

IPhO 2011

Theory problems and solutions · Bangkok, Thailand · 3 theory problems with complete solutions

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Problem 1: A Three-body Problem and LISA

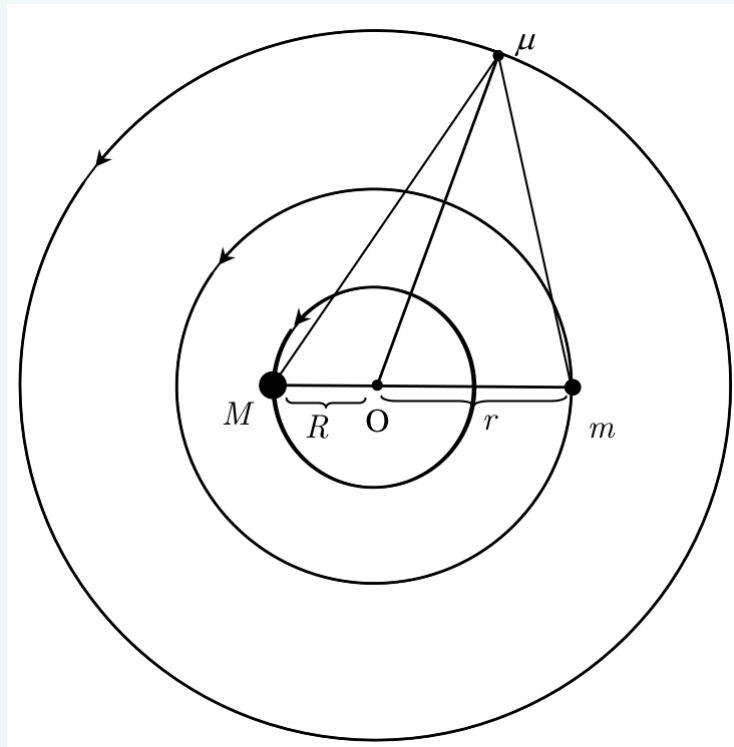


FIGURE 1 Coplanar orbits of three bodies.

1.1 Two gravitating masses M and m are moving in circular orbits of radii R and r , respectively, about their common centre of mass. Find the angular velocity ω_0 of the line joining M and m in terms of R, r, M, m and the universal gravitational constant G .

[1.5 points]

1.2 A third body of infinitesimal mass μ is placed in a coplanar circular orbit about the same centre of mass so that μ remains stationary relative to both M and m as shown in Figure 1. Assume that the infinitesimal mass is not collinear with M and m . Find the values of the following parameters in terms of R and r :

[3.5 points]

1.2.1 distance from μ to M .

1.2.2 distance from μ to m .

1.2.3 distance from μ to the centre of mass.

1.3 Consider the case $M = m$. If μ is now given a small radial perturbation (along $O\mu$), what is the angular frequency of oscillation of μ about the unperturbed position in terms of ω_0 ? Assume that the angular momentum of μ is conserved.

[3.2 points]

The Laser Interferometry Space Antenna (LISA) is a group of three identical spacecrafts for detecting low frequency gravitational waves. Each of the spacecrafts is placed at the corners of an equilateral triangle as shown in Figure 2 and Figure 3. The sides (or 'arms') are about 5.0 million kilometres long. The LISA constellation is in an earth-like orbit around the Sun trailing the Earth by 20° . Each of them moves on a slightly inclined individual orbit around the Sun. Effectively, the three spacecrafts appear to roll about their common centre one revolution per year.

They are continuously transmitting and receiving laser signals between each other. Overall, they detect the gravitational waves by measuring tiny changes in the arm lengths using interferometric means. A collision of massive objects, such as blackholes, in nearby galaxies is an example of the sources of gravitational waves.

