

IPhO 2010

Theory problems and solutions · Zagreb, Croatia · 3 theory problems with complete solutions

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Problem 1: Image of a charge in a metallic object

Introduction – Method of images

A point charge q is placed in the vicinity of a grounded metallic sphere of radius R [see Fig. 1(a)], and consequently a surface charge distribution is induced on the sphere. To calculate the electric field and potential from the distribution of the surface charge is a formidable task. However, the calculation can be considerably simplified by using the so called method of images. In this method, the electric field and potential produced by the charge distributed on the sphere can be represented as an electric field and potential of a single point charge q' placed inside the sphere (you do not have to prove it). Note: **The electric field of this image charge q' reproduces the electric field and the potential only outside the sphere (including its surface).**

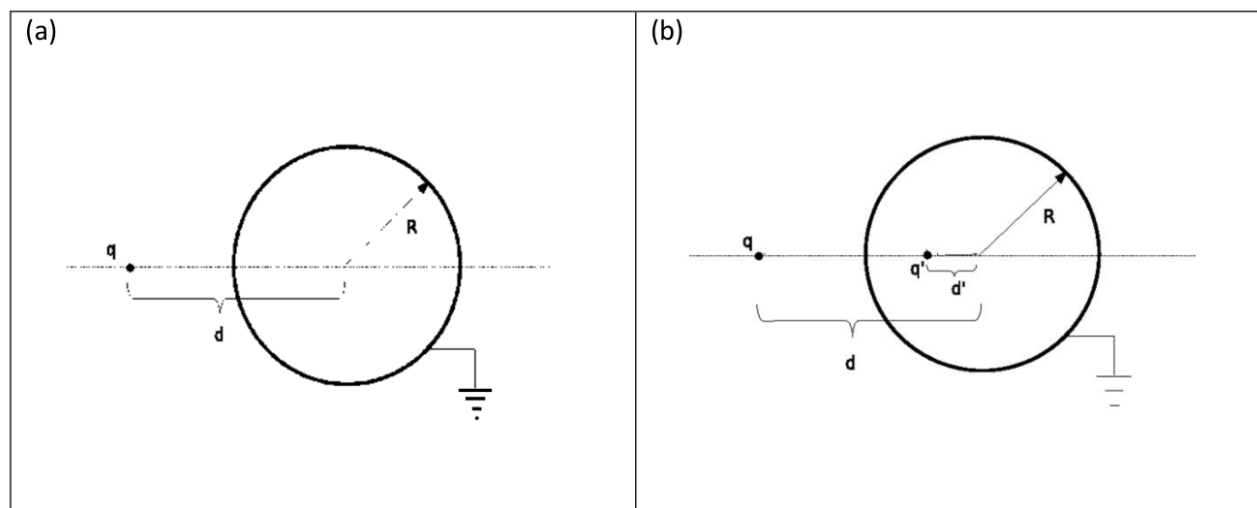


Fig 1. (a) A point charge q in the vicinity of a grounded metallic sphere. (b) The electric field of the charge induced on the sphere can be represented as electric field of an image charge q' .

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Task 1 – The image charge

The symmetry of the problem dictates that the charge q' should be placed on the line connecting the point charge q and the center of the sphere [see Fig. 1(b)].

- What is the value of the potential on the sphere? (0.3 points)
- Express q' and the distance d' of the charge q' from the center of the sphere, in terms of q , d , and R . (1.9 points)
- Find the magnitude of force acting on charge q . Is the force repulsive? (0.5 points)

Task 2 – Shielding of an electrostatic field

Consider a point charge q placed at a distance d from the center of a grounded metallic sphere of radius R . We are interested in how the grounded metallic sphere affects the electric field at point A on the opposite side of the sphere (see Fig. 2). Point A is on the line connecting charge q and the center of the sphere; its distance from the point charge q is r .

- Find the vector of the electric field at point A . (0.6 points)
- For a very large distance $r \gg d$, find the expression for the electric field by using the approximation $(1 + a)^{-2} \approx 1 - 2a$, where $a \ll 1$. (0.6 points)
- In which limit of d does the grounded metallic sphere screen the field of the charge q completely, such that the electric field at point A is exactly zero? (0.3 points)

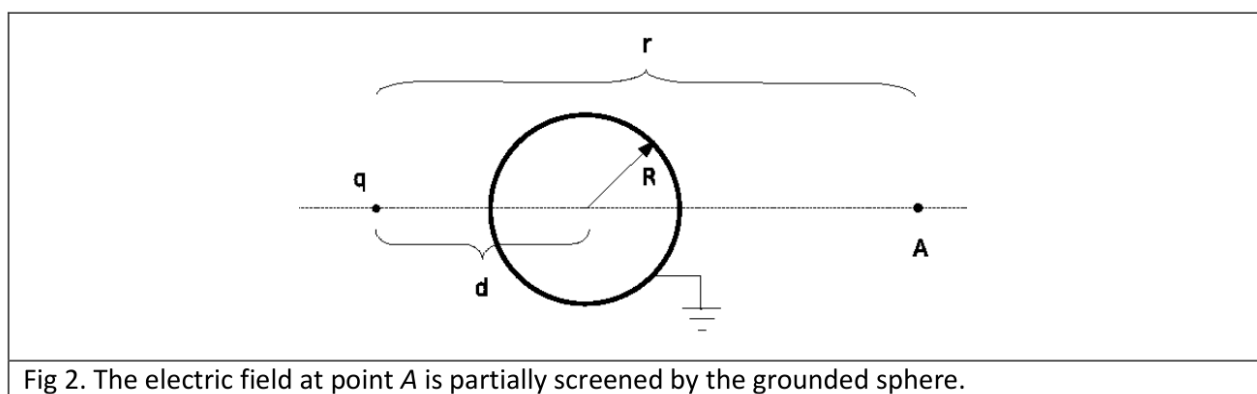


Fig 2. The electric field at point A is partially screened by the grounded sphere.

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Task 3 – Small oscillations in the electric field of the grounded metallic sphere

A point charge q with mass m is suspended on a thread of length L which is attached to a wall, in the vicinity of the grounded metallic sphere. In your considerations, ignore all electrostatic effects of the wall. The point charge makes a mathematical pendulum (see Fig. 3). The point at which the thread is attached to the wall is at a distance l from the center of the sphere. Assume that the effects of gravity are negligible.

- Find the magnitude of the electric force acting on the point charge q for a given angle α and indicate the direction in a clear diagram (0.8 points)
- Determine the component of this force acting in the direction perpendicular to the thread in terms of l , L , R , q and α . (0.8 points)
- Find the frequency for small oscillations of the pendulum. (1.0 points)

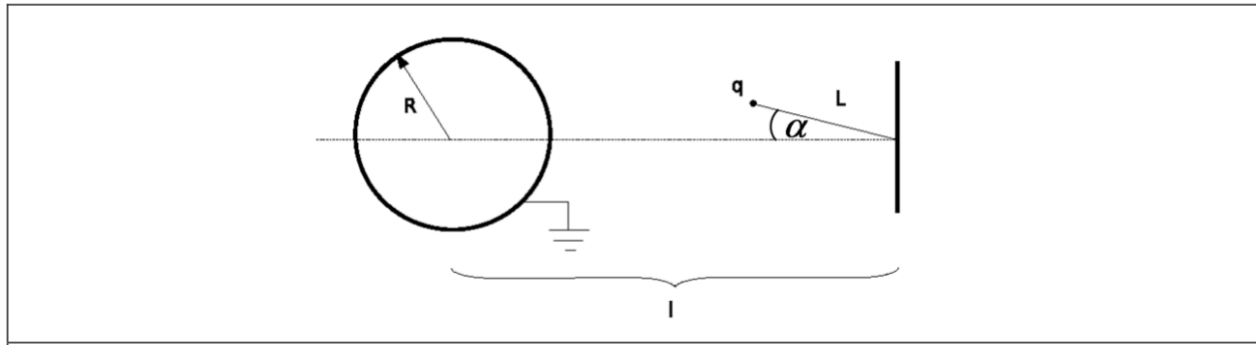


Fig 3. A point charge in the vicinity of a grounded sphere oscillates as a pendulum.

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Task 4 – The electrostatic energy of the system

For a distribution of electric charges it is important to know the electrostatic energy of the system. In our problem (see Fig. 1a), there is an electrostatic interaction between the external charge q and the induced charges on the sphere, and there is an electrostatic interaction among the induced charges on the sphere themselves. In terms of the charge q , radius of the sphere R and the distance d determine the following electrostatic energies:

- the electrostatic energy of the interaction between charge q and the induced charges on the sphere; (1.0 points)
- the electrostatic energy of the interaction among the induced charges on the sphere; (1.2 points)
- the total electrostatic energy of the interaction in the system. (1.0 points)

Hint: There are several ways of solving this problem:

- In one of them, you can use the following integral,

$$\int_d^\infty \frac{x dx}{(x^2 - R^2)^2} = \frac{1}{2} \frac{1}{d^2 - R^2}.$$

- In another one, you can use the fact that for a collection of N charges q_i located at points $\vec{r}_i, i = 1, \dots, N$, the electrostatic energy is a sum over all pairs of charges:

$$V = \frac{1}{2} \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}.$$

Official English text of the problem.

