a P_{\text{heat}}}{c_{\text{wa}}(T_{100} - T_{\text{wa}}) + L_{\text{wa}} + c_{\text{st}}(T_{\text{st}} - T_{100})} = 1.904 \times 10^{-3} \text{ kg s}^{-1}.$$

To the precision of the vessel and concentration data,

$$\boxed{\mu_{\text{st}} \simeq 1.9 \text{ g s}^{-1}}.$$

2.11b

The beam enters normally through the face of area h^2 , so

$$P_{\text{tot}} = Sh^2 = (1.00 \times 10^6)(0.10)^2 = 1.00 \times 10^4 \text{ W}.$$

Therefore

$$\eta = \frac{P_{\text{st}}}{P_{\text{tot}}} = \frac{n_{\text{np}}h^2aP_{\text{heat}}}{Sh^2} = \frac{n_{\text{np}}aP_{\text{heat}}}{S}.$$

Numerically,

$$\eta = 0.4979,$$

or, with sensible rounding,

$$\boxed{\eta \simeq 0.50 = 50\%}.$$

The solution was checked against the official solution.

Problem 3: The Greenlandic Ice Sheet

Introduction

This problem deals with the physics of the Greenlandic ice sheet, the second largest glacier in the world, Fig. 3.1(a). As an idealization, Greenland is modeled as a rectangular island of width $2L$ and length $5L$ with the ground at sea level and completely covered by incompressible ice (constant density ρ_{ice}), see Fig. 3.1(b). The height profile $H(x)$ of the ice sheet does not depend on the y -coordinate and it increases from zero at the coasts $x = \pm L$ to a maximum height H_m along the middle north-south axis (the y -axis), known as the ice divide, see Fig. 3.1(c).

