

IPhO 2012

Theory problems and solutions · Tartu, Estonia · 3 theory problems with complete solutions

eulersolve.org/ipho/2012/

Problem 1: Focus on sketches

Problem T1. Focus on sketches (13 points)

Part A. Ballistics (4.5 points)

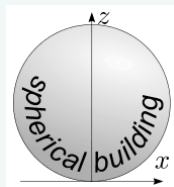
A ball, thrown with an initial speed v_0 , moves in a homogeneous gravitational field in the x - z plane, where the x -axis is horizontal, and the z -axis is vertical and antiparallel to the free fall acceleration g . Neglect the effect of air drag.

i. (0.8 pts) By adjusting the launching angle for a ball thrown with a fixed initial speed v_0 from the origin, targets can be hit within the region given by

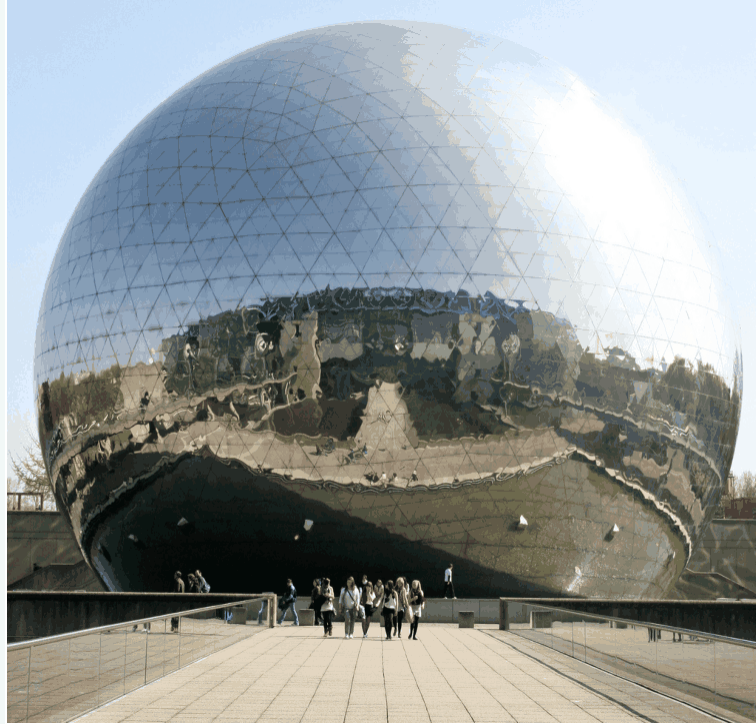
$$z \leq z_0 - kx^2.$$

You can use this fact without proving it. Find the constants z_0 and k .

ii. (1.2 pts) The launching point can now be freely selected on the ground level $z = 0$, and the launching angle can be adjusted as needed. The aim is to hit the topmost point of a spherical building of radius R (see fig.) with the minimal initial speed v_0 . Bouncing off the roof prior to hitting the target is not allowed. Sketch qualitatively the shape of the optimal trajectory of the ball (use the designated box on the answer sheet). Note that the marks are given only for the sketch.



iii. (2.5 pts) What is the minimal launching speed $v_{\min}$ needed to hit the topmost point of a spherical building of radius R ?



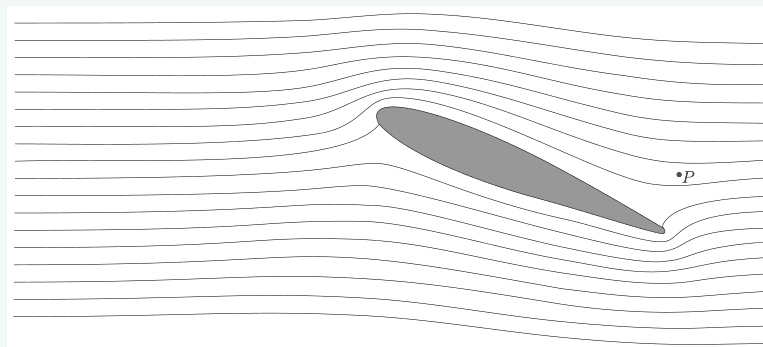
La Géode, Parc de la Villette, Paris. Photo: katchooo/flickr.com

Part B. Air flow around a wing (4 points)

For this part of the problem, the following information may be useful. For a flow of liquid or gas in a tube along a streamline, $p + \rho gh + \frac{1}{2}\rho v^2 = \text{const.}$, assuming that the velocity v is much less than the speed of sound. Here ρ is the density, h is the height, g is free fall acceleration and p is hydrostatic pressure. Streamlines are defined as the trajectories of fluid particles (assuming that the flow pattern is stationary). Note that the term $\frac{1}{2}\rho v^2$ is called the dynamic pressure.

In the fig. shown below, a cross-section of an aircraft wing is depicted together with streamlines of the air flow around the wing, as seen in the wing's reference frame. Assume that (a) the air flow is purely two-dimensional (i.e. that the velocity vectors of air lie in the plane of the figure); (b) the streamline pattern is independent of the aircraft speed; (c) there is no wind; (d) the dynamic pressure is much smaller than the atmospheric pressure, $p_0 = 1.0 \times 10^5 \text{ Pa}$.

You can use a ruler to take measurements from the fig. on the answer sheet.



