

# IPhO 2009

Theory problems and solutions · Mérida, Mexico · 3 theory problems with complete solutions

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## Problem 1: Evolution of the Earth-Moon System

Scientists can determine the distance Earth-Moon with great precision. They achieve this by bouncing a laser beam on special mirrors deposited on the Moon's surface by astronauts in 1969, and measuring the round travel time of the light (see Figure 1).



Figure 1. A laser beam sent from an observatory is used to measure accurately the distance between the Earth and the Moon.

With this observations, they have directly measured that the Moon is slowly receding from the Earth. That is, the Earth-Moon distance is increasing with time. This is happening because due to tidal torques the Earth is transferring angular momentum to the Moon, see Figure 2. In this problem you will derive the basic parameters of the phenomenon.

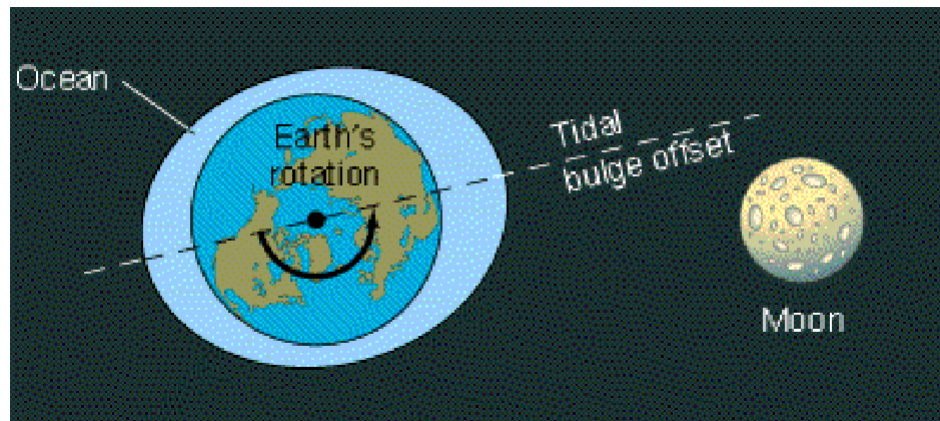


Figure 2. The Moon's gravity produces tidal deformations or "bulges" in the Earth. Because of the Earth's rotation, the line that goes through the bulges is not aligned with the line between the Earth and the Moon. This misalignment produces a torque that transfers angular momentum from the Earth's rotation to the Moon's translation. The drawing is not to scale.

### 1. Conservation of Angular Momentum.

Let  $L_1$  be the present total angular momentum of the Earth-Moon system. Now, make the following assumptions: i)  $L_1$  is the sum of the rotation of the Earth around its axis and the translation of the Moon in its orbit around the Earth only. ii) The Moon's orbit is circular and the Moon can be taken as a point. iii) The Earth's axis of rotation and the Moon's axis of revolution are parallel. iv) To simplify the calculations, we take the motion to be around the center of the Earth and not the center of mass. Throughout the problem, all moments of inertia, torques and angular momenta are defined around the axis of the Earth. v) Ignore the influence of the Sun.

| Part | Question                                                                                                                                                                                                                                                                                                                                                                                                | Points |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 1a   | Write down the equation for the present total angular momentum of the Earth-Moon system. Set this equation in terms of $I_E$ , the moment of inertia of the Earth; $\omega_{E1}$ , the present angular frequency of the Earth's rotation; $I_{M1}$ , the present moment of inertia of the Moon with respect to the Earth's axis; and $\omega_{M1}$ , the present angular frequency of the Moon's orbit. | 0.2    |

This process of transfer of angular momentum will end when the period of rotation of the Earth and the period of revolution of the Moon around the Earth have the same duration. At this point the tidal bulges produced by the Moon on the Earth will be aligned with the line between the Moon and the Earth and the torque will disappear.

| Part | Question                                                                                                                                                                                                                                                                                                                                                             | Points |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 1b   | Write down the equation for the final total angular momentum $L_2$ of the Earth-Moon system. Make the same assumptions as in Question 1a. Set this equation in terms of $I_E$ , the moment of inertia of the Earth; $\omega_2$ , the final angular frequency of the Earth's rotation and Moon's translation; and $I_{M2}$ , the final moment of inertia of the Moon. | 0.2    |

| Part | Question                                                                                                                                                                            | Points |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 1c   | Neglecting the contribution of the Earth's rotation to the final total angular momentum, write down the equation that expresses the angular momentum conservation for this problem. | 0.3    |