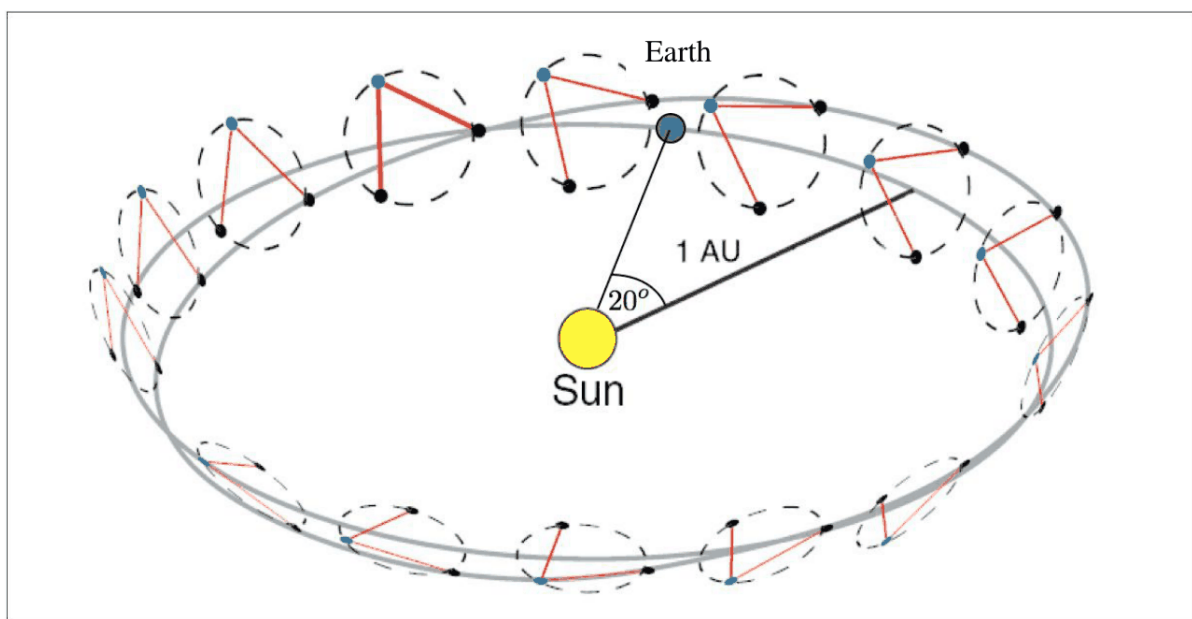


FIGURE 2 Illustration of the LISA orbit. The three spacecrafts roll about their centre of mass with a period of 1 year. Initially, they trail the Earth by 20° (Picture from D.A. Shaddock, "An Overview of the Laser Interferometer Space Antenna", *Publications of the Astronomical Society of Australia*, 2009, **26**, pp.128-132.).

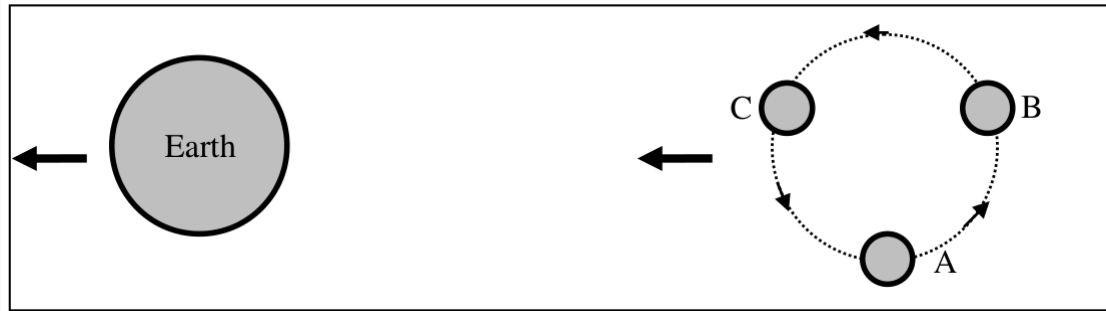


FIGURE 3 Enlarged view of the three spacecraft trailing the Earth. A, B and C are the three spacecraft at the corners of the equilateral triangle.

1.4 In the plane containing the three spacecraft, what is the relative speed of one spacecraft with respect to another?

[1.8 point]

Official English text of the problem.

Answers

- 1.1

$$\omega_0 = \sqrt{\frac{G(M + m)}{(R + r)^3}}$$

- 1.2.1

$$d_{\mu M} = R + r$$

- 1.2.2

$$d_{\mu m} = R + r$$

- 1.2.3

$$d_{\mu O} = \sqrt{R^2 + Rr + r^2}$$

- 1.3, using the stipulated conservation of angular momentum:

$$\omega_{\text{osc}} = \frac{\sqrt{7}}{2} \omega_0$$

- 1.4, with arm length $L = 5.0 \times 10^9$ m and rotation period $T = 1$ year:

$$v_{\text{rel}} = \frac{2\pi L}{T} \simeq 1.0 \times 10^3 \text{ m s}^{-1} = 1.0 \text{ km s}^{-1}$$

Solution

1.1

The distance between the masses is $d = R + r$. Since O is their centre of mass,

$$MR = mr.$$

Gravity supplies the centripetal acceleration of each mass:

$$M\omega_0^2 R = \frac{GMm}{d^2}, \quad m\omega_0^2 r = \frac{GMm}{d^2}.$$

Dividing by the respective masses and adding gives

$$\omega_0^2(R + r) = \frac{G(M + m)}{d^2}.$$

Therefore,

$$\boxed{\omega_0 = \sqrt{\frac{G(M + m)}{(R + r)^3}}}.$$

1.2.1 and 1.2.2

Because μ is infinitesimal, it does not affect the motion of M and m . To remain stationary relative to them, it must revolve about O with angular velocity ω_0 .

At a chosen instant, take

$$O = (0, 0), \quad M = (-R, 0), \quad m = (r, 0), \quad \mu = (x, y),$$

where $y \neq 0$. Write its distances from the two masses as

$$a = \sqrt{(x + R)^2 + y^2}, \quad b = \sqrt{(x - r)^2 + y^2}.$$

Its acceleration is $-\omega_0^2(x, y)$. Newton's law, after cancelling the test mass μ , therefore gives

$$\begin{aligned} \omega_0^2 x &= \frac{GM(x + R)}{a^3} + \frac{Gm(x - r)}{b^3}, \\ \omega_0^2 y &= \frac{GM y}{a^3} + \frac{Gm y}{b^3}. \end{aligned}$$

Since $y \neq 0$, the second equation implies

$$\omega_0^2 = \frac{GM}{a^3} + \frac{Gm}{b^3}.$$

Subtracting x times this equation from the first yields

$$0 = \frac{GMR}{a^3} - \frac{Gmr}{b^3}.$$

Using $MR = mr$, we obtain $a = b$. Consequently,

$$\omega_0^2 = \frac{G(M + m)}{a^3}.$$

Comparison with part 1.1 gives $a = R + r$, and hence

$$\boxed{1.2.1: d_{\mu M} = R + r}, \quad \boxed{1.2.2: d_{\mu m} = R + r}.$$

Thus the three bodies occupy the vertices of an equilateral triangle.