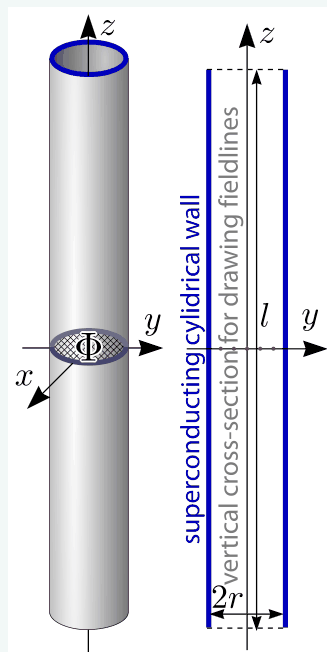
- i. (0.8 pts) If the aircraft's ground speed is $v_0 = 100 \text{ m/s}$, what is the speed of the air, v_P , at the point P (marked in the fig.) with respect to the ground?

ii. (1.2 pts) In the case of high relative humidity, as the ground speed of the aircraft increases over a critical value v_{crit} , a stream of water droplets is created behind the wing. The droplets emerge at a certain point Q . Mark the point Q in the fig. on the answer sheet. Explain qualitatively (using formulae and as little text as possible) how you determined the position of Q .

iii. (2.0 pts) Estimate the critical speed v_{crit} using the following data: relative humidity of the air is $r = 90\%$, specific heat capacity of air at constant pressure $c_p = 1.00 \times 10^3 \text{ J/kg} \cdot \text{K}$, pressure of saturated water vapour: $p_{sa} = 2.31 \text{ kPa}$ at the temperature of the unperturbed air $T_a = 293 \text{ K}$ and $p_{sb} = 2.46 \text{ kPa}$ at $T_b = 294 \text{ K}$. Depending on your approximations, you may also need the specific heat capacity of air at constant volume $c_V = 0.717 \times 10^3 \text{ J/kg} \cdot \text{K}$. Note that the relative humidity is defined as the ratio of the vapour pressure to the saturated vapour pressure at the given temperature. Saturated vapour pressure is defined as the vapour pressure by which vapour is in equilibrium with the liquid.

Part C. Magnetic straws (4.5 points)

Consider a cylindrical tube made of a superconducting material. The length of the tube is l and the inner radius is r with $l \gg r$. The centre of the tube coincides with the origin, and its axis coincides with the z -axis.

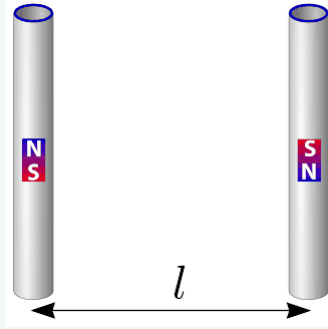


There is a magnetic flux Φ through the central cross-section of the tube, $z = 0$, $x^2 + y^2 < r^2$. A superconductor is a material which expels any magnetic field (the field is zero inside the material).

i. (0.8 pts) Sketch five such magnetic field lines, which pass through the five red dots marked on the axial cross-section of the tube, on the designated diagram on the answer sheet.

ii. (1.2 pts) Find the tension force T along the z -axis in the middle of the tube (i.e. the force by which two halves of the tube, $z > 0$ and $z < 0$, interact with each other).

iii. (2.5 pts) Consider another tube, identical and parallel to the first one.



The second tube has the same magnetic field but in the opposite direction and its centre is placed at $y = l, x = z = 0$ (so that the tubes form opposite sides of a square). Determine the magnetic interaction force F between the two tubes.

Official English text of the problem.

Answers

- **A.i.** $z_0 = \frac{v_0^2}{2g}, \quad k = \frac{g}{2v_0^2}.$

- **A.ii.** The optimal parabola rises from the ground, grazes the upper side of the building tangentially, rises slightly farther, and then descends onto the topmost point. For approach from the left, its equation is

$$z = 2R - \frac{x}{\sqrt{3}} - \frac{4x^2}{3R}.$$

- **A.iii.** $v_{\min} = 3\sqrt{\frac{gR}{2}}.$

- **B.i.** $v_P \approx 20 \text{ m/s}$, approximately in the aircraft's direction of motion.

- **B.ii.** Place Q immediately above the rounded leading edge, where the streamlines on the upper side are closest together: the speed is greatest and the temperature lowest.

- **B.iii.** $v_{\text{crit}} \approx 35 \text{ m/s}$, using a maximum local speed about twice the aircraft speed and including the change in vapour partial pressure. Neglecting that pressure change gives approximately 32 m/s .

- **C.i.** Draw nearly parallel axial lines inside the bore. The four off-axis lines spread out beyond the ends and return around the outside, without entering the superconducting wall. The central axial line remains straight.

- **C.ii.** $T \simeq \frac{\Phi^2}{2\mu_0\pi r^2},$ attracting the two halves.

- **C.iii.** The tubes attract with

$$F \simeq \frac{\Phi^2}{4\pi\mu_0 l^2} \left(2 - \frac{1}{\sqrt{2}} \right).$$

The magnetic results use the long, narrow tube approximation.

Solution

A.i.

For launch angle θ , eliminating time from projectile motion gives

$$z = x \tan \theta - \frac{gx^2}{2v_0^2 \cos^2 \theta}.$$

Writing $u = \tan \theta$,

$$z = xu - \frac{gx^2}{2v_0^2}(1 + u^2).$$

At fixed $x \neq 0$, this quadratic is maximized when

$$u = \frac{v_0^2}{gx}.$$

Substitution gives the envelope

$$z_{\text{env}} = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

At $x = 0$, its height is also the maximum height of a vertical throw. Therefore,

$$\boxed{z_0 = \frac{v_0^2}{2g}}, \quad \boxed{k = \frac{g}{2v_0^2}}.$$

A.ii.

Put the centre of the building at $(0, R)$, so its top is $(0, 2R)$. Choose a launch point to the left; reflection about the z -axis gives an equally good trajectory.

The required sketch is a downward-opening parabola with these features:

1. It starts on the ground to the left of the building and rises steeply.
2. It grazes the upper-left surface tangentially, with no change of velocity from an impact.
3. It continues upward to an apex slightly above and to the left of the building's top.
4. It then descends to the topmost point.