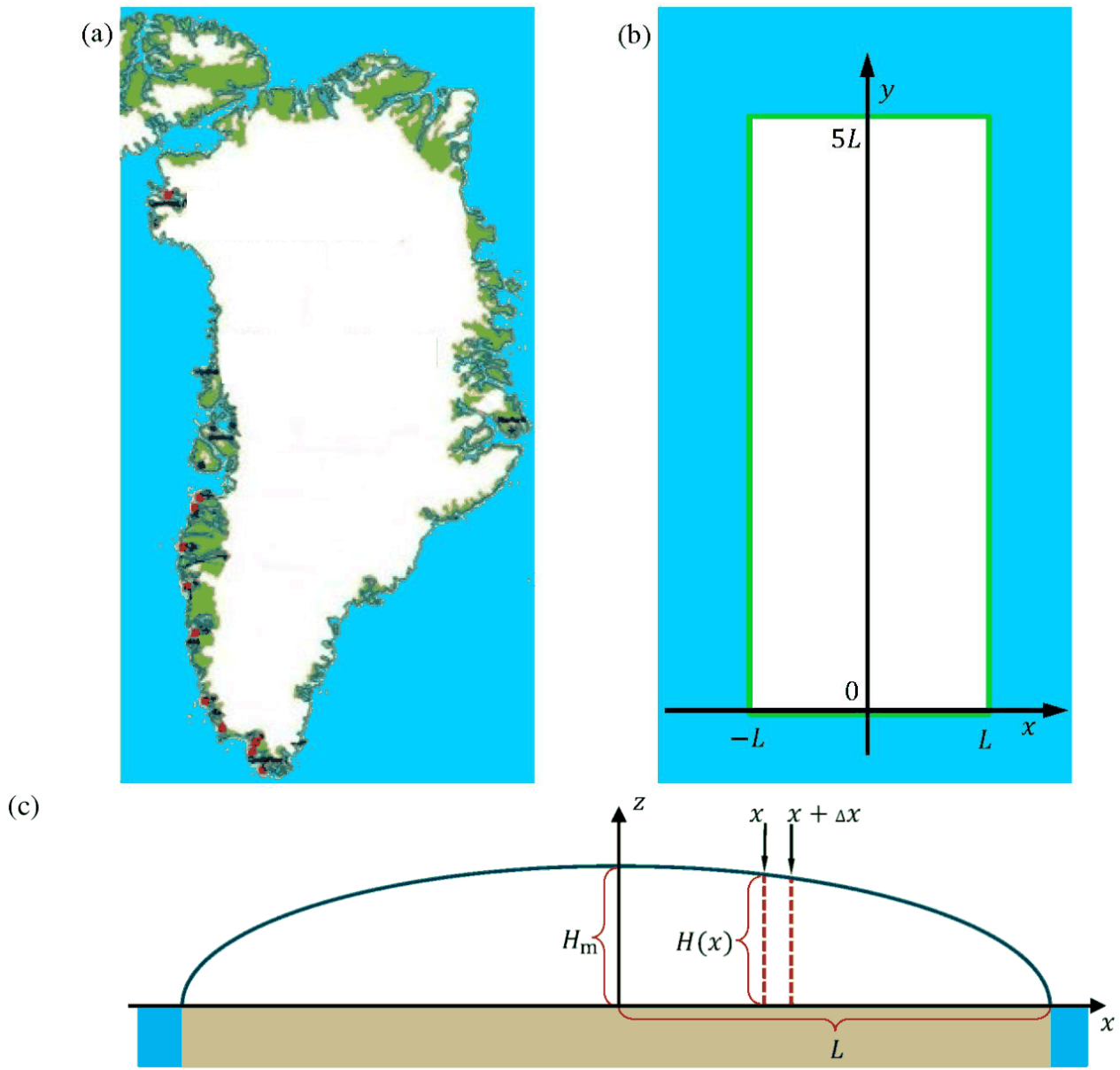


Figure 3.1 (a) A map of Greenland showing the extent of the ice sheet (white), the ice-free, coastal regions (green), and the surrounding ocean (blue). (b) The crude model of the Greenlandic ice sheet as covering a rectangular area in the xy -plane with side lengths $2L$ and $5L$. The ice divide, the line of maximum ice sheet height H_m runs along the y -axis. (c) A vertical cut (xz -plane) through the ice sheet showing the height profile $H(x)$ (blue line). $H(x)$ is independent of the y -coordinate for $0 < y < 5L$, while it drops abruptly to zero at $y = 0$ and $y = 5L$. The z -axis marks the position of the ice divide. For clarity, the vertical dimensions are expanded compared to the horizontal dimensions. The density ρ_{ice} of ice is constant.

Two useful formulas

In this problem you can make use of the integral:

$$\int_0^1 \sqrt{1-x} \, dx = \frac{2}{3}$$

and the approximation $(1+x)^\alpha \approx 1 + \alpha x$, valid for $|\alpha x| \ll 1$.

The height profile of the ice sheet

On short time scales the glacier is an incompressible hydrostatic system with fixed height profile $H(x)$.

3.1	Write down an expression for the pressure $p(x, z)$ inside the ice sheet as a function of vertical height z above the ground and distance x from the ice divide. Neglect the atmospheric pressure.	0.3
-----	--	-----

Consider a given vertical slab of the ice sheet in equilibrium, covering a small horizontal base area $\Delta x \Delta y$ between x and $x + \Delta x$, see the red dashed lines in Fig. 3.1(c). The size of Δy does not matter. The net horizontal force component ΔF on the two vertical sides of the slab, arising from the difference in height on the center-side versus the coastal-side of the slab, is balanced by a friction force $\Delta F = S_b \Delta x \Delta y$ from the ground on the base area $\Delta x \Delta y$, where $S_b = 100 \text{ kPa}$.

3.2a	For a given value of x , show that in the limit $\Delta x \rightarrow 0$, $S_b = kH \, dH/dx$, and determine k .	0.9
3.2b	Determine an expression for the height profile $H(x)$ in terms of ρ_{ice} , g , L , S_b and distance x from the divide. The result will show that the maximum glacier height H_m scales with the half-width L as $H_m \propto L^{1/2}$.	0.8
3.2c	Determine the exponent γ with which the total volume V_{ice} of the ice sheet scales with the area A of the rectangular island, $V_{\text{ice}} \propto A^\gamma$.	0.5

A dynamical ice sheet

On longer time scale, the ice is a viscous incompressible fluid, which by gravity flows from the center part to the coast. In this model, the ice maintains its height profile $H(x)$ in a steady state, where accumulation of ice due to snowfall in the central region is balanced by melting at the coast. In addition to the ice sheet geometry of Fig. 3.1(b) and (c) make the following model assumptions:

- 1) Ice flows in the xz -plane away from the ice divide (the y -axis).
- 2) The accumulation rate c (m/year) in the central region is a constant.
- 3) Ice can only leave the glacier by melting near the coasts at $x = \pm L$.
- 4) The horizontal (x -)component $v_x(x) = dx/dt$ of the ice-flow velocity is independent of z .
- 5) The vertical (z -)component $v_z(z) = dz/dt$ of the ice-flow velocity is independent of x .