Answers

Write $k = 1/(4\pi\epsilon_0)$. In Task 2, let $\hat{\mathbf{e}}$ point from q toward A . For Task 3, define

$$d_\alpha = \sqrt{l^2 + L^2 - 2lL \cos \alpha}, \quad D_0 = (l - L)^2 - R^2,$$

with $l - L > R$ for the depicted exterior equilibrium. Take the signed perpendicular force as positive toward increasing α .

- 1(a)

$$\phi_{\text{sphere}} = 0.$$

- 1(b)

$$q' = -\frac{R}{d}q, \quad d' = \frac{R^2}{d}.$$

The image lies between the center and the surface nearest q .

- 1(c)

$$F = \frac{kq^2 R d}{(d^2 - R^2)^2}.$$

The force is attractive, toward the sphere's center.

- 2(a)

$$\mathbf{E}_A = kq \left[\frac{1}{r^2} - \frac{R}{d(r - d + R^2/d)^2} \right] \hat{\mathbf{e}}.$$

- 2(b) For $r \gg d$, through the first correction:

$$\mathbf{E}_A \simeq \frac{kq}{r^2} \left[1 - \frac{R}{d} - \frac{2R}{r} \left(1 - \frac{R^2}{d^2} \right) \right] \hat{\mathbf{e}}.$$

The leading term is $kq(1 - R/d)\hat{\mathbf{e}}/r^2$.

- 2(c)

$$d \rightarrow R^+ \implies \mathbf{E}_A \rightarrow \mathbf{0}.$$

- 3(a)

$$F(\alpha) = \frac{kq^2 R \sqrt{l^2 + L^2 - 2lL \cos \alpha}}{(l^2 + L^2 - 2lL \cos \alpha - R^2)^2}.$$

Draw the force arrow from the charge directly toward the sphere's center.

- 3(b)

$$F_{\perp} = -\frac{kq^2 R l \sin \alpha}{(l^2 + L^2 - 2lL \cos \alpha - R^2)^2}.$$

- 3(c) The ordinary frequency and angular frequency are

$$f = \frac{1}{2\pi} \sqrt{\frac{kq^2 R l}{m L D_0^2}}, \quad \omega_0 = 2\pi f.$$

- 4(a) Charge–surface interaction:

$$U_{q\sigma} = -\frac{kq^2 R}{d^2 - R^2}.$$

- **4(b)** Mutual energy of the induced charges:

$$U_{\sigma\sigma} = +\frac{kq^2R}{2(d^2 - R^2)}.$$

- **4(c)** Total interaction energy:

$$U_{\text{tot}} = -\frac{kq^2R}{2(d^2 - R^2)}.$$

Solution

1(a)

A conductor in electrostatic equilibrium is an equipotential. Grounding fixes its potential to that of Earth, taken as zero:

$$\phi_{\text{sphere}} = 0.$$

1(b)

Let O be the center. Measure axial positions from O toward the external charge, so that q is at $d > R$ and the image is at d' , inside the sphere.

At the surface point nearest q , the zero-potential condition gives

$$k\left(\frac{q}{d - R} + \frac{q'}{R - d'}\right) = 0.$$

At the opposite surface point,

$$k\left(\frac{q}{d + R} + \frac{q'}{R + d'}\right) = 0.$$

Consequently,

$$\frac{R - d'}{d - R} = \frac{R + d'}{d + R}.$$

Cross-multiplication yields

$$(R - d')(d + R) = (R + d')(d - R),$$

hence

$$d' = \frac{R^2}{d}.$$

Substituting back,

$$q' = -\frac{R}{d}q.$$

These values also satisfy the boundary condition everywhere on the surface. For a surface point whose radius makes angle θ with the direction toward q , let its distances from q and q' be s and s' . Then

$$s^2 = d^2 + R^2 - 2dR \cos \theta,$$

and

$$s'^2 = d'^2 + R^2 - 2d'R \cos \theta = \frac{R^2}{d^2} s^2.$$

Thus $s' = (R/d)s$, and

$$k \left(\frac{q}{s} + \frac{q'}{s'} \right) = 0.$$

1(c)

At the external charge, the image reproduces the field of the induced charges. The separation between q and its image is

$$d - d' = \frac{d^2 - R^2}{d}.$$

Coulomb's law therefore gives

$$F = \frac{k|qq'|}{(d - d')^2} = \frac{kq^2(R/d)}{[(d^2 - R^2)/d]^2}.$$

Hence

$$F = \frac{kq^2 R d}{(d^2 - R^2)^2}.$$

Since q' has the opposite sign to q , the force is always attractive. It points along the line from q toward O .

2(a)

Choose the positive axis from q toward A , with unit vector $\hat{\mathbf{e}}$. Put the sphere's center at $x = 0$. Then

$$x_q = -d, \quad x_{q'} = -d', \quad x_A = r - d.$$

Because A is outside the sphere on the opposite side, $r > d + R$.

The distances from A to the two charges are

$$r, \quad r - d + d' = r - d + \frac{R^2}{d}.$$

By superposition,

$$\mathbf{E}_A = k \left[\frac{q}{r^2} + \frac{q'}{(r - d + d')^2} \right] \hat{\mathbf{e}}.$$

Therefore

$$\mathbf{E}_A = kq \left[\frac{1}{r^2} - \frac{R}{d(r - d + R^2/d)^2} \right] \hat{\mathbf{e}}.$$

The sign of the scalar coefficient determines the direction along the axis.

2(b)

Define

$$\delta = d - \frac{R^2}{d}.$$

Since $0 < \delta < d$, the condition $r \gg d$ implies $\delta/r \ll 1$. We can write

$$\mathbf{E}_A = \frac{kq}{r^2} \left[1 - \frac{R}{d} \left(1 - \frac{\delta}{r} \right)^{-2} \right] \hat{\mathbf{e}}.$$

Using the supplied approximation with $a = -\delta/r$,

$$\left(1 - \frac{\delta}{r} \right)^{-2} \simeq 1 + \frac{2\delta}{r}.$$

It follows that

$$\mathbf{E}_A \simeq \frac{kq}{r^2} \left[1 - \frac{R}{d} - \frac{2R\delta}{dr} \right] \hat{\mathbf{e}}.$$

Substituting δ ,

$$\mathbf{E}_A \simeq \frac{kq}{r^2} \left[1 - \frac{R}{d} - \frac{2R}{r} \left(1 - \frac{R^2}{d^2} \right) \right] \hat{\mathbf{e}}.$$

To leading order, the distant field is that of the effective net charge

$$q + q' = q \left(1 - \frac{R}{d} \right).$$

2(c)

As the external charge approaches the sphere,

$$d \rightarrow R^+, \quad d' \rightarrow R, \quad q' \rightarrow -q.$$

Thus the charge and its image approach the same position with equal and opposite charges. Their fields cancel at every fixed exterior observation point away from the contact point.

In particular, the exact expression in 2(a) gives

$$\lim_{d \rightarrow R^+} \mathbf{E}_A = \mathbf{0}.$$

This is a limit with the charge approaching the surface from outside.

3(a)

Treat the conductor as remaining in electrostatic equilibrium as the charge moves. Let P be the thread's attachment point and Q the charge's position. Choose

$$O = (0, 0), \quad P = (l, 0).$$

For the angle shown in Fig. 3,

$$Q = (l - L \cos \alpha, L \sin \alpha).$$

Therefore the instantaneous center-to-charge distance is

$$d_\alpha = |OQ| = \sqrt{(l - L \cos \alpha)^2 + L^2 \sin^2 \alpha} = \sqrt{l^2 + L^2 - 2lL \cos \alpha}.$$

Applying the result of 1(c),

$$F(\alpha) = \frac{kq^2 R d_\alpha}{(d_\alpha^2 - R^2)^2} = \frac{kq^2 R \sqrt{l^2 + L^2 - 2lL \cos \alpha}}{(l^2 + L^2 - 2lL \cos \alpha - R^2)^2}.$$

The force points from Q toward O .

Diagram description: Draw the sphere centered at O , with P to its right on a horizontal line and $OP = l$. Place Q above that line and to the left of P , with $PQ = L$ and $\angle OPQ = \alpha$. Draw an arrow starting at Q , along QO , pointing toward O , and label it $\mathbf{F}_e$. Label $OQ = d_\alpha$.