The exact landmarks, derived below, are:

Landmark	x/R	z/R
Launch	$-\frac{\sqrt{3} + 3\sqrt{11}}{8} \approx -1.460$	0
Tangential grazing point	$-\frac{\sqrt{3}}{2} \approx -0.866$	$\frac{3}{2}$
Apex	$-\frac{\sqrt{3}}{8} \approx -0.217$	$\frac{33}{16} = 2.0625$
Target	0	2

A.iii.

Let v_t be the speed when the ball reaches the target. Conservation of mechanical energy gives

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_t^2 + mg(2R),$$

so

$$v_0^2 = v_t^2 + 4gR.$$

Thus minimizing the launching speed is equivalent to minimizing v_t .

Reverse the motion. We now launch a projectile from $(0, 2R)$ with speed v_t . By A.i, every such trajectory lies below the envelope

$$z_{\text{env}}(x) = 2R + \frac{v_t^2}{2g} - \frac{gx^2}{2v_t^2}.$$

A trajectory approaching the target from the left must clear the upper surface

$$z_s(x) = R + \sqrt{R^2 - x^2}, \quad -R \leq x \leq 0.$$

Consequently, the envelope must also clear that surface.

Define

$$\lambda = \frac{v_t^2}{gR}, \quad c = \sqrt{1 - \frac{x^2}{R^2}}, \quad 0 \leq c \leq 1.$$

The difference between envelope and roof is

$$\begin{aligned} \frac{z_{\text{env}} - z_s}{R} &= 1 + \frac{\lambda}{2} - \frac{1 - c^2}{2\lambda} - c \\ &= \frac{(c - \lambda)^2 + 2\lambda - 1}{2\lambda}. \end{aligned}$$

If $0 < \lambda < 1/2$, choosing $c = \lambda$ makes this negative. No trajectory can then clear the building. Hence

$$v_t^2 \geq \frac{gR}{2}.$$

At equality, the envelope touches the roof at $c = 1/2$, namely

$$(x, z) = \left(-\frac{\sqrt{3}}{2}R, \frac{3}{2}R \right).$$

We must also show that one actual trajectory attains the bound. Launching the reversed projectile 30° above the horizontal towards the left gives, in the original coordinates,

$$\boxed{z(x) = 2R - \frac{x}{\sqrt{3}} - \frac{4x^2}{3R}}.$$

This parabola passes through the target and is tangent to the sphere at the point above. To verify that it never penetrates the building, set $t = -x/R \geq 0$. Direct substitution gives

$$\frac{x^2 + (z - R)^2 - R^2}{R^2} = \frac{16}{9}t \left(t + \frac{\sqrt{3}}{2} \right) \left(t - \frac{\sqrt{3}}{2} \right)^2 \geq 0.$$

Thus the entire relevant trajectory is outside the sphere, with tangential contact before the target. No bounce is needed.

Finally,

$$v_{\min}^2 = \frac{gR}{2} + 4gR = \frac{9}{2}gR,$$

and therefore

$$v_{\min} = 3\sqrt{\frac{gR}{2}}.$$

The intercept and apex of this parabola give the landmarks listed in A.ii.

B.i.

Let u denote air speed in the wing's frame. Because the pressure changes are small compared with atmospheric pressure, we first treat the density as constant.

For a narrow stream tube of unit span, continuity gives

$$u d = v_0 d_{\infty},$$

where d is the spacing between adjacent streamlines, measured perpendicular to the flow, and d_{∞} is their undisturbed upstream spacing. Thus

$$u = v_0 \frac{d_{\infty}}{d}.$$

From the supplied figure, the spacing near P is approximately 1.25 times the upstream spacing:

$$u_P \approx 0.80v_0 \approx 80 \text{ m/s}.$$

Take the undisturbed flow in the wing's frame to point rightwards. The aircraft then moves leftwards relative to the ground, so the velocity transformation is

$$\mathbf{v}_{P,\text{ground}} = \mathbf{u}_P - v_0 \hat{\mathbf{x}}.$$

The streamlines at P are nearly horizontal. Therefore,

$$v_P \approx |80 - 100| = 20 \text{ m/s},$$

approximately leftwards, in the aircraft's direction of motion.

If the small local inclination is retained, the corresponding expression is

$$v_P = \sqrt{u_P^2 + v_0^2 - 2u_P v_0 \cos \theta_P}.$$

B.ii.

Neglect gravitational potential differences across the wing. Continuity and Bernoulli's equation give

$$u \propto \frac{1}{d}, \quad p = p_0 + \frac{1}{2}\rho(v_0^2 - u^2).$$

For approximately adiabatic flow, conservation of stagnation enthalpy gives

$$c_p T + \frac{u^2}{2} = c_p T_a + \frac{v_0^2}{2}.$$

Hence

$$d \text{ smallest} \implies u \text{ largest} \implies p, T \text{ smallest}.$$

The saturation vapour pressure falls rapidly with temperature, so the relative humidity reaches unity first there.

Place Q just above the rounded leading edge, in the region of minimum streamline spacing on the upper side of the wing. The emerging droplets are subsequently carried downstream.

B.iii.