1.2.3

The midpoint of the segment joining the two finite masses has coordinate

$$x_{\text{mid}} = \frac{r - R}{2}.$$

The third vertex of an equilateral triangle of side $R + r$ lies a perpendicular distance $\sqrt{3}(R + r)/2$ from this midpoint. Therefore the two possible positions of μ are

$$x = \frac{r - R}{2}, \quad y = \pm \frac{\sqrt{3}}{2}(R + r).$$

Its distance from O satisfies

$$d_{\mu O}^2 = x^2 + y^2 = \frac{(r - R)^2 + 3(R + r)^2}{4} = R^2 + Rr + r^2.$$

Thus,

$$\boxed{d_{\mu O} = \sqrt{R^2 + Rr + r^2}}.$$

1.3

When $M = m$, the centre-of-mass condition gives $R = r$. The unperturbed orbital radius of μ is therefore

$$\rho_0 = \sqrt{3}R,$$

and part 1.1 gives

$$\omega_0^2 = \frac{GM}{4R^3}.$$

On the perpendicular bisector of the line joining the equal masses, a point at distance ρ from O is at distance $\sqrt{R^2 + \rho^2}$ from each mass. Their transverse gravitational components cancel, so the inward radial acceleration is

$$g(\rho) = \frac{2GM\rho}{(R^2 + \rho^2)^{3/2}}.$$

This determines the linear radial force: by reflection symmetry, a small angular departure from the perpendicular bisector changes the radial component only at second order.

Use the angular-momentum conservation prescribed in the question. The angular momentum per unit mass is

$$h = \rho^2 \dot{\theta} = \rho_0^2 \omega_0.$$

The radial equation of motion,

$$\ddot{\rho} - \rho \dot{\theta}^2 = -g(\rho),$$

becomes

$$\ddot{\rho} = \frac{h^2}{\rho^3} - \frac{2GM\rho}{(R^2 + \rho^2)^{3/2}}.$$

Put

$$\rho = \rho_0 + \xi, \quad |\xi| \ll \rho_0,$$

and keep only terms linear in ξ . The first term expands as

$$\frac{h^2}{\rho^3} = \rho_0 \omega_0^2 - 3\omega_0^2 \xi + O(\xi^2).$$

For the inward gravitational acceleration,

$$g'(\rho) = \frac{2GM}{(R^2 + \rho^2)^{3/2}} \left(1 - \frac{3\rho^2}{R^2 + \rho^2} \right).$$

Since $\rho_0^2 = 3R^2$,

$$g'(\rho_0) = \omega_0^2 \left(1 - \frac{9}{4} \right) = -\frac{5}{4} \omega_0^2.$$

Thus

$$g(\rho) = \rho_0 \omega_0^2 - \frac{5}{4} \omega_0^2 \xi + O(\xi^2).$$

The equilibrium terms cancel, leaving

$$\ddot{\xi} = \left(-3 + \frac{5}{4} \right) \omega_0^2 \xi = -\frac{7}{4} \omega_0^2 \xi.$$

This is the simple harmonic oscillator equation, so

$$\omega_{\text{osc}} = \frac{\sqrt{7}}{2} \omega_0.$$

This result uses the imposed angular-momentum-conserving radial approximation. Angular momentum about O is not generally conserved for unrestricted motion in the rotating two-mass gravitational field.

1.4

Treat the constellation as an equilateral triangle of constant side length L , rotating in its plane with angular velocity

$$\Omega = \frac{2\pi}{T}.$$

Let $\mathbf{r}_A$ and $\mathbf{r}_B$ be the positions of two spacecraft relative to the triangle's centre. Their velocities are

$$\mathbf{v}_A = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}_A, \quad \mathbf{v}_B = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}_B,$$

where $\mathbf{V}$ is the common translational velocity of the centre.

Subtracting eliminates this common motion:

$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\Omega} \times (\mathbf{r}_B - \mathbf{r}_A).$$

The separation vector lies in the triangle's plane and has magnitude L , so it is perpendicular to $\boldsymbol{\Omega}$. Hence

$$v_{\text{rel}} = \Omega L = \frac{2\pi L}{T}.$$

With

$$L = 5.0 \times 10^6 \text{ km} = 5.0 \times 10^9 \text{ m}, \quad T \simeq 365 \times 24 \times 3600 \text{ s},$$

we obtain

$$v_{\text{rel}} \simeq 996 \text{ m s}^{-1}.$$

To two significant figures,

$$v_{\text{rel}} \simeq 1.0 \times 10^3 \text{ m s}^{-1} = 1.0 \text{ km s}^{-1}.$$

The relative velocity is perpendicular to the arm joining the spacecraft, consistent with the arm's constant length in this idealized motion.

The solution was checked against the official solution.