Consider only the central region $|x| \ll L$ close to the middle of the ice sheet, where height variations of the ice sheet are very small and can be neglected altogether, i.e. $H(x) \approx H_m$.

3.3	Use mass conservation to find an expression for the horizontal ice-flow velocity $v_x(x)$ in terms of c , x , and H_m .	0.6
-----	---	-----

From the assumption of incompressibility, i.e. the constant density ρ_{ice} of the ice, it follows that mass conservation implies the following restriction on the ice flow velocity components

$$\frac{dv_x}{dx} + \frac{dv_z}{dz} = 0.$$

3.4	Write down an expression for the z dependence of the vertical component $v_z(z)$ of the ice-flow velocity.	0.6
-----	--	-----

A small ice particle with the initial surface position (x_i, H_m) will, as time passes, flow as part of the ice sheet along a flow trajectory $z(x)$ in the vertical xz -plane.

3.5	Derive an expression for such a flow trajectory $z(x)$.	0.9
-----	--	-----

Age and climate indicators in the dynamical ice sheet

Based on the ice-flow velocity components $v_x(x)$ and $v_z(z)$, one can estimate the age $\tau(z)$ of the ice in a specific depth $H_m - z$ from the surface of the ice sheet.

3.6	Find an expression for the age $\tau(z)$ of the ice as a function of height z above ground, right at the ice divide $x = 0$.	1.0
-----	---	-----

An ice core drilled in the interior of the Greenlandic ice sheet will penetrate through layers of snow from the past, and the ice core can be analyzed to reveal past climate changes. One of the best indicators is the so-called $\delta^{18}\text{O}$, defined as

$$\delta^{18}\text{O} = \frac{R_{\text{ice}} - R_{\text{ref}}}{R_{\text{ref}}} 1000 \text{‰},$$

where $R = [^{18}\text{O}]/[^{16}\text{O}]$ denotes the relative abundance of the two stable isotopes ^{18}O and ^{16}O of oxygen. The reference R_{ref} is based on the isotopic composition of the oceans around Equator.