We model the air as an ideal gas undergoing approximately adiabatic, reversible expansion before condensation begins. Its specific gas constant is

$$R_{\text{air}} = c_p - c_V = 283 \text{ J kg}^{-1} \text{ K}^{-1}.$$

Let the cooling at Q be $\Delta T = T_a - T_Q > 0$. The supplied saturation-pressure data give the local slope

$$s = \frac{p_{sb} - p_{sa}}{T_b - T_a} = \frac{2460 - 2310}{294 - 293} = 150 \text{ Pa/K}.$$

Using a linear extrapolation,

$$p_s(T_Q) \approx p_{sa} - s\Delta T.$$

Before condensation, the water-vapour mole fraction is unchanged, so its partial pressure scales with total pressure:

$$p_{v,Q} = rp_{sa} \frac{p_Q}{p_0}.$$

The ideal-gas adiabatic relation gives

$$\frac{p_Q}{p_0} = \left(\frac{T_Q}{T_a} \right)^{c_p/R_{\text{air}}} \approx 1 - \frac{c_p}{R_{\text{air}}T_a} \Delta T.$$

At the onset of condensation, $p_{v,Q} = p_s(T_Q)$. Therefore,

$$rp_{sa} \left(1 - \frac{c_p \Delta T}{R_{\text{air}}T_a} \right) = p_{sa} - s\Delta T,$$

which yields

$$\Delta T = \frac{(1-r)p_{sa}}{s - rp_{sa}c_p/(R_{\text{air}}T_a)}.$$

Numerically,

$$\Delta T = \frac{0.10(2310)}{150 - \frac{0.90(2310)(1000)}{283(293)}} \approx 1.85 \text{ K}.$$

Thus condensation begins at approximately 291.2 K.

Let

$$\alpha = \frac{u_Q}{v_0} \approx \frac{d_\infty}{d_Q}.$$

The narrowest spacing in the drawing is approximately half the undisturbed spacing, giving $\alpha \approx 2$. The energy equation then gives

$$c_p \Delta T = \frac{1}{2}(u_Q^2 - v_{\text{crit}}^2) = \frac{1}{2}(\alpha^2 - 1)v_{\text{crit}}^2.$$

Consequently,

$$v_{\text{crit}} = \sqrt{\frac{2c_p \Delta T}{\alpha^2 - 1}} \approx \sqrt{\frac{2(1000)(1.85)}{4 - 1}} \approx 35 \text{ m/s}.$$

The corresponding pressure decrease is about 2.2%, consistent with treating density changes as small for the graphical velocity estimate.

A simpler approximation holds the vapour partial pressure constant. It gives

$$\Delta T \approx \frac{(1 - r)p_{sa}}{s} = 1.54 \text{ K}, \quad v_{\text{crit}} \approx 32 \text{ m/s}.$$

Both estimates place the threshold at a few tens of metres per second; the pressure-corrected estimate is about 35 m/s.

C.i.

Choose the flux direction to be $+z$ inside the tube; reversing every arrow gives the opposite convention.

Far from the ends, the field in the bore is nearly uniform and axial. Draw five nearly straight, parallel lines through the marked dots.

At the openings:

- The two off-axis lines on each side fan outward beyond the upper end, curve down around the exterior, and return through the lower opening.
- The pattern is symmetric about the axis and about the middle plane.
- Lines passing closer to the axis form larger exterior loops.
- No line crosses the superconducting wall; the field is tangent to its surfaces.

For the dot on the axis, axial symmetry requires a straight line along the z -axis. It extends to infinity in both directions; it has no finite endpoint. Arrows on the other lines point upward inside the bore and downward along their exterior return portions.

C.ii.

Write the bore area as

$$A = \pi r^2.$$

Since $l \gg r$, the central field is approximately uniform:

$$B = \frac{\Phi}{A}.$$

The Maxwell stress along a magnetic field is tensile, with magnitude

$$\sigma_{\parallel} = \frac{B^2}{2\mu_0}.$$

Integrating this stress over the central bore gives the attraction between the halves:

$$T \simeq \frac{B^2 A}{2\mu_0} = \boxed{\frac{\Phi^2}{2\mu_0 \pi r^2}}.$$

The weak exterior return field produces a small correction in the long-tube limit.

The result also follows from magnetic energy. At fixed trapped flux and cross-section,

$$U \simeq \frac{B^2}{2\mu_0} Al = \frac{\Phi^2 l}{2\mu_0 A}.$$

Increasing the tube's length requires work $T dl = dU$, giving the same tension.

C.iii.

At distances large compared with the tube radius, the flux emerging from an end behaves as the field of an effective magnetic pole. Its strength is

$$q_m = \frac{\Phi}{\mu_0},$$

because its radial field is

$$B(s) = \frac{\mu_0 q_m}{4\pi s^2} = \frac{\Phi}{4\pi s^2}.$$

These are equivalent sources for the exterior field; no actual magnetic monopoles are required. The two ends have opposite signs.

The force between two such effective poles separated by s has magnitude

$$\frac{\mu_0 q_m^2}{4\pi s^2} = \frac{\Phi^2}{4\pi\mu_0 s^2}.$$

Set

$$K = \frac{\Phi^2}{4\pi\mu_0}.$$

For the antiparallel tubes, all four end-to-end interactions must be included:

End pairs	Separation	Interaction	Combined component towards the other tube
Upper–upper and lower–lower	l	Attraction	$\frac{2K}{l^2}$
Upper–lower and lower–upper	$\sqrt{2}l$	Repulsion	$-\frac{K}{\sqrt{2}l^2}$

For each diagonal pair, the force magnitude is $K/(2l^2)$, and its transverse component is this magnitude multiplied by $\cos 45^\circ$. Their vertical components cancel.

Therefore,

$$F \simeq \frac{\Phi^2}{4\pi\mu_0 l^2} \left(2 - \frac{1}{\sqrt{2}} \right).$$

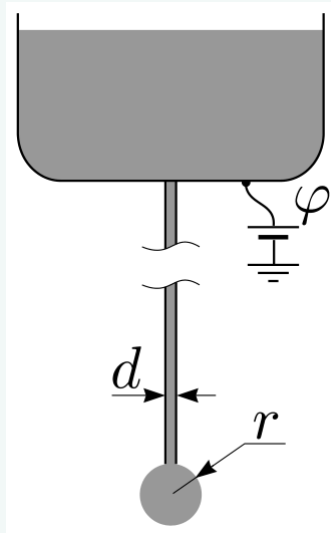
The force is attractive: it acts in the $+y$ direction on the first tube and in the $-y$ direction on the second.