3(b)

A unit vector perpendicular to the thread, in the direction of increasing α , is

$$\hat{\mathbf{e}}_\alpha = (\sin \alpha, \cos \alpha).$$

Since the force points toward O ,

$$\mathbf{F}_e = -\frac{F}{d_\alpha}(l - L \cos \alpha, L \sin \alpha).$$

Its tangential component is

$$\begin{aligned} F_\perp &= \mathbf{F}_e \cdot \hat{\mathbf{e}}_\alpha \\ &= -\frac{F}{d_\alpha}[(l - L \cos \alpha) \sin \alpha + L \sin \alpha \cos \alpha] \\ &= -\frac{Fl \sin \alpha}{d_\alpha}. \end{aligned}$$

Substitution gives

$$F_\perp = -\frac{kq^2 R l \sin \alpha}{(l^2 + L^2 - 2lL \cos \alpha - R^2)^2}.$$

For positive α , this component acts toward decreasing α , so it is restoring. Its magnitude is obtained by replacing $-\sin \alpha$ with $|\sin \alpha|$.

3(c)

At the equilibrium $\alpha = 0$,

$$d_0 = l - L.$$

The depicted equilibrium requires $l - L > R$, so the charge lies outside the sphere.

Tension has no component perpendicular to the thread. Newton's second law in the tangential direction is therefore

$$mL\ddot{\alpha} = F_\perp.$$

For small α ,

$$\sin \alpha \simeq \alpha,$$

and

$$d_\alpha^2 - R^2 = (l - L)^2 - R^2 + lL\alpha^2 + O(\alpha^4).$$

Writing $D_0 = (l - L)^2 - R^2$, the linearized equation becomes

$$mL\ddot{\alpha} = -\frac{kq^2 R l}{D_0^2} \alpha,$$

or

$$\ddot{\alpha} + \omega_0^2 \alpha = 0, \quad \omega_0^2 = \frac{kq^2 Rl}{mLD_0^2}.$$

Thus the ordinary frequency is

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{kq^2 Rl}{mL[(l-L)^2 - R^2]^2}}.$$

The approximation requires sufficiently small amplitude that both

$$|\alpha| \ll 1, \quad lL\alpha^2 \ll D_0$$

hold.

4(a)

Omit the external point charge's self-energy and take the interaction energy to vanish as $d \rightarrow \infty$.

Let ϕ_σ denote the potential produced by the induced surface charges. At the position of q , the image gives

$$\phi_\sigma(q) = \frac{kq'}{d-d'} = -\frac{kqR}{d^2 - R^2}.$$

The interaction energy between q and the induced charges is

$$U_{q\sigma} = q\phi_\sigma(q).$$

There is no factor $1/2$ here: each pair consisting of q and an induced charge is counted once. Hence

$$U_{q\sigma} = -\frac{kq^2 R}{d^2 - R^2}.$$

4(b)

Let σ be the induced surface charge density, and let ϕ_q be the potential due to the external charge. The mutual energy of the induced charges is

$$U_{\sigma\sigma} = \frac{1}{2} \int_S \sigma \phi_\sigma dS.$$

The factor $1/2$ prevents double counting of pairs within the surface distribution.

On the grounded sphere,

$$\phi_q + \phi_\sigma = 0.$$

Therefore

$$U_{\sigma\sigma} = -\frac{1}{2} \int_S \sigma \phi_q dS.$$

The remaining integral equals the charge-surface interaction energy:

$$\int_S \sigma \phi_q dS = kq \int_S \frac{\sigma dS}{|\mathbf{r} - \mathbf{r}_q|} = q\phi_\sigma(q) = U_{q\sigma}.$$

Thus

$$U_{\sigma\sigma} = -\frac{1}{2} U_{q\sigma},$$

giving

$$U_{\sigma\sigma} = +\frac{kq^2R}{2(d^2 - R^2)}.$$

4(c)

Adding the two physical contributions,

$$U_{\text{tot}} = U_{q\sigma} + U_{\sigma\sigma} = \frac{1}{2}U_{q\sigma}.$$

Consequently,

$$U_{\text{tot}} = -\frac{kq^2R}{2(d^2 - R^2)}.$$

This also follows from the work required to bring q quasistatically from infinity:

$$U_{\text{tot}}(d) = -\int_d^\infty \frac{kq^2R x}{(x^2 - R^2)^2} dx = -\frac{kq^2R}{2(d^2 - R^2)}.$$

As a consistency check, the force in the direction of increasing d is

$$-\frac{dU_{\text{tot}}}{dd} = -\frac{kq^2Rd}{(d^2 - R^2)^2},$$

which reproduces the attractive force found in 1(c).

The solution was checked against the official solution.

Problem 2: Chimney physics

Introduction

Gaseous products of burning are released into the atmosphere of temperature T_{Air} through a high chimney of cross-section A and height h (see Fig. 1). The solid matter is burned in the furnace which is at temperature T_{Smoke} . The volume of gases produced per unit time in the furnace is B .

Assume that:

- the velocity of the gases in the furnace is negligibly small
- the density of the gases (smoke) does not differ from that of the air at the same temperature and pressure; while in furnace, the gases can be treated as ideal
- the pressure of the air changes with height in accordance with the hydrostatic law; the change of the density of the air with height is negligible
- the flow of gases fulfills the Bernoulli equation which states that the following quantity is conserved in all points of the flow:

$$\frac{1}{2}\rho v^2(z) + \rho g z + p(z) = \text{const},$$

where ρ is the density of the gas, $v(z)$ is its velocity, $p(z)$ is pressure, and z is the height - the change of the density of the gas is negligible throughout the chimney

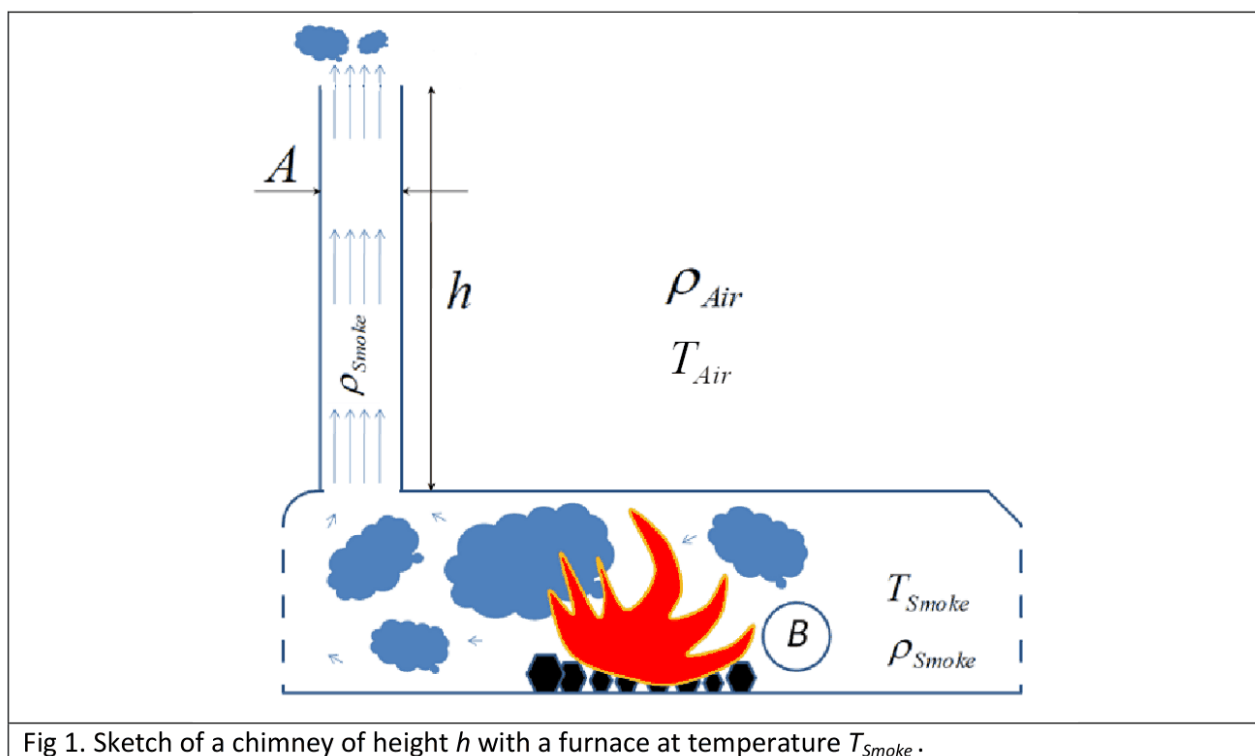


Fig 1. Sketch of a chimney of height h with a furnace at temperature T_{Smoke} .