Problem 2: An Electrified Soap Bubble

A spherical soap bubble with internal air density ρ_i , temperature T_i and radius R_0 is surrounded by air with density ρ_a , atmospheric pressure P_a and temperature T_a . The soap film has surface tension γ , density ρ_s and thickness t . The mass and the surface tension of the soap do not change with the temperature. Assume that $R_0 \gg t$.

The increase in energy, dE , that is needed to increase the surface area of a soap-air interface by dA , is given by $dE = \gamma dA$ where γ is the surface tension of the film.

2.1 Find the ratio $\frac{\rho_i T_i}{\rho_a T_a}$ in terms of γ , P_a and R_0 . **[1.7 point]**

2.2 Find the numerical value of $\frac{\rho_i T_i}{\rho_a T_a} - 1$ using $\gamma = 0.0250 \text{ N m}^{-1}$, $R_0 = 1.00 \text{ cm}$, and $P_a = 1.013 \times 10^5 \text{ N m}^{-2}$. **[0.4 point]**

2.3 The bubble is initially formed with warmer air inside. Find the minimum numerical value of T_i such that the bubble can float in still air. Use $T_a = 300 \text{ K}$, $\rho_s = 1000 \text{ kg m}^{-3}$, $\rho_a = 1.30 \text{ kg m}^{-3}$, $t = 100 \text{ nm}$ and $g = 9.80 \text{ m s}^{-2}$. **[2.0 points]**

After the bubble is formed for a while, it will be in thermal equilibrium with the surroundings. This bubble in still air will naturally fall towards the ground.

2.4 Find the minimum velocity u of an updraught (air flowing upwards) that will keep the bubble from falling at thermal equilibrium. Give your answer in terms of ρ_s , R_0 , g , t and the air's coefficient of viscosity η . You may assume that the velocity is small such that Stokes's law applies, and ignore the change in the radius when the temperature lowers to the equilibrium. The drag force from Stokes's Law is $F = 6\pi\eta R_0 u$. **[1.6points]**

2.5 Calculate the numerical value for u using $\eta = 1.8 \times 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1}$. **[0.4 point]**

The above calculations suggest that the terms involving the surface tension γ add very little to the accuracy of the result. In all of the questions below, you can neglect the surface tension terms.

2.6 If this spherical bubble is now electrified uniformly with a total charge q , find an equation describing the new radius R_1 in terms of R_0 , P_a , q and the permittivity of free space ϵ_0 . **[2.0points]**

2.7 Assume that the total charge is not too large (i.e. $\frac{q^2}{\epsilon_0 R_0^4} \ll P_a$) and the bubble only experiences a small increase in its radius, find ΔR where $R_1 = R_0 + \Delta R$.

Given that $(1 + x)^n \approx 1 + nx$ where $x \ll 1$. **[0.7 point]**

2.8 What must be the magnitude of this charge q in terms of $t, \rho_a, \rho_s, \varepsilon_0, R_0, P_a$ in order that the bubble will float motionlessly in still air? Calculate also the numerical value of q . The permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}$ farad/m. **[1.2 points]**

Official English text of the problem.

Answers

• 2.1

$$\frac{\rho_i T_i}{\rho_a T_a} = 1 + \frac{4\gamma}{P_a R_0}.$$

• 2.2

$$\frac{\rho_i T_i}{\rho_a T_a} - 1 = 9.87 \times 10^{-5}.$$

• 2.3

$$T_{i,\min} = T_a \frac{1 + \frac{4\gamma}{P_a R_0}}{1 - \frac{3\rho_s t}{\rho_a R_0}} = 307 \text{ K}.$$

• 2.4, neglecting the small surface-tension correction:

$$u_{\min} \simeq \frac{2\rho_s R_0 t g}{3\eta}.$$

• 2.5

$$u_{\min} \simeq 0.36 \text{ m s}^{-1}.$$

• 2.6

$$R_1^4 - R_0^3 R_1 - \frac{q^2}{32\pi^2 \varepsilon_0 P_a} = 0, \quad R_1 \geq R_0.$$

• 2.7

$$\Delta R \simeq \frac{q^2}{96\pi^2 \varepsilon_0 P_a R_0^3}.$$

- **2.8**, using the small-expansion approximation:

$$|q| \simeq \sqrt{\frac{96\pi^2 \varepsilon_0 P_a \rho_s t R_0^3}{\rho_a}} = 2.56 \times 10^{-7} \text{ C}.$$

Without linearizing the expansion, the thin-film result is

$$|q| = \sqrt{\frac{96\pi^2 \varepsilon_0 P_a \rho_s t R_0^3}{\rho_a}} \left(1 + \frac{3\rho_s t}{\rho_a R_0}\right)^{1/6} = 2.57 \times 10^{-7} \text{ C}.$$

Solution

We treat the air as an ideal gas with the same composition inside and outside. In the thin-film approximation, the bubble volume and soap mass are

$$V = \frac{4}{3}\pi R^3, \quad m_s \simeq 4\pi R^2 t \rho_s.$$

The film has two soap-air interfaces, but its mass is that of one shell.