The solution was checked against the official solution.

Problem 2: Kelvin water dropper

Problem T2. Kelvin water dropper (8 points)

The following facts about the surface tension may turn out to be useful for this problem. For the molecules of a liquid, the positions at the liquid-air interface are less favourable as compared with the positions in the bulk of the liquid. This interface is described by the so-called surface energy, $U = \sigma S$, where S is the surface area of the interface and σ is the surface tension coefficient of the liquid. Moreover, two fragments of the liquid surface pull each other with a force $F = \sigma l$, where l is the length of a straight line separating the fragments.



A long metallic pipe with internal diameter d is pointing directly downwards. Water is slowly dripping from a nozzle at its lower end, see fig. Water can be considered to be electrically conducting; its surface tension is σ and its density is ρ . A droplet of radius r hangs below the nozzle. The radius grows slowly in time until the droplet separates from the nozzle due to the free fall acceleration g . Always assume that $d \ll r$.

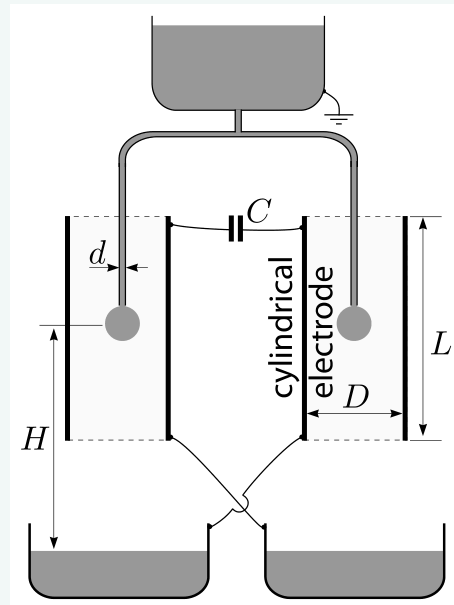
Part A. Single pipe (4 points)

- i. (1.2 pts) Find the radius $r_{\max}$ of a drop just before it separates from the nozzle.
- ii. (1.2 pts) Relative to the far-away surroundings, the pipe's electrostatic potential is φ . Find the charge Q of a drop when its radius is r .
- iii. (1.6 pts) Consider the situation in which r is kept constant and φ is slowly increased. The droplet becomes unstable and breaks into pieces if the hydrostatic pressure inside the droplet becomes smaller than the atmospheric pressure. Find the critical potential $\varphi_{\max}$ at which this will happen.

Part B. Two pipes (4 points)

An apparatus called the "Kelvin water dropper" consists of two pipes, each identical to the one described in Part A, connected via a T-junction, see fig. The ends of both pipes are at the centres of two cylindrical electrodes (with height L and diameter D with $L \gg D \gg r$). For both tubes, the dripping rate is n droplets per unit time. Droplets fall from height H into conductive bowls underneath the nozzles, cross-connected to the electrodes as shown in the diagram. The

electrodes are connected via a capacitance C . There is no net charge on the system of bowls and electrodes. Note that the top water container is earthed as shown. The first droplet to fall will have some microscopic charge which will cause an imbalance between the two sides and a small charge separation across the capacitor.



- i. (1.2 pts) Express the absolute value of the charge Q_0 of the drops as they separate from the tubes, and at the instant when the capacitor's charge is q . Express Q_0 in terms of $r_{\max}$ (from Part A-i) and neglect the effect described in Part A-iii.
- ii. (1.5 pts) Find the dependence of q on time t by approximating it with a continuous function $q(t)$ and assuming that $q(0) = q_0$.
- iii. (1.3 pts) The dropper's functioning can be hindered by the effect shown in Part A-iii. In addition, a limit $U_{\max}$ to the achievable potential between the electrodes is set by the electrostatic push between a droplet and the bowl beneath it. Find $U_{\max}$.

Official English text of the problem.

Answers

Here ε_0 is the vacuum permittivity, and q denotes the magnitude of the charge on either capacitor plate.

- A-i:

$$r_{\max} = \left(\frac{3\sigma d}{4\rho g} \right)^{1/3}.$$

- A-ii:

$$Q = 4\pi\varepsilon_0 r \varphi.$$

- **A-iii:**

$$\varphi_{\max} = 2\sqrt{\frac{\sigma r}{\varepsilon_0}}.$$

The same threshold applies to the magnitude of a negative potential.

- **B-i:**

$$Q_0 = \frac{2\pi\varepsilon_0 r_{\max}}{C} q.$$

- **B-ii:** In the continuous approximation with negligible flight delay,

$$q(t) = q_0 \exp\left(\frac{2\pi\varepsilon_0 n r_{\max}}{C} t\right).$$

- **B-iii:** The limit due to electrostatic repulsion is

$$U_{\max} = r_{\max} \sqrt{\frac{2\rho g H}{3\varepsilon_0}} = \sqrt{\frac{\sigma d H}{2\varepsilon_0 r_{\max}}}.$$

No numerical parameter values are supplied.

Solution

Part A-i

The slowly growing drop is in mechanical equilibrium until its weight exceeds the maximum supporting force of surface tension.