Task 1

- What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas into the atmosphere? Express your result in terms of B , A , T_{Air} , $g = 9.81 \text{ m/s}^2$, $\Delta T = T_{Smoke} - T_{Air}$. **Important: in all subsequent tasks assume that this minimal height is the height of the chimney.** (3.5 points)
- Assume that two chimneys are built to serve exactly the same purpose. Their cross sections are identical, but are designed to work in different parts of the world: one in cold regions, designed to work at an average atmospheric temperature of -30°C and the other in warm regions, designed to work at an average atmospheric temperature of 30°C . The temperature of the furnace is 400°C . It was calculated that the height of the chimney designed to work in cold regions is 100 m. How high is the other chimney? (0.5 points)
- How does the velocity of the gases vary along the height of the chimney? Make a sketch/diagram assuming that the chimney cross-section does not change along the height. Indicate the point where the gases enter the chimney. (0.6 points)
- How does the pressure of the gases vary along the height of the chimney? (0.5 points)

Solar power plant

The flow of gases in a chimney can be used to construct a particular kind of solar power plant (solar chimney). The idea is illustrated in Fig. 2. The Sun heats the air underneath the collector of area S with an open periphery to allow the undisturbed inflow of air (see Fig. 2). As the heated air rises through the chimney (thin solid arrows), new cold air enters the collector from its

surrounding (thick dotted arrows) enabling a continuous flow of air through the power plant. The flow of air through the chimney powers a turbine, resulting in the production of electrical energy. The energy of solar radiation per unit time per unit of horizontal area of the collector is G . Assume that all that energy can be used to heat the air in the collector (the mass heat capacity of the air is c , and one can neglect its dependence on the air temperature). We define the efficiency of the solar chimney as the ratio of the kinetic energy of the gas flow and the solar energy absorbed in heating of the air prior to its entry into the chimney.

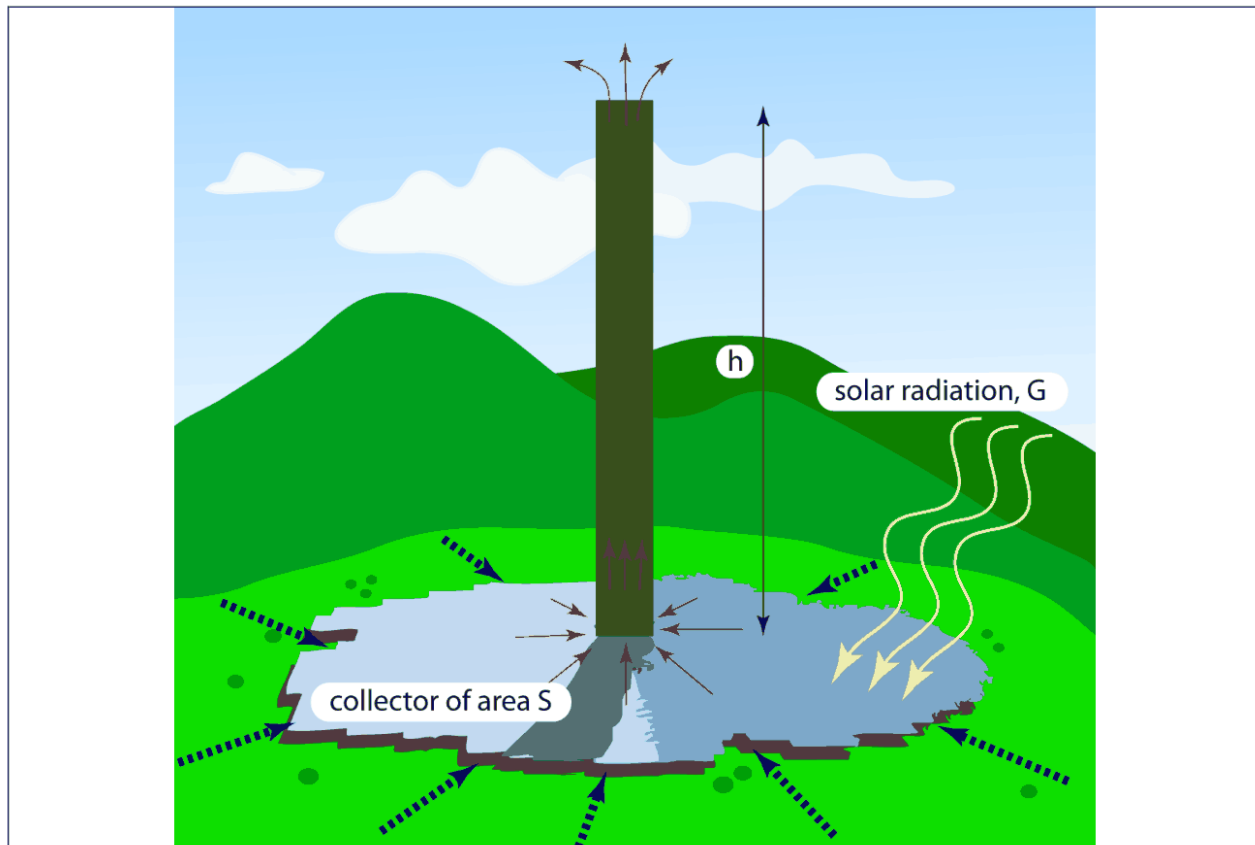


Fig 2. Sketch of a solar power plant.

Fig 2. Sketch of a solar power plant.

Task 2

- What is the efficiency of the solar chimney power plant? (2.0 points)
- Make a diagram showing how the efficiency of the chimney changes with its height. (0.4 points)

Manzanares prototype

The prototype chimney built in Manzanares, Spain, had a height of 195 m, and a radius 5 m. The collector is circular with diameter of 244 m. The specific heat of the air under typical operational conditions of the prototype solar chimney is 1012 J/kg K, the density of the hot air is about 0.9 kg/m³, and the typical temperature of the atmosphere $T_{Air} = 295$ K. In Manzanares, the solar power per unit of horizontal surface is typically 150 W/m² during a sunny day.

Task 3

- What is the efficiency of the prototype power plant? Write down the numerical estimate. (0.3 points)

- b) How much power could be produced in the prototype power plant? (0.4 points)
- c) How much energy could the power plant produce during a typical sunny day? (0.3 points)

Task 4

- a) How large is the rise in the air temperature as it enters the chimney (warm air) from the surrounding (cold air)? Write the general formula and evaluate it for the prototype chimney. (1.0 points)
- b) What is the mass flow rate of air through the system? (0.5 points)

Official English text of the problem.

Answers

Take $z = 0$ at the chimney entrance and $z = h$ at its outlet. Below, ρ_h denotes the density of the smoke or heated air.

- **1(a)** For $\Delta T > 0$,

$$h_{\min} = \frac{B^2 T_{Air}}{2gA^2 \Delta T}.$$

- **1(b)**

$$h_{\text{warm}} \simeq 145 \text{ m}.$$

- **1(c)**

$$v(z) = \frac{B}{A} \quad (0 < z < h).$$

The velocity rises from nearly zero in the furnace to B/A in the entrance region, then remains constant throughout the chimney.

- **1(d)**

$$p(z) = p_{Air}(h) + \rho_h g(h - z).$$

The pressure decreases linearly with height, with slope $-\rho_h g$, reaching the external atmospheric pressure at the outlet.

- **2(a)**

$$\eta = \frac{gh}{cT_{Air}}.$$

- **2(b)** The graph of η against h is a straight line through the origin, with slope $g/(cT_{Air})$.

- **3(a)**

$$\eta = 0.00641 \simeq 0.64\%.$$

- **3(b)** The ideal available power is

$$P \simeq 4.49 \times 10^4 \text{ W} \simeq 45 \text{ kW}.$$

- **3(c)** For an effective illumination time τ ,

$$E = P\tau.$$

For 12 h at the stated irradiance, $E \simeq 0.54 \text{ MWh} = 1.9 \text{ GJ}$. If the stated irradiance is a full-day average, the 24 h result is $1.08 \text{ MWh} = 3.88 \text{ GJ}$.

- **4(a)**

$$\Delta T = \left[\frac{(GS)^2 T_{Air}}{2gh \rho_h^2 A^2 c^2} \right]^{1/3} \simeq 9.1 \text{ K}.$$

- **4(b)**

$$\dot{m} = \frac{GS}{c\Delta T} = \left[\frac{2gh(\rho_h A)^2 GS}{cT_{Air}} \right]^{1/3} \simeq 7.7 \times 10^2 \text{ kg s}^{-1}.$$

Solution

Write $\rho_a = \rho_{Air}$, $\rho_h = \rho_{Smoke}$, and $p_0 = p_{Air}(0)$. For the solar chimney, ρ_h instead denotes the heated-air density. Absolute temperatures in all formulas are in kelvin.

1(a)

The chimney operates by natural draft: the external air is denser than the hot gas, so its pressure falls more rapidly with height. At the limiting operating condition, the nearly stationary gas in the furnace is at the atmospheric pressure p_0 . This is the boundary condition for discharging the gas without requiring furnace overpressure.

By the hydrostatic law outside the chimney,

$$p_{Air}(h) = p_0 - \rho_a gh.$$

At the open outlet, the gas pressure equals the local atmospheric pressure. Bernoulli's equation between the furnace, where the velocity is negligible, and the outlet gives

$$p_0 = p_{Air}(h) + \rho_h gh + \frac{1}{2} \rho_h v^2.$$

Consequently,

$$\frac{1}{2} \rho_h v^2 = (\rho_a - \rho_h) gh.$$

The smoke and air have the same density at equal temperature and pressure. Applying the ideal-gas law at the common base pressure therefore gives

$$\frac{\rho_a}{\rho_h} = \frac{T_{Smoke}}{T_{Air}} = 1 + \frac{\Delta T}{T_{Air}}.$$

Thus

$$v^2 = 2gh \frac{\Delta T}{T_{Air}}.$$

Because the density change between the furnace and chimney is neglected, the volumetric discharge capacity is Av . Releasing all the gas requires

$$Av \geq B.$$

At the minimum height, equality holds:

$$\left(\frac{B}{A}\right)^2 = 2gh_{\min} \frac{\Delta T}{T_{Air}}.$$

Hence

$$h_{\min} = \frac{B^2 T_{Air}}{2gA^2 \Delta T}.$$

This result requires $\Delta T > 0$. For $B > 0$ and $\Delta T = 0$, no finite height produces the required natural draft under these assumptions.