2.1

Both interfaces contribute to the surface energy. Neglecting their small difference in radius,

$$E_s = 2\gamma(4\pi R^2) = 8\pi\gamma R^2.$$

For a small outward displacement dR , mechanical equilibrium requires

$$(P_i - P_a) dV = dE_s.$$

Since

$$dV = 4\pi R^2 dR, \quad dE_s = 16\pi\gamma R dR,$$

the excess internal pressure is

$$P_i - P_a = \frac{4\gamma}{R_0}.$$

The ideal-gas relation $P = \rho \mathcal{R}_{\text{air}} T$, with the same specific gas constant on both sides, gives

$$\frac{\rho_i T_i}{\rho_a T_a} = \frac{P_i}{P_a} = 1 + \frac{4\gamma}{P_a R_0}.$$

2.2

With $R_0 = 1.00 \times 10^{-2} \text{ m}$,

$$P_i - P_a = \frac{4(0.0250)}{1.00 \times 10^{-2}} = 10.0 \text{ Pa}.$$

Thus

$$\frac{\rho_i T_i}{\rho_a T_a} - 1 = \frac{10.0}{1.013 \times 10^5} = 9.87 \times 10^{-5}.$$

2.3

The upward buoyant force is the weight of the displaced ambient air:

$$F_B = \rho_a V_0 g, \quad V_0 = \frac{4}{3} \pi R_0^3.$$

At the minimum temperature that permits floating, buoyancy just balances the combined weight of the internal air and soap:

$$\rho_a V_0 g = \rho_i V_0 g + 4\pi R_0^2 t \rho_s g.$$

Therefore,

$$\rho_i = \rho_a - \frac{3\rho_s t}{R_0}.$$

Numerically,

$$\frac{3\rho_s t}{R_0} = \frac{3(1000)(1.00 \times 10^{-7})}{1.00 \times 10^{-2}} = 0.0300 \text{ kg m}^{-3},$$

so neutral buoyancy requires

$$\rho_i = 1.27 \text{ kg m}^{-3}.$$

Using part 2.1,

$$T_{i,\min} = \frac{\rho_a T_a}{\rho_i} \left(1 + \frac{4\gamma}{P_a R_0} \right) = T_a \frac{1 + \frac{4\gamma}{P_a R_0}}{1 - \frac{3\rho_s t}{\rho_a R_0}}.$$

Consequently,

$$T_{i,\min} = 300 \frac{1 + 9.8717 \times 10^{-5}}{1 - 0.0300/1.30} = 307.117 \text{ K},$$

or

$$\boxed{T_{i,\min} = 307 \text{ K}}.$$

A higher temperature gives a lower internal air density and hence an upward net force.

2.4

At thermal equilibrium, $T_i = T_a$. Part 2.1 then gives

$$\rho_i - \rho_a = \rho_a \frac{4\gamma}{P_a R_0}.$$

For a stationary bubble, the upward Stokes drag must balance its weight minus buoyancy:

$$6\pi\eta R_0 u_{\min} = 4\pi R_0^2 t \rho_s g + \frac{4}{3} \pi R_0^3 (\rho_i - \rho_a) g.$$

Hence, retaining surface tension,

$$u_{\min} = \frac{2\rho_s R_0 t g}{3\eta} + \frac{8\rho_a \gamma R_0 g}{9\eta P_a}.$$

The ratio of the second term to the first is

$$\frac{4\rho_a \gamma}{3P_a \rho_s t} = 4.28 \times 10^{-3}.$$

Thus surface tension changes the required speed by only about 0.43%. Neglecting this correction gives the expression in the requested parameters:

$$u_{\min} \simeq \frac{2\rho_s R_0 t g}{3\eta}.$$

This uses the stipulated Stokes drag law and the instruction to ignore the radius change during cooling.

2.5

Substituting the given values,

$$u_{\min} \simeq \frac{2(1000)(1.00 \times 10^{-2})(1.00 \times 10^{-7})(9.80)}{3(1.8 \times 10^{-5})} = 0.362963 \text{ m s}^{-1}.$$

Therefore,

$$u_{\min} \simeq 0.36 \text{ m s}^{-1}.$$

Including surface tension gives $0.364516 \text{ m s}^{-1}$, which rounds to the same answer at two significant figures.

2.6

We now neglect surface tension as instructed. Before charging, the bubble is at temperature T_a , with internal pressure P_a . The enclosed air mass is conserved, and the final state is also at temperature T_a . The ideal-gas law therefore gives

$$P_i V_1 = P_a V_0, \quad P_i = P_a \frac{R_0^3}{R_1^3}.$$

A uniformly charged spherical shell has electric field

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2} \quad (r > R),$$

and zero field inside. Its electrostatic energy is

$$U_e = \frac{\epsilon_0}{2} \int_R^\infty E(r)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0 R}.$$

At fixed charge, an increase in radius lowers this energy. The corresponding outward electric pressure is

$$p_e = -\frac{dU_e}{dV} = -\frac{dU_e/dR}{4\pi R^2} = \frac{q^2}{32\pi^2\epsilon_0 R^4}.$$

Mechanical equilibrium requires

$$P_i + p_e = P_a.$$

At $R = R_1$,

$$P_a \frac{R_0^3}{R_1^3} + \frac{q^2}{32\pi^2\epsilon_0 R_1^4} = P_a.$$

Multiplying by R_1^4/P_a gives

$$R_1^4 - R_0^3 R_1 - \frac{q^2}{32\pi^2\epsilon_0 P_a} = 0.$$

For nonzero charge, the physical solution has $R_1 > R_0$.