The nozzle circumference is πd . When the surface tension acts vertically around the neck, its maximum upward force is

$$F_\sigma = \pi d \sigma.$$

The neck volume and the pressure-force correction at the small nozzle opening are negligible for $d \ll r$. Approximating the drop as spherical, its weight is

$$mg = \frac{4}{3}\pi r^3 \rho g.$$

At detachment,

$$\frac{4}{3}\pi r_{\max}^3 \rho g = \pi d \sigma,$$

hence

$$r_{\max} = \left(\frac{3\sigma d}{4\rho g}\right)^{1/3}.$$

The spherical approximation is consistent with this result: the ratio of gravitational to capillary stresses is

$$\frac{\rho g r_{\max}^2}{\sigma} = \frac{3d}{4r_{\max}} \ll 1.$$

Part A-ii

The conducting drop and pipe have the same potential. Neglecting the thin pipe's perturbation to the drop's electric field, the drop behaves as a conducting sphere.

Outside a sphere carrying charge Q , Gauss's law gives

$$E(s) = \frac{Q}{4\pi\epsilon_0 s^2}.$$

Taking the distant surroundings to have zero potential,

$$\varphi = \int_r^\infty E(s) ds = \frac{Q}{4\pi\epsilon_0 r}.$$

Therefore

$$\boxed{Q = 4\pi\epsilon_0 r\varphi}.$$

The sign of the drop's charge is the sign of φ .

Part A-iii

Surface tension produces an inward capillary pressure. From the given surface energy,

$$W_\sigma = 4\pi r^2 \sigma,$$

the pressure required to increase the surface area is

$$p_\sigma = \frac{dW_\sigma}{dV} = \frac{8\pi r\sigma}{4\pi r^2} = \frac{2\sigma}{r}.$$

The electric field produces an outward pressure. For a conductor, the surface charge density is $\epsilon_0 E$, and the field acting on that charge due to the other charges is $E/2$. Thus

$$p_e = \frac{\epsilon_0 E^2}{2}.$$

Using Part A-ii,

$$E = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{\varphi}{r}.$$

Consequently, mechanical equilibrium gives

$$p_{\text{in}} - p_{\text{atm}} = p_\sigma - p_e = \frac{2\sigma}{r} - \frac{\epsilon_0 \varphi^2}{2r^2}.$$

According to the specified instability criterion, the threshold occurs when the internal pressure equals atmospheric pressure:

$$\frac{2\sigma}{r} = \frac{\epsilon_0 \varphi_{\text{max}}^2}{2r^2}.$$

Therefore

$$\boxed{\varphi_{\text{max}} = 2\sqrt{\frac{\sigma r}{\epsilon_0}}}.$$

Equivalently, the corresponding charge magnitude is

$$Q_{\text{crit}} = 4\pi\epsilon_0 r\varphi_{\text{max}} = 8\pi\sqrt{\epsilon_0 \sigma r^3}.$$

Part B-i

Write

$$R = r_{\max}, \quad U = \frac{q}{C}.$$

We neglect stray capacitances compared with C . By symmetry and zero net charge, the two electrode potentials relative to earth are

$$V_L = +\frac{U}{2}, \quad V_R = -\frac{U}{2}.$$

Since $L \gg D$, the potential near the centre of each cylinder, before accounting for the small drop, is approximately the cylinder's potential. Since $D \gg R$, the drop's own contribution to its surface potential is approximately that of an isolated sphere.

The water and pipes are grounded. Thus, while the left drop remains attached,

$$0 = V_L + \frac{Q_L}{4\pi\epsilon_0 R}.$$

It follows that

$$Q_L = -2\pi\epsilon_0 R U.$$

Similarly,

$$Q_R = +2\pi\epsilon_0 R U.$$

The charge is retained upon separation, so its magnitude is

$$Q_0 = 2\pi\epsilon_0 R U = \frac{2\pi\epsilon_0 r_{\max}}{C} q.$$

Because of the cross-connections, each drop falls into a bowl already carrying charge of the same sign as the drop. Its arrival therefore reinforces the charge separation.

Part B-ii

Use the continuous approximation and neglect the delay between detachment and collection. This assumes that the voltage changes little during a drop's flight.

Each tube produces n drops per unit time. The positive capacitor plate receives positive charge at rate nQ_0 , while the negative plate receives negative charge at the same rate. Hence the magnitude q on either plate obeys

$$\frac{dq}{dt} = nQ_0 = \frac{2\pi\epsilon_0 n R}{C} q.$$

There is no additional factor of two: a pair of drops increases the plate charges from $+q, -q$ to $+(q + Q_0), -(q + Q_0)$.

Integrating and imposing $q(0) = q_0$,

$$\ln \frac{q(t)}{q_0} = \frac{2\pi\epsilon_0 n R}{C} t,$$

so

$$q(t) = q_0 \exp\left(\frac{2\pi\epsilon_0 n r_{\max}}{C} t\right).$$

The characteristic growth time is

$$\tau = \frac{C}{2\pi\epsilon_0 n r_{\max}}.$$

The negligible-delay approximation requires the flight time to be small compared with τ . The exponential growth applies before breakup or electrostatic repulsion significantly interferes with collection.

Part B-iii

Consider the drop formed inside the positive electrode. It carries charge $-Q_0$, and its collecting bowl is connected to the negative electrode.

Treat U as approximately constant during the fall. The potential produced by the apparatus changes from $+U/2$ near the nozzle to $-U/2$ at the bowl. Therefore the increase in the drop's external electrostatic potential energy is

$$\Delta W_e = (-Q_0) \left(-\frac{U}{2} - \frac{U}{2} \right) = Q_0 U.$$

The potential used here is the potential produced by the apparatus. The drop's zero total potential while attached also includes its own field. During flight, its charge and radius remain approximately constant, so its self-energy does not contribute to the translational work.