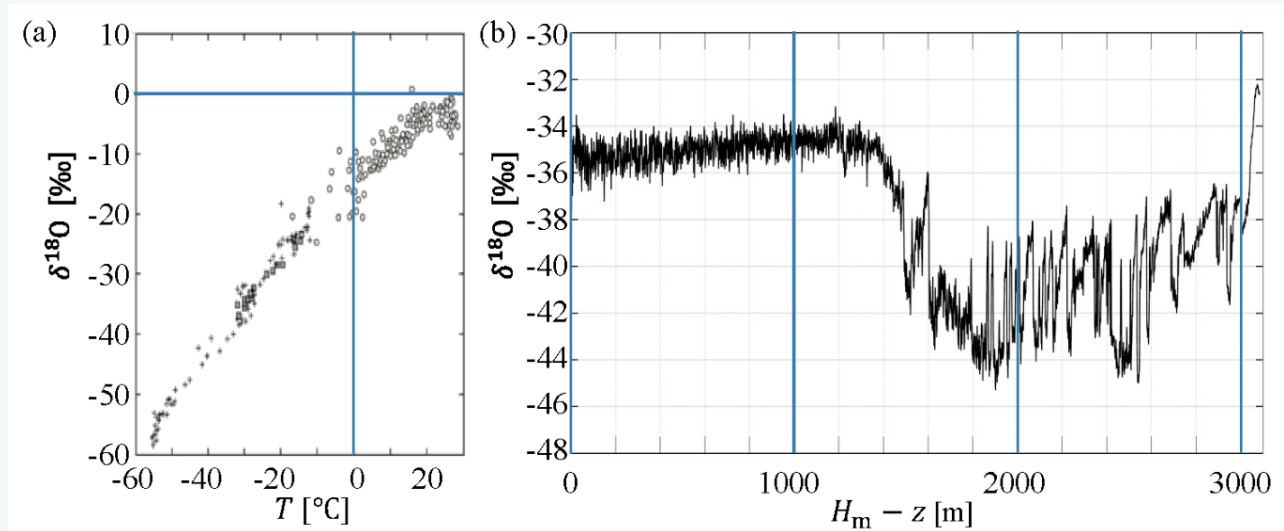


Figure 3.2 (a) Observed relationship between $\delta^{18}\text{O}$ in snows versus the mean annual surface temperature T . **(b)** Measurements of $\delta^{18}\text{O}$ versus depth $H_m - z$ from the surface, taken from an ice core drilled from surface to bedrock at a specific place along the Greenlandic ice divide where $H_m = 3060$ m.

Observations from the Greenland ice sheet show that $\delta^{18}\text{O}$ in the snow varies approximately linearly with temperature, Fig. 3.2(a). Assuming that this has always been the case, $\delta^{18}\text{O}$ retrieved from an ice core at depth $H_m - z$ leads to an estimate of the temperature T near Greenland at the age $\tau(z)$.

Measurements of $\delta^{18}\text{O}$ in a 3060 m long Greenlandic ice core show an abrupt change in $\delta^{18}\text{O}$ at a depth of 1492 m, Fig. 3.2(b), marking the end of the last ice age. The ice age began 120,000 years ago, corresponding to a depth of 3040 m, and the current interglacial age began 11,700 years ago, corresponding to a depth of 1492 m. Assume that these two periods can be described by two different accumulation rates, c_{ia} (ice age) and c_{ig} (interglacial age), respectively. You can assume H_m to be constant throughout these 120,000 years.

3.7a	Determine the accumulation rates c_{ia} and c_{ig} .	0.8
------	--	-----

3.7b	Use the data in Fig. 3.2 to find the temperature change at the transition from the ice age to the interglacial age.	0.2
------	---	-----

Sea level rise from melting of the Greenland ice sheet

A complete melting of the Greenlandic ice sheet will cause a sea level rise in the global ocean. As a crude estimate of this sea level rise, one may simply consider a uniform rise throughout a global ocean with constant area $A_O = 3.61 \times 10^{14} \text{ m}^2$.

3.8	Calculate the average global sea level rise, which would result from a complete melting of the Greenlandic ice sheet, given its present area of $A_G = 1.71 \times 10^{12} \text{ m}^2$ and $S_b = 100 \text{ kPa}$.	0.6
-----	---	-----

The massive Greenland ice sheet exerts a gravitational pull on the surrounding ocean. If the ice sheet melts, this local high tide is lost and the sea level will drop close to Greenland, an effect which partially counteracts the sea level rise calculated above.

To estimate the magnitude of this gravitational pull on the water, the Greenlandic ice sheet is now modeled as a point mass located at the ground level and having the total mass of the Greenlandic ice sheet. Copenhagen lies at a distance of 3500 km along the Earth surface from the center of the point mass. One may consider the Earth, without the point mass, to be spherically symmetric and having a global ocean spread out over the entire surface of the Earth of area $A_E = 5.10 \times 10^{14} \text{ m}^2$. All effects of rotation of the Earth may be neglected.

3.9	Within this model, determine the difference $h_{\text{CPH}} - h_{\text{OPP}}$ between sea levels in Copenhagen (h_{CPH}) and diametrically opposite to Greenland (h_{OPP}).	1.8
-----	---	-----