1(b)

Both chimneys have the same B , A , and furnace temperature, so

$$h \propto \frac{T_{Air}}{T_{Smoke} - T_{Air}}.$$

The relevant temperatures and temperature differences are

$$\begin{aligned} T_c &= 243.15 \text{ K}, & T_w &= 303.15 \text{ K}, \\ \Delta T_c &= 430 \text{ K}, & \Delta T_w &= 370 \text{ K}. \end{aligned}$$

Therefore,

$$h_w = h_c \frac{T_w}{T_c} \frac{\Delta T_c}{\Delta T_w} = (100 \text{ m}) \frac{303.15}{243.15} \frac{430}{370}.$$

This gives

$$h_w = 144.9 \text{ m} \simeq 145 \text{ m}.$$

The warmer surroundings provide a smaller relative density difference, requiring a taller chimney.

1(c)

Conservation of mass in steady flow gives

$$\dot{m} = \rho_h A v(z).$$

Since both ρ_h and A are constant throughout the chimney, $v(z)$ must also be constant. With the stipulated minimum height, the volume flow is exactly B , so

$$v(z) = \frac{B}{A} = \sqrt{\frac{2gh\Delta T}{T_{Air}}}.$$

For the requested sketch, put z on the horizontal axis and v on the vertical axis. Mark the chimney entrance at $z = 0$. Show a rapid rise from the negligible furnace velocity to B/A in the entrance region, followed by a horizontal line at $v = B/A$ from the entrance to $z = h$. If the entrance region is idealized as having negligible extent, its acceleration can be represented by a step at $z = 0$.

1(d)

Within the chimney, Bernoulli's equation is

$$\frac{1}{2} \rho_h v^2 + \rho_h g z + p(z) = \text{constant}.$$

Since v is constant,

$$\frac{dp}{dz} = -\rho_h g.$$

Using the outlet boundary condition $p(h) = p_{Air}(h)$,

$$\boxed{p(z) = p_{Air}(h) + \rho_h g(h - z)}.$$

Equivalently,

$$p(z) = p_0 - \rho_a g h + \rho_h g(h - z).$$

The exterior pressure at height z is $p_{Air}(z) = p_0 - \rho_a g z$. Subtracting,

$$\boxed{p(z) - p_{Air}(z) = -(\rho_a - \rho_h)g(h - z)}.$$

Thus the gas pressure is below the surrounding atmospheric pressure everywhere inside the chimney below its outlet, and the two pressures become equal at the outlet.

Immediately after the entrance,

$$p(0^+) = p_0 - (\rho_a - \rho_h)gh = p_0 - \frac{1}{2}\rho_h v^2.$$

This entrance pressure reduction accompanies the acceleration from the furnace into the chimney. Thereafter, the pressure decreases linearly with height.

2(a)

Consider a mass m of air heated by ΔT in the collector. The absorbed heat is

$$Q = mc\Delta T.$$

Its kinetic energy in the chimney is

$$K = \frac{1}{2}mv^2.$$

According to the definition in the question,

$$\eta = \frac{K}{Q} = \frac{v^2}{2c\Delta T}.$$

Applying the result of 1(a),

$$v^2 = \frac{2gh\Delta T}{T_{Air}},$$

gives

$$\boxed{\eta = \frac{gh}{cT_{Air}}}.$$

Within the stated model, the efficiency is independent of the temperature rise, collector area, and chimney cross-section. It measures the kinetic power available relative to the absorbed solar power.

2(b)

At fixed T_{Air} and c ,

$$\eta(h) = \frac{g}{cT_{Air}}h.$$

The requested diagram has h on the horizontal axis and η on the vertical axis. It is a straight line through $(0, 0)$, with positive slope

$$\boxed{\frac{d\eta}{dh} = \frac{g}{cT_{Air}}}.$$

Doubling the height doubles the efficiency. This linear relation applies only within the approximations of the problem; it should not be extrapolated to heights where changes of density become significant.

3(a)

Substituting the prototype data,

$$\eta = \frac{(9.81 \text{ m s}^{-2})(195 \text{ m})}{(1012 \text{ J kg}^{-1} \text{ K}^{-1})(295 \text{ K})} = 0.0064077.$$

Therefore,

$$\boxed{\eta \simeq 0.64\%}.$$

3(b)

The collector radius is $244/2 = 122 \text{ m}$, so its area is

$$S = \pi(122 \text{ m})^2 = 4.67595 \times 10^4 \text{ m}^2.$$

All incident solar power is assumed to heat the air:

$$P_{\text{solar}} = GS = (150 \text{ W m}^{-2})(4.67595 \times 10^4 \text{ m}^2) = 7.01392 \times 10^6 \text{ W}.$$

The available kinetic power is

$$P = \eta GS = (0.0064077)(7.01392 \times 10^6 \text{ W}),$$

hence

$$\boxed{P = 4.4943 \times 10^4 \text{ W} \simeq 45 \text{ kW}}.$$

This is the ideal power available for conversion under the problem's definition; no additional turbine or generator efficiency is specified.

3(c)

For an effective illumination time τ at the stated irradiance,

$$\boxed{E = P\tau = \frac{ghGS}{cT_{Air}}\tau}.$$

Thus each hour at the stated conditions supplies approximately 44.9 kWh.

If a typical sunny day is estimated as 12 h at the stated irradiance,

$$E = (44.943 \text{ kW})(12 \text{ h}) = 539.3 \text{ kWh},$$

or

$$\boxed{E \simeq 0.54 \text{ MWh} \simeq 1.9 \times 10^9 \text{ J}}.$$

If 150 W m^{-2} is instead interpreted as an average over the entire 24 h day,

$$\boxed{E = (44.943 \text{ kW})(24 \text{ h}) \simeq 1.08 \text{ MWh} = 3.88 \times 10^9 \text{ J}}.$$

The statement does not specify the effective illumination time or averaging convention.

4(a)

In steady operation, the solar heating power equals the rate at which heat is carried away by the air:

$$GS = \dot{m} c \Delta T.$$

Mass conservation gives

$$\dot{m} = \rho_h A v.$$

Combining these with the chimney speed,

$$v = \sqrt{\frac{2gh\Delta T}{T_{Air}}},$$

we obtain

$$GS = \rho_h A c \Delta T \sqrt{\frac{2gh\Delta T}{T_{Air}}} = \rho_h A c \sqrt{\frac{2gh}{T_{Air}}} (\Delta T)^{3/2}.$$

Solving for ΔT ,

$$\Delta T = \left[\frac{(GS)^2 T_{Air}}{2gh \rho_h^2 A^2 c^2} \right]^{1/3}.$$

For the prototype,

$$A = \pi(5 \text{ m})^2 = 78.5398 \text{ m}^2.$$

Using the given hot-air density,

$$\Delta T = \left[\frac{(7.01392 \times 10^6)^2 (295)}{2(9.81)(195)(0.9)^2 (78.5398)^2 (1012)^2} \right]^{1/3} \text{ K}.$$

Therefore,

$$\Delta T = 9.05 \text{ K} \simeq 9.1 \text{ K}.$$

The air enters the chimney at approximately 304 K.

4(b)

From the heating balance,

$$\dot{m} = \frac{GS}{c \Delta T}.$$

Numerically,

$$\dot{m} = \frac{7.01392 \times 10^6 \text{ W}}{(1012 \text{ J kg}^{-1} \text{ K}^{-1})(9.0503 \text{ K})} = 765.8 \text{ kg s}^{-1}.$$

Thus

$$\dot{m} \simeq 7.7 \times 10^2 \text{ kg s}^{-1}.$$

Eliminating ΔT also gives the general expression

$$\dot{m} = \left[\frac{2gh(\rho_h A)^2 GS}{cT_{Air}} \right]^{1/3}.$$

As a numerical check, the corresponding speed is

$$v = \sqrt{\frac{2(9.81)(195)(9.0503)}{295}} = 10.83 \text{ m s}^{-1},$$

which gives $\rho_h A v = 765.8 \text{ kg s}^{-1}$ and kinetic power $\frac{1}{2} \dot{m} v^2 = 44.94 \text{ kW}$, consistent with 3(b).

The solution was checked against the official solution.