Neglecting air resistance and initial kinetic energy, the gravitational energy available is

$$mgH = \frac{4}{3}\pi R^3 \rho g H.$$

For collection to remain possible, this must be sufficient to overcome the electrostatic energy increase:

$$mgH \geq Q_0 U.$$

At the energy limit, using $Q_0 = 2\pi\epsilon_0 R U$,

$$\frac{4}{3}\pi R^3 \rho g H = 2\pi\epsilon_0 R U_{\max}^2.$$

Thus

$$U_{\max} = r_{\max} \sqrt{\frac{2\rho g H}{3\epsilon_0}}.$$

Using the detachment condition $mg = \pi d\sigma$, this can also be written

$$U_{\max} = \sqrt{\frac{\sigma d H}{2\epsilon_0 r_{\max}}}.$$

The separate breakup threshold from Part A-iii, evaluated for a drop of radius $r_{\max}$, corresponds to

$$U_{\text{break}} = 2\varphi_{\max}(r_{\max}) = 4\sqrt{\frac{\sigma r_{\max}}{\epsilon_0}}.$$

This provides an additional restriction on operation with intact drops.

The solution was checked against the official solution.

Problem 3: Protostar formation

Problem T3. Protostar formation (9 points)

Let us model the formation of a star as follows. A spherical cloud of sparse interstellar gas, initially at rest, starts to collapse due to its own gravity. The initial radius of the ball is r_0 and the mass is m . The temperature of the surroundings (much sparser than the gas) and the initial temperature of the gas is uniformly T_0 . The gas may be assumed to be ideal. The average molar mass of the gas is μ and its adiabatic index is $\gamma > \frac{4}{3}$. Assume that $G \frac{m\mu}{r_0} \gg RT_0$, where R is the gas constant and G is the gravitational constant.

- i. (0.8 pts) During much of the collapse, the gas is so transparent that any heat generated is immediately radiated away, i.e. the ball stays in thermodynamic equilibrium with its surroundings. What is the number of times, n , by which the pressure increases when the radius is halved to $r_1 = 0.5r_0$? Assume that the gas density remains uniform.
- ii. (1 pt) Estimate the time t_2 needed for the radius to shrink from r_0 to $r_2 = 0.95r_0$. Neglect the change of the gravity field at the position of a falling gas particle.
- iii. (2.5 pts) Assuming that the pressure remains negligible, find the time $t_{r \rightarrow 0}$ needed for the ball to collapse from r_0 down to a much smaller radius, using Kepler's Laws.
- iv. (1.7 pts) At some radius $r_3 \ll r_0$, the gas becomes dense enough to be opaque to the heat radiation. Calculate the amount of heat Q radiated away during the collapse from the radius r_0 down to r_3 .
- v. (1 pt) For radii smaller than r_3 you may neglect heat loss due to radiation. Determine how the temperature T of the ball depends on its radius for $r < r_3$.
- vi. (2 pts) Eventually we cannot neglect the effect of the pressure on the dynamics of the gas and the collapse stops at $r = r_4$ (with $r_4 \ll r_3$). However, the radiation loss can still be neglected and the temperature is not yet high enough to ignite nuclear fusion. The pressure of such a protostar is not uniform anymore, but rough estimates with inaccurate numerical prefactors can still be done. Estimate the final radius r_4 and the respective temperature T_4 .

Official English text of the problem.

Answers

- i.

$$n = 8.$$

- ii.

$$t_2 \simeq \sqrt{\frac{0.10 r_0^3}{Gm}} \simeq 0.32 \sqrt{\frac{r_0^3}{Gm}}.$$

• iii.

$$t_{r \rightarrow 0} = \frac{\pi}{2\sqrt{2}} \sqrt{\frac{r_0^3}{Gm}} \simeq 1.11 \sqrt{\frac{r_0^3}{Gm}}.$$

• iv.

$$Q = \frac{3mRT_0}{\mu} \ln\left(\frac{r_0}{r_3}\right).$$

• v.

$$T(r) = T_0 \left(\frac{r_3}{r}\right)^{3(\gamma-1)}.$$

• vi. Suppressing numerical factors in the energy balance, the estimates are

$$r_4 \sim r_3 \left(\frac{RT_0 r_3}{Gm\mu}\right)^{1/(3\gamma-4)}$$

and

$$T_4 \sim T_0 \left(\frac{Gm\mu}{RT_0 r_3}\right)^{3(\gamma-1)/(3\gamma-4)}.$$

An expression retaining the heat-capacity factor and an approximate gravitational structure factor is derived below.

Solution

Let the amount of gas be $N = m/\mu$ moles. Its volume and ideal-gas pressure are

$$V = \frac{4\pi r^3}{3}, \quad pV = NRT.$$

The given inequality means that the initial gravitational energy scale greatly exceeds the thermal energy scale, justifying neglect of pressure during the early collapse.

i. Pressure increase during isothermal contraction

Thermodynamic equilibrium with the surroundings keeps the temperature at T_0 . Thus

$$p = \frac{NRT_0}{V} \propto r^{-3}.$$

Consequently,

$$n = \frac{p_1}{p_0} = \left(\frac{r_0}{r_1}\right)^3 = \left(\frac{1}{0.5}\right)^3 = \boxed{8}.$$

ii. Time for the first five per cent decrease in radius

By the shell theorem, a particle at the surface initially experiences the inward gravitational acceleration

$$g_0 = \frac{Gm}{r_0^2}.$$

Keeping this acceleration constant, as instructed, and using the initial condition of rest,

$$r_0 - r_2 = \frac{1}{2}g_0 t_2^2.$$

Since $r_0 - r_2 = 0.05r_0$,

$$t_2 \simeq \sqrt{\frac{2(0.05r_0)}{Gm/r_0^2}} = \sqrt{\frac{0.10r_0^3}{Gm}}.$$

iii. Collapse time from Kepler's laws

A surface particle moves under the gravitational field of the enclosed mass m :

$$\ddot{r} = -\frac{Gm}{r^2}.$$

Regard its radial fall as the limit of a very eccentric bound Kepler orbit. The apocentre is r_0 , while the pericentre tends to zero, so the semimajor axis tends to

$$a = \frac{r_0}{2}.$$

Kepler's third law gives the orbital period

$$P = 2\pi\sqrt{\frac{a^3}{Gm}}.$$

The journey from apocentre to pericentre takes half a period. Therefore,

$$t_{r \rightarrow 0} = \frac{P}{2} = \pi\sqrt{\frac{(r_0/2)^3}{Gm}} = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{r_0^3}{Gm}}.$$