Data sheet: Table of physical parameters

Speed of light in vacuum	$c = 2.998 \times 10^8 \text{ m s}^{-1}$
Planck's constant over 2π	$\hbar = 1.055 \times 10^{-34} \text{ J s}$
Gravitational constant	$G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Gravitational acceleration	$g = 9.82 \text{ m s}^{-2}$
Elementary charge	$e = 1.602 \times 10^{-19} \text{ C}$
Dielectric permittivity of vacuum	$\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2 \text{ J}^{-1} \text{ m}^{-1}$
Electron mass	$m_e = 9.109 \times 10^{-31} \text{ kg}$
Avogadro constant	$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	$k_B = 1.381 \times 10^{-23} \text{ J K}^{-1}$
Stony meteorite, specific heat	$c_{\text{sm}} = 1.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Stony meteorite, thermal conductivity	$k_{\text{sm}} = 2.0 \text{ W m}^{-1} \text{ K}^{-1}$
Stony meteorite, density	$\rho_{\text{sm}} = 3.3 \times 10^3 \text{ kg m}^{-3}$
Stony meteorite, melting point	$T_{\text{sm}} = 1.7 \times 10^3 \text{ K}$
Stony meteorite, specific melting heat	$L_{\text{sm}} = 2.6 \times 10^5 \text{ J kg}^{-1}$
Silver, molar mass	$M_{\text{Ag}} = 1.079 \times 10^{-1} \text{ kg mol}^{-1}$
Silver, density	$\rho_{\text{Ag}} = 1.049 \times 10^4 \text{ kg m}^{-3}$
Silver, specific heat capacity	$c_{\text{Ag}} = 2.40 \times 10^2 \text{ J kg}^{-1} \text{ K}^{-1}$

Water, molar mass	$M_{\text{wa}} = 1.801 \times 10^{-2} \text{ kg mol}^{-1}$
Water, density	$\rho_{\text{wa}} = 0.998 \times 10^3 \text{ kg m}^{-3}$
Water, specific heat capacity	$c_{\text{wa}} = 4.181 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Water, heat of vaporization	$L_{\text{wa}} = 2.260 \times 10^6 \text{ J kg}^{-1}$
Water, boiling temperature	$T_{100} = 100^\circ\text{C} = 373.15 \text{ K}$
Ice, density of glacier	$\rho_{\text{ice}} = 0.917 \times 10^3 \text{ kg m}^{-3}$
Steam, specific heat capacity	$c_{\text{st}} = 2.080 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$
Earth, mass of the	$m_{\text{E}} = 5.97 \times 10^{24} \text{ kg}$
Earth, radius of the	$R_{\text{E}} = 6.38 \times 10^6 \text{ m}$
Sun, mass of the	$m_{\text{S}} = 1.99 \times 10^{30} \text{ kg}$
Sun, radius of the	$R_{\text{S}} = 6.96 \times 10^8 \text{ m}$
Average Sun-Earth distance	$a_{\text{E}} = 1.50 \times 10^{11} \text{ m}$

Official English text of the problem.

Answers

- **3.1:** $p(x, z) = \rho_{\text{ice}} g [H(x) - z]$, for $0 \leq z \leq H(x)$.
- **3.2a:** $S_b = -\rho_{\text{ice}} g H \frac{dH}{dx}$, so $k = -\rho_{\text{ice}} g$ on the $x > 0$ half.
- **3.2b:** $H(x) = \sqrt{\frac{2S_b}{\rho_{\text{ice}} g} (L - |x|)}$, with $H_m = \sqrt{\frac{2S_b L}{\rho_{\text{ice}} g}}$.
- **3.2c:** $\gamma = \frac{5}{4}$; specifically, $V_{\text{ice}} = \frac{2}{3} A H_m$.
- **3.3:** $v_x(x) = \frac{c}{H_m} x$.
- **3.4:** $v_z(z) = -\frac{c}{H_m} z$.
- **3.5:** $z(x) = H_m \frac{x_i}{x}$ for $x_i \neq 0$. For $x_i = 0$, the trajectory is vertical along $x = 0$.
- **3.6:** $\tau(z) = \frac{H_m}{c} \ln \frac{H_m}{z}$.
- **3.7a:** $c_{\text{ia}} = 0.123 \text{ m/year}$, $c_{\text{ig}} = 0.175 \text{ m/year}$.
- **3.7b:** $\Delta T \approx +10 \text{ K}$, approximately 10°C of warming.
- **3.8:** $\Delta h_{\text{mean}} \approx 8.8 \text{ m}$.
- **3.9:** $h_{\text{CPH}} - h_{\text{OPP}} \approx 4.5 \text{ m}$, with Copenhagen higher.