Problem 3: Simple model of an atomic nucleus

Introduction

Although atomic nuclei are quantum objects, a number of phenomenological laws for their basic properties (like radius or binding energy) can be deduced from simple assumptions: (i) nuclei are built from nucleons (i.e. protons and neutrons); (ii) strong nuclear interaction holding these nucleons together has a very short range (it acts only between neighboring nucleons); (iii) the number of protons (Z) in a given nucleus is approximately equal to the number of neutrons (N), i.e. $Z \approx N \approx A/2$, where A is the total number of nucleons ($A \gg 1$). **Important: Use these assumptions in Tasks 1-4 below.**

Task 1 - Atomic nucleus as closely packed system of nucleons

In a simple model, an atomic nucleus can be thought of as a ball consisting of closely packed nucleons [see Fig. 1(a)], where the nucleons are hard balls of radius $r_N = 0.85 \text{ fm}$ ($1 \text{ fm} = 10^{-15} \text{ m}$). The nuclear force is present only for two nucleons in contact. The volume of the nucleus V is larger than the volume of all nucleons AV_N , where $V_N = \frac{4}{3}r_N^3\pi$. The ratio $f = AV_N/V$ is called the packing factor and gives the percentage of space filled by the nuclear matter.

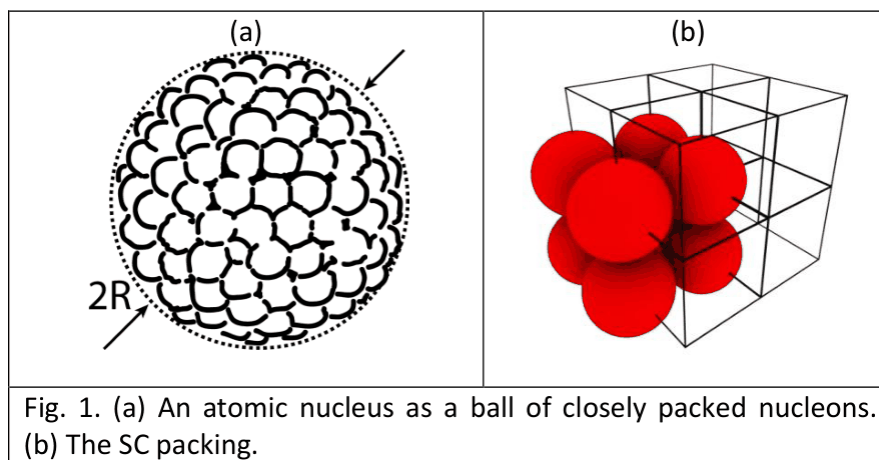


Fig. 1. (a) An atomic nucleus as a ball of closely packed nucleons. (b) The SC packing.

a) Calculate what would be the packing factor f if nucleons were arranged in a "simple cubic" (SC) crystal system, where each nucleon is centered on a lattice point of an infinite cubic lattice [see Fig. 1(b)]. (0.3 points)

Important: In all subsequent tasks, assume that the actual packing factor for nuclei is equal to the one from Task 1a. If you are not able to calculate it, in subsequent tasks use $f = 1/2$.

b) Estimate the average mass density ρ_m , charge density ρ_c , and the radius R for a nucleus having A nucleons. The average mass of a nucleon is $1.67 \cdot 10^{-27}$ kg. (1.0 points)

Task 2 - Binding energy of atomic nuclei – volume and surface terms

Binding energy of a nucleus is the energy required to disassemble it into separate nucleons and it essentially comes from the attractive nuclear force of each nucleon with its neighbors. If a given nucleon is not on the surface of the nucleus, it contributes to the total binding energy with $a_V = 15.8$ MeV ($1 \text{ MeV} = 1.602 \cdot 10^{-13}$ J). The contribution of one surface nucleon to the binding energy is approximately $a_V/2$. Express the binding energy E_b of a nucleus with A nucleons in terms of A , a_V , and f , and by including the surface correction. (1.9 points)

Task 3 - Electrostatic (Coulomb) effects on the binding energy

The electrostatic energy of a homogeneously charged ball (with radius R and total charge Q_0) is

$$U_c = \frac{3Q_0^2}{20\pi\epsilon_0 R},$$

where $\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2\text{N}^{-1}\text{m}^{-2}$.

a) Apply this formula to get the electrostatic energy of a nucleus. In a nucleus, each proton is not acting upon itself (by Coulomb force), but only upon the rest of the protons. One can take this into account by replacing $Z^2 \rightarrow Z(Z - 1)$ in the obtained formula. Use this correction in subsequent tasks. (0.4 points)

b) Write down the complete formula for binding energy, including the main (volume) term, the surface correction term and the obtained electrostatic correction. (0.3 points)

Task 4 - Fission of heavy nuclei

Fission is a nuclear process in which a nucleus splits into smaller parts (lighter nuclei). Suppose that a nucleus with A nucleons splits into only two equal parts as depicted in Fig. 2.

a) Calculate the total kinetic energy of the fission products E_{kin} when the center of two lighter nuclei are separated by the distance $d \geq 2R(A/2)$, where $R(A/2)$ is their radius. The large nucleus was initially at rest. (1.3 points)

b) Assume that $d = 2R(A/2)$ and evaluate the expression for E_{kin} obtained in part a) for $A = 100, 150, 200$ and 250 (express the results in units of MeV). Estimate the values of A for which fission is possible in the model described above? (1.0 points)

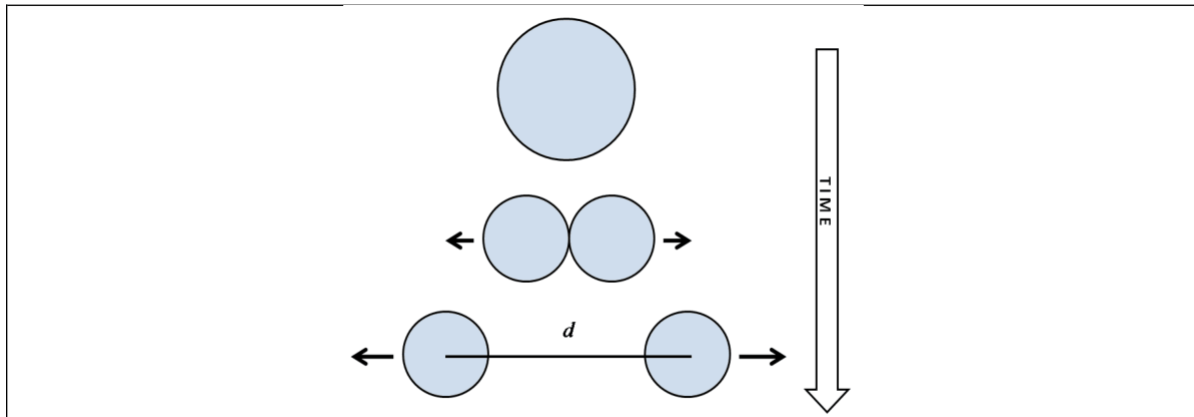


Fig. 2. A schematic description of nuclear fission in our model.

Fig. 2. A schematic description of nuclear fission in our model.

Task 5 – Transfer reactions

a) In modern physics, the energetics of nuclei and their reactions is described in terms of masses. For example, if a nucleus (with zero velocity) is in an excited state with energy E_{exc} above the ground state, its mass is $m = m_0 + E_{\text{exc}}/c^2$, where m_0 is its mass in the ground state at rest. The nuclear reaction $^{16}\text{O} + ^{54}\text{Fe} \rightarrow ^{12}\text{C} + ^{58}\text{Ni}$ is an example of the so-called “transfer reactions”, in which a part of one nucleus (“cluster”) is transferred to the other (see Fig. 3). In our example the transferred part is a ^4He -cluster (α -particle). The transfer reactions occur with maximum probability if the velocity of the projectile-like reaction product (in our case: ^{12}C) is equal both in magnitude and direction to the velocity of projectile (in our case: ^{16}O). The target ^{54}Fe is initially at rest. In the reaction, ^{58}Ni is excited into one of its higher-lying states. Find the excitation energy of that state (and express it units of MeV) if the kinetic energy of the projectile ^{16}O is 50 MeV. The speed of light is $c = 3 \cdot 10^8 \text{ m/s}$. (2.2 points)

1.	$M(^{16}\text{O})$	15.99491 a.m.u.
2.	$M(^{54}\text{Fe})$	53.93962 a.m.u.
3.	$M(^{12}\text{C})$	12.00000 a.m.u.
4.	$M(^{58}\text{Ni})$	57.93535 a.m.u.