## 2. Final Separation and Final Angular Frequency of the Earth-Moon System.

Assume that the gravitational equation for a circular orbit (of the Moon around the Earth) is always valid. Neglect the contribution of the Earth's rotation to the final total angular momentum.

| Part | Question                                                                                                                                                                                                                                                                                 | Points |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 2a   | Write down the gravitational equation for the circular orbit of the Moon around the Earth, at the final state, in terms of $M_E$ , $\omega_2$ , $G$ and the final separation $D_2$ between the Earth and the Moon. $M_E$ is the mass of the Earth and $G$ is the gravitational constant. | 0.2    |

| Part | Question                                                                                                                                                                                                                                              | Points |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 2b   | Write down the equation for the final separation $D_2$ between the Earth and the Moon in terms of the known parameters, $L_1$ , the total angular momentum of the system, $M_E$ and $M_M$ , the masses of the Earth and Moon, respectively, and $G$ . | 0.5    |

| Part | Question                                                                                                                                                     | Points |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 2c   | Write down the equation for the final angular frequency $\omega_2$ of the Earth-Moon system in terms of the known parameters $L_1$ , $M_E$ , $M_M$ and $G$ . | 0.5    |

Below you will be asked to find the numerical values of  $D_2$  and  $\omega_2$ . For this you need to know the moment of inertia of the Earth.

| Part | Question                                                                                                                                                                                                                                                            | Points |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 2d   | Write down the equation for the moment of inertia of the Earth $I_E$ assuming it is a sphere with inner density $\rho_i$ from the center to a radius $r_i$ , and with outer density $\rho_o$ from the radius $r_i$ to the surface at a radius $r_o$ (see Figure 3). | 0.5    |

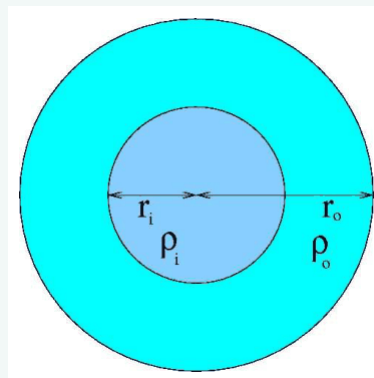


Figure 3. The Earth as a sphere with two densities,  $\rho_i$  and  $\rho_o$ .

Determine the numerical values requested in this problem always to *two significant digits*.

| Part | Question                                                                                                                                                                                                                            | Points |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 2e   | Evaluate the moment of inertia of the Earth $I_E$ , using $\rho_i = 1.3 \times 10^4 \text{ kg m}^{-3}$ , $r_i = 3.5 \times 10^6 \text{ m}$ , $\rho_o = 4.0 \times 10^3 \text{ kg m}^{-3}$ , and $r_o = 6.4 \times 10^6 \text{ m}$ . | 0.2    |

The masses of the Earth and Moon are  $M_E = 6.0 \times 10^{24} \text{ kg}$  and  $M_M = 7.3 \times 10^{22} \text{ kg}$ , respectively. The present separation between the Earth and the Moon is  $D_1 = 3.8 \times 10^8 \text{ m}$ . The present angular frequency of the Earth's rotation is  $\omega_{E1} = 7.3 \times 10^{-5} \text{ s}^{-1}$ . The present angular frequency of the Moon's translation around the Earth is  $\omega_{M1} = 2.7 \times 10^{-6} \text{ s}^{-1}$ , and the gravitational constant is  $G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ .

| Part | Question                                                                          | Points |
|------|-----------------------------------------------------------------------------------|--------|
| 2f   | Evaluate the numerical value of the total angular momentum of the system, $L_1$ . | 0.2    |

| Part | Question                                                                                 | Points |
|------|------------------------------------------------------------------------------------------|--------|
| 2g   | Find the final separation $D_2$ in meters and in units of the present separation $D_1$ . | 0.3    |

| Part | Question                                                                                                                     | Points |
|------|------------------------------------------------------------------------------------------------------------------------------|--------|
| 2h   | Find the final angular frequency $\omega_2$ in $s^{-1}$ , as well as the final duration of the day in units of present days. | 0.3    |