Solution

3.1

Hydrostatic balance in the vertical direction gives

$$\frac{\partial p}{\partial z} = -\rho_{\text{ice}}g.$$

The pressure at the surface is zero when atmospheric pressure is neglected:

$$p(x, H(x)) = 0.$$

Integrating downward from the surface therefore gives

$$\boxed{p(x, z) = \rho_{\text{ice}}g[H(x) - z]}.$$

3.2a

Consider the right-hand half of the ice sheet, $x > 0$. The pressure force on a vertical face of height $H(x)$ and width Δy is

$$F(x) = \Delta y \int_0^{H(x)} \rho_{\text{ice}}g[H(x) - z] dz = \frac{1}{2}\rho_{\text{ice}}gH(x)^2\Delta y.$$

The force on the center-side face acts toward increasing x , while that on the coastal-side face acts toward decreasing x . Their resultant is

$$\Delta F = \frac{1}{2}\rho_{\text{ice}}g [H(x)^2 - H(x + \Delta x)^2] \Delta y.$$

To first order in Δx ,

$$\Delta F = -\rho_{\text{ice}}gH(x)\frac{dH}{dx} \Delta x \Delta y.$$

Balancing this outward force against the basal friction $S_b\Delta x\Delta y$ yields

$$\boxed{S_b = -\rho_{\text{ice}}gH\frac{dH}{dx}}, \quad \boxed{k = -\rho_{\text{ice}}g}.$$

The derivative is negative on this half, so $S_b > 0$, as required. On either half, the corresponding magnitude relation is $S_b = \rho_{\text{ice}}gH|dH/dx|$.

3.2b

On $0 \leq x \leq L$, the result above can be written

$$\frac{d(H^2)}{dx} = -\frac{2S_b}{\rho_{\text{ice}}g}.$$

Using the coastal boundary condition $H(L) = 0$,

$$H(x)^2 = \frac{2S_b}{\rho_{\text{ice}}g}(L - x).$$

Reflection about the ice divide gives the complete profile:

$$\boxed{H(x) = \sqrt{\frac{2S_b}{\rho_{\text{ice}}g}(L - |x|)}}, \quad -L \leq x \leq L.$$

At the divide,

$$\boxed{H_m = \sqrt{\frac{2S_bL}{\rho_{\text{ice}}g}}},$$

so $H_m \propto L^{1/2}$ when the basal friction, density, and gravitational acceleration are fixed.

3.2c

The island area is

$$A = (2L)(5L) = 10L^2.$$

Since the profile is independent of y , the ice volume is

$$V_{\text{ice}} = 5L \int_{-L}^L H(x) dx = 10LH_m \int_0^L \sqrt{1 - \frac{x}{L}} dx.$$

With $u = x/L$, the supplied integral gives

$$V_{\text{ice}} = 10L^2H_m \int_0^1 \sqrt{1-u} du = \frac{20}{3}L^2H_m = \boxed{\frac{2}{3}AH_m}.$$

Thus

$$V_{\text{ice}} \propto L^{2+1/2} = L^{5/2} \propto A^{5/4},$$

and

$$\boxed{\gamma = \frac{5}{4}}.$$

3.3

Take a central control volume between the divide and a position $x > 0$, with width Δy . In steady state, the accumulation over its surface must equal its horizontal outflow.

The accumulated mass per unit time is

$$\dot{m}_{\text{in}} = \rho_{\text{ice}}cx\Delta y.$$

There is no horizontal flow through the divide by symmetry. Since v_x is independent of height, the outflow through the face at x is

$$\dot{m}_{\text{out}} = \rho_{\text{ice}}v_x(x)H_m\Delta y.$$

Equating these gives

$$\boxed{v_x(x) = \frac{c}{H_m}x}.$$

The same signed expression applies for $x < 0$, where the flow is toward negative x . This result uses the central-region approximation $H(x) \approx H_m$.

3.4

Incompressibility requires

$$\frac{dv_z}{dz} = -\frac{dv_x}{dx} = -\frac{c}{H_m}.$$

Integration gives

$$v_z(z) = -\frac{c}{H_m}z + C.$$

There is no flow through the ground, so $v_z(0) = 0$, implying $C = 0$. Therefore,

$$\boxed{v_z(z) = -\frac{c}{H_m}z}.$$

In particular, $v_z(H_m) = -c$: downward motion at the surface balances the addition of new ice.