2.7

Let

$$x = \frac{\Delta R}{R_0} \ll 1, \quad R_1 = R_0(1 + x).$$

The equation from part 2.6 becomes

$$R_0^4 [(1 + x)^4 - (1 + x)] = \frac{q^2}{32\pi^2 \epsilon_0 P_a}.$$

To first order,

$$(1 + x)^4 - (1 + x) \simeq (1 + 4x) - (1 + x) = 3x.$$

Thus

$$3R_0^4 x = \frac{q^2}{32\pi^2 \epsilon_0 P_a},$$

and

$$\boxed{\Delta R \simeq \frac{q^2}{96\pi^2 \epsilon_0 P_a R_0^3}}.$$

2.8

Charging increases the displaced air volume while leaving the total bubble mass unchanged.

Before charging, the internal air density equals ρ_a , because surface tension is now neglected and both its pressure and temperature equal the ambient values. Its mass is therefore

$$m_i = \rho_a V_0.$$

The conserved soap mass is

$$m_s = 4\pi R_0^2 t \rho_s.$$

Here t is the original thickness; the film becomes thinner as it expands.

Neutral buoyancy after charging requires

$$\rho_a V_1 g = (\rho_a V_0 + m_s) g.$$

Hence

$$\rho_a \frac{4}{3} \pi (R_1^3 - R_0^3) = 4\pi R_0^2 t \rho_s,$$

or

$$R_1^3 - R_0^3 = \frac{3\rho_s t R_0^2}{\rho_a}.$$

For a small expansion,

$$R_1^3 - R_0^3 \simeq 3R_0^2 \Delta R,$$

so the required radius increase is

$$\Delta R \simeq \frac{\rho_s t}{\rho_a}.$$

Numerically,

$$\Delta R \simeq \frac{1000(1.00 \times 10^{-7})}{1.30} = 7.69 \times 10^{-5} \text{ m}, \quad \frac{\Delta R}{R_0} \simeq 0.00769.$$

Equating this increase to the result of part 2.7 gives

$$\frac{q^2}{96\pi^2 \varepsilon_0 P_a R_0^3} = \frac{\rho_s t}{\rho_a},$$

and therefore

$$|q| \simeq \sqrt{\frac{96\pi^2 \varepsilon_0 P_a \rho_s t R_0^3}{\rho_a}}.$$

Substitution yields

$$|q| \simeq \sqrt{\frac{96\pi^2 (8.85 \times 10^{-12}) (1.013 \times 10^5) (1000) (1.00 \times 10^{-7}) (1.00 \times 10^{-2})^3}{1.30}},$$

so

$$|q| \simeq 2.56 \times 10^{-7} \text{ C} = 0.256 \text{ } \mu\text{C}.$$

The result can also be obtained without linearizing the expansion. Neutral buoyancy gives

$$R_1 = R_0 \left(1 + \frac{3\rho_s t}{\rho_a R_0} \right)^{1/3}.$$

Combining this with part 2.6,

$$q^2 = 32\pi^2 \varepsilon_0 P_a R_1 (R_1^3 - R_0^3) = \frac{96\pi^2 \varepsilon_0 P_a \rho_s t R_0^2 R_1}{\rho_a}.$$

Thus the unexpanded thin-film result is

$$|q| = \sqrt{\frac{96\pi^2 \varepsilon_0 P_a \rho_s t R_0^3}{\rho_a}} \left(1 + \frac{3\rho_s t}{\rho_a R_0} \right)^{1/6} = 2.57 \times 10^{-7} \text{ C}.$$

Either sign of charge produces the same expansion because the electric pressure depends on q^2 .

The solution was checked against the official solution.

Problem 3: To Commemorate the Centenary of Rutherford's Atomic

Nucleus: the Scattering of an Ion by a Neutral Atom

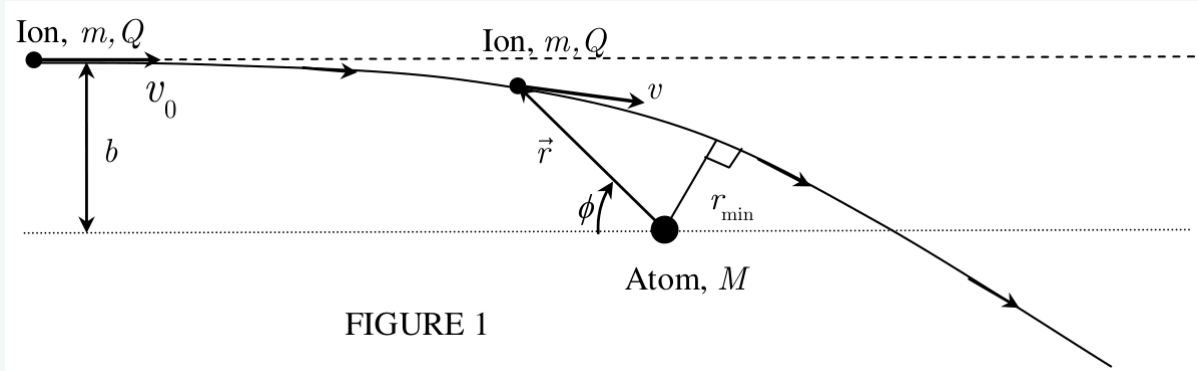


FIGURE 1

FIGURE 1

An ion of mass m , charge Q , is moving with an initial non-relativistic speed v_0 from a great distance towards the vicinity of a neutral atom of mass $M \gg m$ and of electrical polarisability α . The impact parameter is b as shown in Figure 1.