Verify that the assumption of neglecting the contribution of the Earth's rotation to the final total angular momentum is justified by finding the ratio of the final angular momentum of the Earth to that of the Moon. This should be a small quantity.

| Part | Question                                                                       | Points |
|------|--------------------------------------------------------------------------------|--------|
| 2i   | Find the ratio of the final angular momentum of the Earth to that of the Moon. | 0.2    |

### 3. How much is the Moon receding per year?

Now, you will find how much the Moon is receding from the Earth each year. For this, you will need to know the equation for the torque acting at present on the Moon. Assume that the tidal bulges can be approximated by two point masses, each of mass  $m$ , located on the surface of the Earth, see Fig. 4. Let  $\theta$  be the angle between the line that goes through the bulges and the line that joins the centers of the Earth and the Moon.

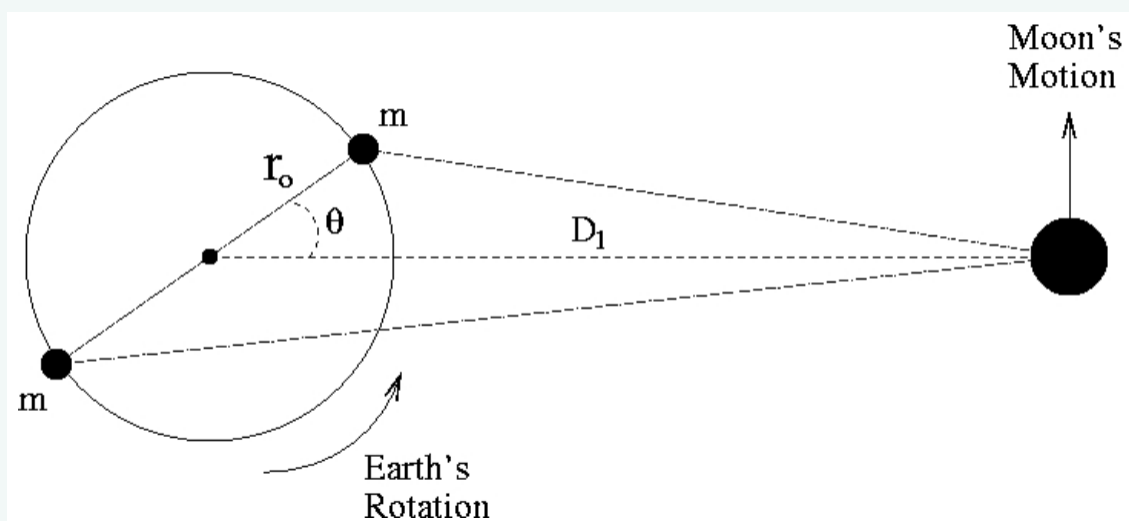


Figure 4. Schematic diagram to estimate the torque produced on the Moon by the bulges on the Earth. The drawing is not to scale.

| Part | Question                                                                                | Points |
|------|-----------------------------------------------------------------------------------------|--------|
| 3a   | Find $F_c$ , the magnitude of the force produced on the Moon by the closest point mass. | 0.4    |

| Part | Question                                                                                 | Points |
|------|------------------------------------------------------------------------------------------|--------|
| 3b   | Find $F_f$ , the magnitude of the force produced on the Moon by the farthest point mass. | 0.4    |

You may now evaluate the torques produced by the point masses.

| Part | Question                                                                        | Points |
|------|---------------------------------------------------------------------------------|--------|
| 3c   | Find the magnitude of $\tau_c$ , the torque produced by the closest point mass. | 0.4    |

| Part | Question                                                                         | Points |
|------|----------------------------------------------------------------------------------|--------|
| 3d   | Find the magnitude of $\tau_f$ , the torque produced by the farthest point mass. | 0.4    |

| Part | Question                                                                                                                                                                                                                                 | Points |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 3e   | Find the magnitude of the total torque $\tau$ produced by the two masses. Since $r_o \ll D_1$ you should approximate your expression to lowest significant order in $r_o/D_1$ . You may use that $(1+x)^a \approx 1+ax$ , if $x \ll 1$ . | 1.0    |

| Part | Question                                                                                                                                                                                                                    | Points |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 3f   | Calculate the numerical value of the total torque $\tau$ , taking into account that $\theta = 3^\circ$ and that $m = 3.6 \times 10^{16}$ kg (note that this mass is of the order of $10^{-8}$ times the mass of the Earth). | 0.5    |