Table 1. The rest masses of the reactants in their ground states. $1 \text{ a. m. u.} = 1.6605 \cdot 10^{-27} \text{ kg}$.

b) The ^{58}Ni nucleus produced in the excited state discussed in the part a), deexcites into its ground state by emitting a gamma-photon in the direction of its motion. Consider this decay in the frame of reference in which ^{58}Ni is at rest to find the recoil energy of ^{58}Ni (i.e. kinetic energy which ^{58}Ni acquires after the emission of the photon). What is the photon energy in that system? What is the photon energy in the lab system of reference (i.e. what would be the energy of the photon measured in the detector which is positioned in the direction in which the ^{58}Ni nucleus moves)? (1.6 points)

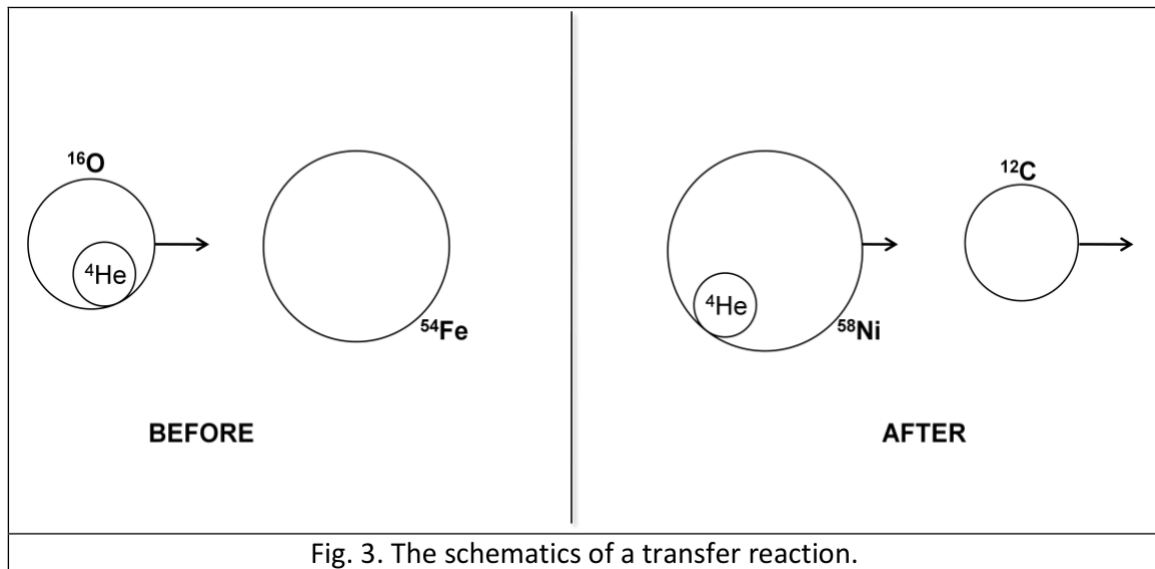


Fig. 3. The schematics of a transfer reaction.

Fig. 3. The schematics of a transfer reaction.

Official English text of the problem.

Answers

For compactness, define

$$S(A) = A - \left(A^{1/3} - 2f^{1/3}\right)^3, \quad r_0 = r_N f^{-1/3} = 1.0546 \text{ fm},$$

and

$$a_C = \frac{3e^2}{20\pi\epsilon_0 r_0} = 0.81955 \text{ MeV}.$$

- 1(a).

$$f = \frac{\pi}{6} \approx 0.5236 = 52.36\%.$$

- 1(b).

$$\rho_m \approx 3.40 \times 10^{17} \text{ kg m}^{-3}, \quad \rho_c \approx 1.63 \times 10^{25} \text{ C m}^{-3}$$

$$R = r_N (A/f)^{1/3} \approx 1.05 A^{1/3} \text{ fm}.$$

- 2.

$$E_b^{(\text{nuc})} = a_V \left[A - \frac{S(A)}{2} \right] = a_V \left[A - 3f^{1/3} A^{2/3} + 6f^{2/3} A^{1/3} - 4f \right].$$

- 3(a).

$$U_c = \frac{a_C Z(Z-1)}{A^{1/3}} \simeq \frac{a_C}{4} \left(A^{5/3} - 2A^{2/3} \right).$$

- **3(b).** Writing the complete binding energy as $B(A)$,

$$B(A) = a_V \left[A - 3f^{1/3} A^{2/3} + 6f^{2/3} A^{1/3} - 4f \right] - \frac{a_C}{4} \left(A^{5/3} - 2A^{2/3} \right).$$

- **4(a).**

$$E_{\text{kin}}(d) = 2B(A/2) - B(A) - \frac{e^2 A^2}{64\pi\epsilon_0 d}.$$

- **4(b).** At contact, the calculated energies for $A = 100, 150, 200, 250$ are, respectively,

$$-29.3, \quad -24.9, \quad -6.77, \quad +23.6 \text{ MeV}.$$

The heavy-nucleus threshold is approximately $A_{\text{crit}} \simeq 213$. Of the four examples, only $A = 250$ permits fission from rest through the contact configuration.

- **5(a).**

$$E_{\text{exc}} \approx 10.86 \text{ MeV}.$$

- **5(b).**

$$T_{\text{recoil}} \approx 1.09 \text{ keV}, \quad E_{\gamma}^* \approx 10.86 \text{ MeV}, \quad E_{\gamma, \text{lab}} \approx 10.92 \text{ MeV}.$$

Here the star denotes the rest frame of the excited nickel nucleus before emission.

Solution

1(a). Packing factor

In simple cubic packing, neighboring spheres touch along a cube edge, so the lattice spacing is $2r_N$. A cubic cell contains eight corner nucleons, each shared by eight cells: one complete nucleon per cell.

Consequently,

$$f = \frac{\text{nucleon volume per cell}}{\text{cell volume}} = \frac{\frac{4}{3}\pi r_N^3}{(2r_N)^3} = \frac{\pi}{6}.$$

1(b). Densities and nuclear radius

The packing-factor definition gives

$$V = \frac{AV_N}{f} = \frac{4\pi r_N^3 A}{3f}.$$

Equating this to $4\pi R^3/3$,

$$R = r_N \left(\frac{A}{f} \right)^{1/3} = r_0 A^{1/3}, \quad r_0 = 1.0546 \text{ fm}.$$

At the accuracy of this model, the nuclear mass is Am_N , where $m_N = 1.67 \times 10^{-27} \text{ kg}$. Thus

$$\rho_m = \frac{Am_N}{V} = \frac{3fm_N}{4\pi r_N^3}.$$

With $f = \pi/6$,

$$\rho_m = \frac{m_N}{8r_N^3} = 3.40 \times 10^{17} \text{ kg m}^{-3}.$$

Using $Z \simeq A/2$, the nuclear charge is approximately $Ae/2$. The elementary charge consistent with the given electron-volt conversion is $e = 1.602 \times 10^{-19} \text{ C}$. Therefore,

$$\rho_c = \frac{Ae/2}{V} = \frac{3fe}{8\pi r_N^3} = \frac{e}{16r_N^3},$$

so

$$\rho_c = 1.63 \times 10^{25} \text{ C m}^{-3}.$$

Both average densities are independent of A .

2. Volume and full surface-layer contributions

The nucleon number density is

$$n = \frac{A}{V} = \frac{f}{V_N}.$$

Using the specified surface layer of thickness $2r_N$, its volume is

$$V_{\text{surface}} = \frac{4\pi}{3} [R^3 - (R - 2r_N)^3].$$

Within the continuum packing model, the number of surface nucleons is therefore

$$\begin{aligned} S(A) &= \frac{f}{r_N^3} [R^3 - (R - 2r_N)^3] \\ &= A - f \left[\left(\frac{A}{f} \right)^{1/3} - 2 \right]^3 \\ &= \boxed{A - \left(A^{1/3} - 2f^{1/3} \right)^3}. \end{aligned}$$

This expression applies when $R \geq 2r_N$, as is appropriate to the large nuclei considered here. Expanding without discarding any term,

$$S(A) = 6f^{1/3}A^{2/3} - 12f^{2/3}A^{1/3} + 8f.$$

There are $A - S(A)$ interior nucleons contributing a_V each, and $S(A)$ surface nucleons contributing $a_V/2$ each. Hence

$$E_b^{(\text{nuc})} = a_V[A - S(A)] + \frac{a_V}{2}S(A),$$

or

$$E_b^{(\text{nuc})} = a_V \left[A - 3f^{1/3}A^{2/3} + 6f^{2/3}A^{1/3} - 4f \right].$$

Numerically,

$$\frac{E_b^{(\text{nuc})}}{\text{MeV}} = 15.8A - 38.2042A^{2/3} + 61.5849A^{1/3} - 33.0914.$$

The $A^{1/3}$ and constant terms are retained in all subsequent calculations.

3(a). Coulomb self-energy

Substituting $Q_0 = Ze$ into the supplied formula, and then applying the prescribed self-interaction correction, gives

$$U_c = \frac{3e^2 Z(Z-1)}{20\pi\epsilon_0 R}.$$

Since $R = r_0 A^{1/3}$, define

$$a_C = \frac{3e^2}{20\pi\epsilon_0 r_0}.$$

The supplied constants give

$$\frac{e^2}{4\pi\epsilon_0} = 1.44049 \text{ MeV fm}, \quad a_C = 0.81955 \text{ MeV}.$$

Thus

$$U_c = \frac{a_C Z(Z-1)}{A^{1/3}}.$$

Applying $Z \simeq A/2$ while retaining the subtraction of one,

$$Z(Z-1) \simeq \frac{A(A-2)}{4},$$

and therefore

$$U_c(A) = \frac{a_C}{4} (A^{5/3} - 2A^{2/3}).$$

3(b). Complete binding energy

The positive electrostatic energy reduces the energy required to disassemble the nucleus. Therefore,

$$B(A) = E_b^{(\text{nuc})}(A) - U_c(A),$$

which gives

$$B(A) = a_V \left[A - 3f^{1/3} A^{2/3} + 6f^{2/3} A^{1/3} - 4f \right] - \frac{a_C}{4} (A^{5/3} - 2A^{2/3}).$$

4(a). Kinetic energy at separation d

Each fragment has $A/2$ nucleons and charge approximately $Ae/4$. For two nonoverlapping, spherically symmetric charge distributions, their mutual electrostatic energy equals that of point charges at their centers:

$$U_{12}(d) = \frac{1}{4\pi\epsilon_0} \frac{(Ae/4)^2}{d} = \frac{e^2 A^2}{64\pi\epsilon_0 d}.$$

The $Z(Z-1)$ correction has already been included in each nucleus's self-energy. It does not apply to this interaction between two distinct sets of protons.