For an initially uniform cloud, this is also the collapse time of every interior shell. Indeed, a shell initially at radius s_0 encloses

$$M(s_0) = m\left(\frac{s_0}{r_0}\right)^3,$$

so its collapse time is

$$\frac{\pi}{2\sqrt{2}}\sqrt{\frac{s_0^3}{GM(s_0)}} = \frac{\pi}{2\sqrt{2}}\sqrt{\frac{r_0^3}{Gm}},$$

independent of s_0 . This supports the uniform, homologous collapse approximation.

iv. Heat radiated during the transparent stage

The internal energy of an ideal gas depends only on temperature. Since $T = T_0$, its internal energy remains constant.

Taking $\delta Q_{\text{out}} > 0$ for heat leaving the gas, the thermal first law is

$$dU = -p dV - \delta Q_{\text{out}}.$$

Thus, with $dU = 0$,

$$\delta Q_{\text{out}} = -p dV = -NRT_0 \frac{dV}{V}.$$

Integrating from V_0 to V_3 ,

$$Q = NRT_0 \ln\left(\frac{V_0}{V_3}\right).$$

Since $V \propto r^3$,

$$Q = \frac{3mRT_0}{\mu} \ln\left(\frac{r_0}{r_3}\right).$$

The decrease in gravitational potential energy is **not all radiated away**: during this rapid collapse, most becomes coherent inward kinetic energy. The radiated heat calculated here is the compression work that would otherwise increase the gas's internal energy.

v. Temperature during the opaque stage

Once radiation losses become negligible, the compression is adiabatic within the model. For an ideal gas,

$$pV^\gamma = \text{constant}.$$

Combining this with $pV = NRT$ gives

$$TV^{\gamma-1} = \text{constant}.$$

At the beginning of this stage, $r = r_3$ and $T = T_0$. Hence

$$T(r)V(r)^{\gamma-1} = T_0V_3^{\gamma-1},$$

and therefore

$$T(r) = T_0 \left(\frac{V_3}{V(r)}\right)^{\gamma-1} = T_0 \left(\frac{r_3}{r}\right)^{3(\gamma-1)}.$$

vi. Radius and temperature at the arrest of collapse

The molar heat capacity is

$$C_V = \frac{R}{\gamma - 1},$$

so the internal energy is

$$U = \frac{mRT}{\mu(\gamma - 1)}.$$

Write the final gravitational potential energy as

$$W_4 = -\alpha \frac{Gm^2}{r_4},$$

where α is a dimensionless structure factor of order unity. A uniform sphere has $\alpha = 3/5$; the actual nonuniform protostar need not have precisely this value.

At the arrest of the bulk contraction, its bulk kinetic energy is approximately zero. Conservation of energy gives

$$U_4 + W_4 = U_0 + W_0 - Q.$$

The right-hand side is small compared with Gm^2/r_4 . The initial gravitational energy is smaller by a factor of order r_4/r_0 . The radiated energy is already small compared with Gm^2/r_3 , since

$$\frac{Q}{Gm^2/r_3} = 3 \frac{RT_0 r_0}{Gm\mu} \frac{r_3}{r_0} \ln\left(\frac{r_0}{r_3}\right) \ll 1.$$

Thus, to the requested accuracy,

$$U_4 \simeq |W_4|,$$

or

$$\frac{mRT_4}{\mu(\gamma-1)} \simeq \alpha \frac{Gm^2}{r_4}.$$

Equivalently,

$$T_4 \simeq \frac{\alpha(\gamma-1)Gm\mu}{Rr_4}.$$

Combining this with the adiabatic temperature law,

$$T_0 \left(\frac{r_3}{r_4}\right)^{3(\gamma-1)} \simeq \frac{\alpha(\gamma-1)Gm\mu}{Rr_4},$$

yields

$$\left(\frac{r_3}{r_4}\right)^{3\gamma-4} \simeq \frac{\alpha(\gamma-1)Gm\mu}{RT_0 r_3}.$$

Therefore, retaining these factors,

$$r_4 \simeq r_3 \left[\frac{RT_0 r_3}{\alpha(\gamma-1)Gm\mu} \right]^{1/(3\gamma-4)}$$

and

$$T_4 \simeq T_0 \left[\frac{\alpha(\gamma-1)Gm\mu}{RT_0 r_3} \right]^{3(\gamma-1)/(3\gamma-4)}.$$

Taking $\alpha = 3/5$ supplies one concrete rough estimate; suppressing the numerical factors gives the scaling expressions in **Answers**.

Finally,

$$U \propto r^{-3(\gamma-1)}, \quad |W| \propto r^{-1}.$$

Because $\gamma > 4/3$, the internal energy grows faster during contraction than the magnitude of the gravitational energy. This allows the collapse to be arrested. Also,

$$\frac{Gm\mu}{RT_0 r_3} \gg 1,$$

so the estimates give $r_4 \ll r_3$ and $T_4 \gg T_0$, consistently with the assumptions.

The solution was checked against the official solution.