3.5

For a particle with $x_i \neq 0$, its trajectory satisfies

$$\frac{dz}{dx} = \frac{dz/dt}{dx/dt} = \frac{v_z}{v_x} = -\frac{z}{x}.$$

Separating variables and applying the initial condition $z(x_i) = H_m$,

$$\int_{H_m}^z \frac{dz'}{z'} = - \int_{x_i}^x \frac{dx'}{x'},$$

so

$$\ln \frac{z}{H_m} = - \ln \left| \frac{x}{x_i} \right|.$$

The particle remains on its original side of the divide; hence $x/x_i > 0$, and

$$\boxed{xz = x_i H_m}, \quad \boxed{z(x) = H_m \frac{x_i}{x}}.$$

The trajectory is a branch of a rectangular hyperbola, along which the particle moves outward and downward. Equivalently,

$$x(t) = x_i e^{ct/H_m}, \quad z(t) = H_m e^{-ct/H_m}.$$

For $x_i = 0$, the particle remains at $x = 0$ and descends vertically according to the same expression for $z(t)$. These trajectories apply while the particle remains in the central region.

3.6

At the ice divide, the horizontal velocity vanishes. The age is the elapsed time since the particle was deposited at $z = H_m$:

$$\tau(z) = \int_{H_m}^z \frac{dz'}{v_z(z')} = -\frac{H_m}{c} \int_{H_m}^z \frac{dz'}{z'}.$$

Therefore,

$$\boxed{\tau(z) = \frac{H_m}{c} \ln \frac{H_m}{z}}.$$

This is zero at the surface and increases with depth. In this idealized model, it diverges as $z \rightarrow 0$, because the vertical velocity approaches zero at the bed.

3.7a

For this ice core, $H_m = 3060$ m. The present heights above the bed of the two dated layers are

$$\begin{aligned} z_1 &= 3060 - 1492 = 1568 \text{ m}, \\ z_2 &= 3060 - 3040 = 20 \text{ m}. \end{aligned}$$

The transition layer has experienced only the 11 700 years of interglacial flow. Thus

$$\ln \frac{3060}{1568} = \frac{c_{ig}(11\,700 \text{ year})}{3060 \text{ m}},$$

giving

$$c_{ig} = \frac{3060}{11\,700} \ln \frac{3060}{1568} = 0.175 \text{ m/year}.$$

For the older layer, the accumulation rate changed during its descent. With constant H_m ,

$$\frac{d \ln z}{dt} = -\frac{c(t)}{H_m}.$$

It experienced ice-age flow for

$$120\,000 - 11\,700 = 108\,300 \text{ years},$$

followed by interglacial flow for 11 700 years. Consequently,

$$\ln \frac{3060}{20} = \frac{c_{ia}(108\,300 \text{ year}) + c_{ig}(11\,700 \text{ year})}{3060 \text{ m}}.$$

Subtracting the equation for the transition layer eliminates the interglacial contribution:

$$\ln \frac{1568}{20} = \frac{c_{ia}(108\,300 \text{ year})}{3060 \text{ m}}.$$

Hence

$$c_{ia} = \frac{3060}{108\,300} \ln \frac{1568}{20} = 0.123 \text{ m/year}.$$

Accounting for the interglacial compression of the older ice is essential.

3.7b

Figure 3.2(b) gives representative isotope values of approximately

$$\delta_{ia}^{18}\text{O} \approx -42 \text{ ‰}, \quad \delta_{ig}^{18}\text{O} \approx -35 \text{ ‰}.$$

Thus the transition corresponds to an increase of roughly 7 ‰.

The approximately linear trend in Fig. 3.2(a), in the relevant temperature range, has slope about

$$\frac{\Delta(\delta^{18}\text{O})}{\Delta T} \approx 0.7 \text{ ‰ K}^{-1}.$$

Therefore,

$$\Delta T \approx \frac{7 \text{ ‰}}{0.7 \text{ ‰ K}^{-1}} \approx 10 \text{ K}.$$

The change is a warming:

$$T_{ig} - T_{ia} \approx +10 \text{ K}.$$

The precision is limited by the graphical readings.