Take the separated free nucleons as the common internal-energy reference. The initial nucleus has energy $-B(A)$, while the two separated fragments have total energy

$$-2B(A/2) + U_{12}(d) + E_{\text{kin}}(d).$$

Energy conservation consequently gives

$$E_{\text{kin}}(d) = 2B(A/2) - B(A) - U_{12}(d).$$

Expanding the binding energies,

$$\begin{aligned} E_{\text{kin}}(d) = & -3a_V f^{1/3} (2^{1/3} - 1) A^{2/3} \\ & + 6a_V f^{2/3} (2^{2/3} - 1) A^{1/3} - 4a_V f \\ & + \frac{a_C}{4} \left[(1 - 2^{-2/3}) A^{5/3} + 2(2^{1/3} - 1) A^{2/3} \right] \\ & - \frac{e^2 A^2}{64\pi\epsilon_0 d}. \end{aligned}$$

The volume contributions cancel because the total number of nucleons is conserved. As the fragments separate, their mutual electrostatic energy decreases and their total kinetic energy increases.

4(b). Contact energies and fission threshold

At contact,

$$d_0 = 2R(A/2) = 2^{2/3} r_0 A^{1/3}.$$

Using $e^2/(4\pi\epsilon_0 r_0) = 5a_C/3$,

$$U_{12}(d_0) = \frac{5a_C}{48 \cdot 2^{2/3}} A^{5/3}.$$

Combining this with the expression above,

$$\begin{aligned} E_{\text{kin}}(d_0) = & \frac{a_C}{4} \left(1 - \frac{17}{12} 2^{-2/3} \right) A^{5/3} \\ & + (2^{1/3} - 1) \left(\frac{a_C}{2} - 3a_V f^{1/3} \right) A^{2/3} \\ & + 6a_V f^{2/3} (2^{2/3} - 1) A^{1/3} - 4a_V f. \end{aligned}$$

Numerically,

$$\frac{E_{\text{kin}}(d_0)}{\text{MeV}} = 0.0220368 A^{5/3} - 9.82357 A^{2/3} + 36.1750 A^{1/3} - 33.0914.$$

A	Contact separation d_0 (fm)	$E_{\text{kin}}(d_0)$ (MeV)
100	7.77	-29.3
150	8.89	-24.9
200	9.79	-6.77
250	10.55	+23.6

A negative calculated kinetic energy means that the contact configuration cannot be reached from the initially unexcited nucleus under the assumed energy balance.

For heavy nuclei, solving $E_{\text{kin}}(d_0) = 0$ gives

$$A_{\text{crit}} \simeq 213.0.$$

Thus the model permits fission through the contact configuration for approximately

$$A \gtrsim 213.$$

Among the four specified nuclei, only $A = 250$ satisfies this condition. Once contact is accessible, further separation releases additional Coulomb energy.

5(a). Excitation energy in the transfer reaction

Let m_1, m_2, m_3, m_4 denote the tabulated ground-state masses of oxygen, iron, carbon and nickel, respectively. Let

$$m_4^* = m_4 + \frac{E_{\text{exc}}}{c^2}$$

be the excited nickel mass.

Using the given constants,

$$1 \text{ a. m. u. } c^2 = \frac{(1.6605 \times 10^{-27})(3 \times 10^8)^2}{1.602 \times 10^{-13}} \text{ MeV} = 932.865 \text{ MeV}.$$

The ground-state reaction energy is

$$\begin{aligned} Q &= (m_1 + m_2 - m_3 - m_4)c^2 \\ &= (15.99491 + 53.93962 - 12.00000 - 57.93535)(932.865) \text{ MeV} \\ &= -0.764949 \text{ MeV}. \end{aligned}$$

We can retain exact relativistic kinematics. Oxygen and carbon have the same velocity v , hence the same Lorentz factor

$$\gamma = 1 + \frac{50 \text{ MeV}}{m_1 c^2}, \quad \beta = \frac{v}{c} = \sqrt{1 - \gamma^{-2}}.$$

Conservation of energy and momentum gives the excited nickel energy and momentum:

$$E_4^{\text{lab}} = \gamma(m_1 - m_3)c^2 + m_2 c^2, \quad p_4 = \gamma(m_1 - m_3)v.$$

Its invariant rest energy therefore satisfies

$$(m_4^* c^2)^2 = (E_4^{\text{lab}})^2 - (p_4 c)^2.$$

Writing $\Delta m = m_1 - m_3$ and using $\gamma^2(1 - \beta^2) = 1$,

$$m_4^* = \sqrt{m_2^2 + \Delta m^2 + 2\gamma m_2 \Delta m}.$$

Consequently,

$$E_{\text{exc}} = c^2 \left[\sqrt{m_2^2 + (m_1 - m_3)^2 + 2\gamma m_2 (m_1 - m_3)} - m_4 \right].$$

Numerically,

$$E_{\text{exc}} = 10.86074 \dots \text{ MeV} \approx 10.86 \text{ MeV}.$$

For an energy-balance check, the outgoing kinetic energies are

$$\begin{aligned} T_C &= \frac{m_3}{m_1} (50 \text{ MeV}) = 37.51193 \text{ MeV}, \\ T_{\text{Ni}} &= 0.86237 \text{ MeV}. \end{aligned}$$

They satisfy

$$E_{\text{exc}} = 50 + Q - T_C - T_{\text{Ni}}.$$

The usual nonrelativistic approximation,

$$T_{\text{Ni}} \simeq \frac{(m_1 - m_3)^2}{m_1 m_4} (50 \text{ MeV}),$$

would give $E_{\text{exc}} \simeq 10.862 \text{ MeV}$, the same result at the quoted precision.

For use in part (b), the excited nickel speed is

$$\beta_4 = \frac{V}{c} = \frac{p_4 c}{E_4^{\text{lab}}} = \frac{\gamma(m_1 - m_3)\beta}{m_2 + \gamma(m_1 - m_3)} = 0.00564849,$$

so $V \approx 1.69 \times 10^6 \text{ m s}^{-1}$.

5(b). Recoil and photon energies

First work in the rest frame of the excited nickel nucleus. Momentum conservation requires the final nickel nucleus to recoil opposite to the photon, with momentum magnitude

$$p_{\text{recoil}} = \frac{E_{\gamma}^*}{c}.$$

Set

$$W = m_4 c^2, \quad E = E_{\text{exc}}.$$

Energy conservation then gives

$$W + E = E_{\gamma}^* + \sqrt{W^2 + (E_{\gamma}^*)^2}.$$

Solving,

$$E_{\gamma}^* = \frac{(W + E)^2 - W^2}{2(W + E)} = E - \frac{E^2}{2(W + E)}.$$

Thus the recoil kinetic energy is

$$T_{\text{recoil}} = E - E_{\gamma}^* = \frac{E^2}{2(W + E)} \simeq \frac{E^2}{2W}.$$

The last approximation is valid because $E \ll W$; it is equivalent to the nonrelativistic recoil expression $p_{\text{recoil}}^2 / (2m_4)$.

Here

$$W = 54045.8700 \text{ MeV}.$$

Using the excitation energy from part (a),

$$T_{\text{recoil}} = 0.00109104 \text{ MeV} \approx 1.09 \text{ keV},$$

and

$$E_{\gamma}^* = 10.85965 \dots \text{ MeV} \approx 10.86 \text{ MeV}.$$

The extra digits display the small recoil subtraction.

For forward emission, Lorentz transformation of the photon energy to the laboratory gives

$$E_{\gamma, \text{lab}} = \Gamma_4 (E_{\gamma}^* + V p_{\gamma}^*) = \Gamma_4 E_{\gamma}^* (1 + \beta_4),$$

where $\Gamma_4 = (1 - \beta_4^2)^{-1/2}$. Hence

$$E_{\gamma,\text{lab}} = E_{\gamma}^* \sqrt{\frac{1 + \beta_4}{1 - \beta_4}}.$$

The forward Doppler factor is 1.00566453, so

$$E_{\gamma,\text{lab}} = 10.92117 \dots \text{ MeV} \approx 10.92 \text{ MeV}.$$

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.