The atom is instantaneously polarised by the electric field $\vec{E}$ of the in-coming (approaching) ion. The resulting electric dipole moment of the atom is $\vec{p} = \alpha \vec{E}$. Ignore any radiative losses in this problem.

3.1 Calculate the electric field intensity $\vec{E}_p$ at a distance r from an ideal electric dipole $\vec{p}$ at the origin O along the direction of $\vec{p}$ in Figure 2. **[1.2 points]**

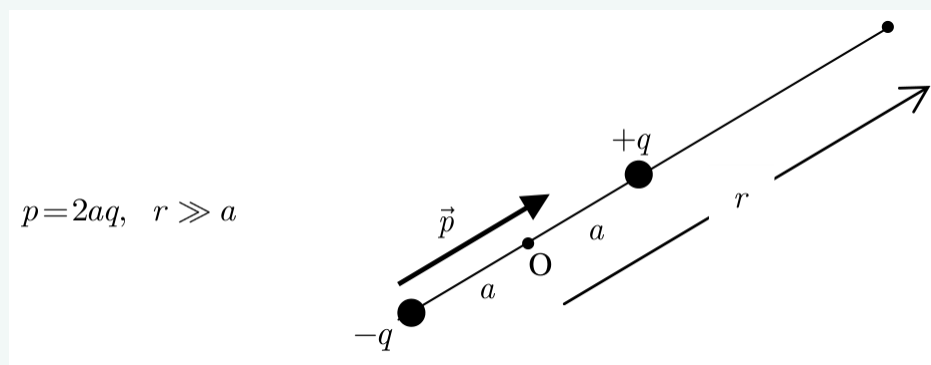


FIGURE 2

3.2 Find the expression for the force $\vec{f}$ acting on the ion due to the polarised atom. Show that this force is attractive regardless of the sign of the charge of the ion. **[3.0 points]**

3.3 What is the electric potential energy of the ion-atom interaction in terms of α , Q and r ? **[0.9 points]**

3.4 Find the expression for $r_{\min}$, the distance of the closest approach, as shown in Figure 1. **[2.4 points]**

3.5 If the impact parameter b is less than a critical value b_0 , the ion will descend along a spiral to the atom. In such a case, the ion will be neutralized and the atom is, in turn, charged. This process is known as the "charge exchange" interaction. What is the cross sectional area $A = \pi b_0^2$ of this "charge exchange" collision of the atom as seen by the ion? **[2.5 points]**

Answers

Let $k = 1/(4\pi\epsilon_0)$, and let $\hat{\mathbf{r}}$ point from the atom towards the ion. Set $U(\infty) = 0$.

- 3.1

$$\vec{E}_p = \frac{2k}{r^3} \vec{p}.$$

- 3.2

$$\vec{f} = -\frac{2\alpha k^2 Q^2}{r^5} \hat{\mathbf{r}}.$$

The force is attractive for either sign of Q .

- 3.3

$$U(r) = -\frac{\alpha k^2 Q^2}{2r^4} = -\frac{\alpha Q^2}{32\pi^2 \epsilon_0^2 r^4}.$$

- 3.4 For a scattered trajectory, $b > b_0$,

$$r_{\min} = \left[\frac{b^2 + \sqrt{b^4 - \frac{4\alpha k^2 Q^2}{mv_0^2}}}{2} \right]^{1/2}.$$

At $b = b_0$, the ion asymptotically approaches $r = b_0/\sqrt{2}$. For $b < b_0$, there is no positive radial turning point.

- 3.5

$$b_0 = \left(\frac{4\alpha k^2 Q^2}{mv_0^2} \right)^{1/4},$$

$$A = \frac{2\pi k |Q|}{v_0} \sqrt{\frac{\alpha}{m}} = \frac{|Q|}{2\epsilon_0 v_0} \sqrt{\frac{\alpha}{m}}.$$

Solution

Since $M \gg m$, we neglect the recoil of the atom and treat its position as fixed. We use the ideal induced-dipole model specified in the question, with $\alpha > 0$.

3.1

Represent the dipole by charges $+q$ and $-q$, separated by $2a$, so that $p = 2aq$. At a point on the axis in the direction of $\vec{p}$, the distances from the two charges are $r - a$ and $r + a$.

The fields of the two charges point in opposite directions, giving

$$\vec{E}_p = kq \left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2} \right] \hat{\mathbf{p}},$$

where $\hat{\mathbf{p}}$ is the unit vector along the dipole moment.

Combining the fractions,

$$\vec{E}_p = \frac{4kqar}{(r^2 - a^2)^2} \hat{\mathbf{p}}.$$

In the ideal-dipole limit $a/r \rightarrow 0$, with $p = 2aq$ fixed,

$$\boxed{\vec{E}_p = \frac{2kp}{r^3} \hat{\mathbf{p}} = \frac{2k}{r^3} \vec{p}.}$$

The field on the opposite side of the dipole axis also points along $\vec{p}$ and has the same magnitude at the same distance.