3.8

Use the island geometry and height profile to determine the present model volume. Since $A_G = 10L^2$,

$$L = \sqrt{\frac{A_G}{10}} = \sqrt{\frac{1.71 \times 10^{12}}{10}} = 4.14 \times 10^5 \text{ m.}$$

The maximum height predicted by the model is

$$H_m = \sqrt{\frac{2S_b L}{\rho_{\text{ice}} g}} = \sqrt{\frac{2(1.00 \times 10^5)(4.1352 \times 10^5)}{(917)(9.82)}} = 3.03 \times 10^3 \text{ m.}$$

Using part 3.2c,

$$V_{\text{ice}} = \frac{2}{3} A_G H_m = 3.45 \times 10^{15} \text{ m}^3.$$

Melting conserves mass, so the resulting water volume is

$$V_{\text{water}} = \frac{\rho_{\text{ice}}}{\rho_{\text{wa}}} V_{\text{ice}}.$$

Spreading this water uniformly over the constant ocean area gives

$$\Delta h_{\text{mean}} = \frac{V_{\text{water}}}{A_O} = \frac{\rho_{\text{ice}}}{\rho_{\text{wa}}} \frac{2A_G H_m}{3A_O}.$$

Numerically,

$$\Delta h_{\text{mean}} = \frac{917}{998} \frac{3.4548 \times 10^{15}}{3.61 \times 10^{14}} = 8.79 \text{ m.}$$

Thus

$$\boxed{\Delta h_{\text{mean}} \approx 8.8 \text{ m}.}$$

3.9

An ocean at rest has an equipotential free surface. We therefore compare the gravitational potentials at Copenhagen and at the point opposite Greenland.

The point mass representing the ice sheet is

$$M = \rho_{\text{ice}} V_{\text{ice}} = (917)(3.4548 \times 10^{15}) = 3.17 \times 10^{18} \text{ kg.}$$

Let R denote Earth's radius and h a small sea-surface displacement from a reference sphere. Earth's contribution to the gravitational potential per unit mass is

$$\Phi_E(R + h) = -\frac{Gm_E}{R + h} \approx -\frac{Gm_E}{R} + gh,$$

where $h \ll R$. The ice mass contributes

$$\Phi_{\text{ice}} = -\frac{GM}{d},$$

with d the straight-line distance to the ice mass.

To first order, d can be evaluated on the reference sphere: changes of a few metres are negligible compared with distances of millions of metres. Equal surface potentials therefore require

$$gh_{\text{CPH}} - \frac{GM}{d_{\text{CPH}}} = gh_{\text{OPP}} - \frac{GM}{d_{\text{OPP}}}.$$

Thus

$$h_{\text{CPH}} - h_{\text{OPP}} = \frac{GM}{g} \left(\frac{1}{d_{\text{CPH}}} - \frac{1}{d_{\text{OPP}}} \right).$$

The given 3500 km is an arc length s , so the central angle and chord distance are

$$\theta = \frac{s}{R}, \quad d_{\text{CPH}} = 2R \sin \frac{\theta}{2} = 2R \sin \frac{s}{2R}.$$

At the opposite point,

$$d_{\text{OPP}} = 2R.$$

Consequently,

$$\boxed{h_{\text{CPH}} - h_{\text{OPP}} = \frac{GM}{2gR} \left[\frac{1}{\sin(s/(2R))} - 1 \right]}.$$

The stated Earth area gives $R = \sqrt{A_E/(4\pi)} \approx 6.37 \times 10^6$ m, consistent with the supplied radius. Using $R = 6.38 \times 10^6$ m,

$$\theta = 0.5486, \quad d_{\text{CPH}} = 3.456 \times 10^6 \text{ m}, \quad d_{\text{OPP}} = 1.276 \times 10^7 \text{ m}.$$

Hence

$$h_{\text{CPH}} - h_{\text{OPP}} = \frac{(6.67 \times 10^{-11})(3.168 \times 10^{18})}{9.82} \left(\frac{1}{3.456 \times 10^6} - \frac{1}{1.276 \times 10^7} \right) = 4.54 \text{ m}.$$

Therefore,

$$\boxed{h_{\text{CPH}} - h_{\text{OPP}} \approx 4.5 \text{ m}}.$$

The positive sign means that Greenland's gravitational attraction raises the equilibrium ocean surface at Copenhagen relative to the opposite side of Earth. Any common offset needed to conserve ocean volume cancels from this difference.

The solution was checked against the official solution.