3.2

Let the ion be at $\vec{r} = r\hat{\mathbf{r}}$ relative to the atom. The electric field produced by the ion at the atom is

$$\vec{E}_{\text{ion}} = -\frac{kQ}{r^2} \hat{\mathbf{r}}.$$

Consequently, the induced dipole moment is

$$\vec{p} = \alpha \vec{E}_{\text{ion}} = -\frac{\alpha kQ}{r^2} \hat{\mathbf{r}}.$$

The ion lies on the axis of this dipole. Using the result of 3.1,

$$\vec{E}_p = \frac{2k}{r^3} \vec{p} = -\frac{2\alpha k^2 Q}{r^5} \hat{\mathbf{r}}.$$

Thus the force on the ion is

$$\boxed{\vec{f} = Q\vec{E}_p = -\frac{2\alpha k^2 Q^2}{r^5} \hat{\mathbf{r}} = -\frac{\alpha Q^2}{8\pi^2 \epsilon_0^2 r^5} \hat{\mathbf{r}}.}$$

Because $\alpha > 0$ and the charge enters as Q^2 , the force always points towards the atom, regardless of the sign of Q .

3.3

For a central conservative force,

$$f_r = -\frac{dU}{dr}.$$

Choosing zero potential energy at infinity,

$$U(r) = -\int_{\infty}^r f_r(s) ds = -\int_{\infty}^r \left(-\frac{2\alpha k^2 Q^2}{s^5} \right) ds.$$

Therefore,

$$\boxed{U(r) = -\frac{\alpha k^2 Q^2}{2r^4}.}$$

Equivalently,

$$U = -\frac{1}{2}\alpha E_{\text{ion}}^2.$$

The factor $1/2$ accounts for the energy required to induce the dipole. Indeed, the polarization energy $p^2/(2\alpha)$, together with the coupling energy $-pE_{\text{ion}}$, gives $-\alpha E_{\text{ion}}^2/2$ when $p = \alpha E_{\text{ion}}$.

3.4

The force is central, so angular momentum about the atom is conserved. Its magnitude is determined by the incoming trajectory:

$$L = mv_0 b.$$

Mechanical energy is also conserved. At infinity, the potential energy vanishes, so

$$\mathcal{E} = \frac{1}{2}mv_0^2.$$

Separating the kinetic energy into radial and transverse contributions gives

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{\alpha k^2 Q^2}{2r^4} = \frac{1}{2}mv_0^2.$$

At the closest approach of a scattered trajectory, $\dot{r} = 0$. Substituting $L = mv_0 b$,

$$\frac{mv_0^2 b^2}{2r_{\min}^2} - \frac{\alpha k^2 Q^2}{2r_{\min}^4} = \frac{1}{2}mv_0^2.$$

Multiplication by $2r_{\min}^4/(mv_0^2)$ gives

$$r_{\min}^4 - b^2 r_{\min}^2 + \frac{\alpha k^2 Q^2}{mv_0^2} = 0.$$

This is a quadratic equation in $r_{\min}^2$, whose roots are

$$r_{\pm}^2 = \frac{b^2 \pm \sqrt{b^4 - \frac{4\alpha k^2 Q^2}{mv_0^2}}}{2}.$$

An ion arriving from infinity encounters the larger root first. Hence

$$r_{\min} = \left[\frac{b^2 + \sqrt{b^4 - \frac{4\alpha k^2 Q^2}{mv_0^2}}}{2} \right]^{1/2}.$$

The two roots are distinct for $b > b_0$, with b_0 determined below. As a check, when the interaction tends to zero, this expression gives $r_{\min} \rightarrow b$, as expected for straight-line motion.

3.5

The radial motion is governed by the effective potential

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{\alpha k^2 Q^2}{2r^4}.$$

For nonzero angular momentum, the positive centrifugal term dominates at large distances, while the attractive term dominates at short distances. The resulting potential barrier can prevent the ion from reaching the atom.

The critical trajectory occurs when the two radial turning points merge. Thus the discriminant found in 3.4 must vanish:

$$b_0^4 - \frac{4\alpha k^2 Q^2}{mv_0^2} = 0.$$

It follows that

$$b_0 = \left(\frac{4\alpha k^2 Q^2}{mv_0^2} \right)^{1/4}.$$

For $b < b_0$, the radial-energy equation

$$\dot{r}^2 = v_0^2 \left(1 - \frac{b^2}{r^2} + \frac{\alpha k^2 Q^2}{mv_0^2 r^4} \right)$$

has no positive zero. The incoming ion therefore continues inward and, under the assumption in the question, undergoes charge exchange.

At exactly $b = b_0$,

$$\dot{r}^2 = v_0^2 \left(1 - \frac{b_0^2}{2r^2} \right)^2.$$

The incoming ion approaches the unstable circular-orbit radius

$$r_c = \frac{b_0}{\sqrt{2}}$$

asymptotically. This boundary has zero area in the impact-parameter plane and does not affect the cross section.

The captured trajectories occupy a disk of radius b_0 in that plane. Therefore,

$$A = \pi b_0^2 = \pi \sqrt{\frac{4\alpha k^2 Q^2}{mv_0^2}},$$

or

$$A = \frac{2\pi k|Q|}{v_0} \sqrt{\frac{\alpha}{m}} = \frac{|Q|}{2\varepsilon_0 v_0} \sqrt{\frac{\alpha}{m}}.$$

With SI inputs, A is in m^2 . No numerical parameter values are supplied.

The solution was checked against the official solution.