Since the torque is the rate of change of angular momentum with time, find the increase in the distance Earth-Moon at present, per year. For this step, express the angular momentum of the Moon in terms of  $M_M$ ,  $M_E$ ,  $D_1$  and  $G$  only.

| Part | Question                                                           | Points |
|------|--------------------------------------------------------------------|--------|
| 3g   | Find the increase in the distance Earth-Moon at present, per year. | 1.0    |

Finally, estimate how much the length of the day is increasing each year.

| Part | Question                                                                                                           | Points |
|------|--------------------------------------------------------------------------------------------------------------------|--------|
| 3h   | Find the decrease of $\omega_{E1}$ per year and how much is the length of the day at present increasing each year. | 1.0    |

#### 4. Where is the energy going?

In contrast to the angular momentum, that is conserved, the total (rotational plus gravitational) energy of the system is not. We will look into this in this last section.

| Part | Question                                                                                                                                                                                                             | Points |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 4a   | Write down an equation for the total (rotational plus gravitational) energy of the Earth-Moon system at the present, $E$ . Put this equation in terms of $I_E$ , $\omega_{E1}$ , $M_M$ , $M_E$ , $D_1$ and $G$ only. | 0.4    |

| Part | Question                                                                                                                                                                                                                                                                | Points |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 4b   | Write down an equation for the change in $E$ , $\Delta E$ , as a function of the changes in $D_1$ and in $\omega_{E1}$ . Evaluate the numerical value of $\Delta E$ for a year, using the values of changes in $D_1$ and in $\omega_{E1}$ found in questions 3g and 3h. | 0.4    |

Verify that this loss of energy is consistent with an estimate for the energy dissipated as heat in the tides produced by the Moon on the Earth. Assume that the tides rise, on the average by 0.5 m, a layer of water  $h = 0.5$  m deep that covers the surface of the Earth (for simplicity assume that all the surface of the Earth is covered with water). This happens twice a day. Further assume that

10% of this gravitational energy is dissipated as heat due to viscosity when the water descends. Take the density of water to be  $\rho_{\text{water}} = 10^3 \text{ kg m}^{-3}$  and the gravitational acceleration on the surface of the Earth to be  $g = 9.8 \text{ m s}^{-2}$ .

| Part | Question                                         | Points |
|------|--------------------------------------------------|--------|
| 4c   | What is the mass of this surface layer of water? | 0.2    |

| Part | Question                                                                                                                                    | Points |
|------|---------------------------------------------------------------------------------------------------------------------------------------------|--------|
| 4d   | Calculate how much energy is dissipated in a year? How does this compare with the energy lost per year by the Earth-Moon system at present? | 0.3    |

Official English text of the problem.

## Answers

For the bulge forces and torques, define the distances

$$d_c = \sqrt{D_1^2 + r_o^2 - 2D_1r_o \cos \theta}, \quad d_f = \sqrt{D_1^2 + r_o^2 + 2D_1r_o \cos \theta}.$$

Let  $T_1$  and  $T_2$  denote the present and final day lengths. Numerical results are given to two significant figures.

- **1a:**  $L_1 = I_E \omega_{E1} + I_{M1} \omega_{M1}$ .
- **1b:**  $L_2 = (I_E + I_{M2}) \omega_2$ .
- **1c:**  $I_E \omega_{E1} + I_{M1} \omega_{M1} = L_1 \simeq I_{M2} \omega_2$ .
- **2a:**  $\omega_2^2 D_2^3 = GM_E$ .
- **2b:**  $D_2 = \frac{L_1^2}{GM_E M_M^2}$ .
- **2c:**  $\omega_2 = \frac{G^2 M_E^2 M_M^3}{L_1^3}$ .
- **2d:**  $I_E = \frac{8\pi}{15} [\rho_i r_i^5 + \rho_o (r_o^5 - r_i^5)]$ .
- **2e:**  $I_E = 8.0 \times 10^{37} \text{ kg m}^2$ .
- **2f:**  $L_1 = 3.4 \times 10^{34} \text{ kg m}^2 \text{ s}^{-1}$ .
- **2g:**  $D_2 = 5.5 \times 10^8 \text{ m} = 1.4 D_1$ .
- **2h:**  $\omega_2 = 1.6 \times 10^{-6} \text{ s}^{-1}$ ,  $T_2/T_1 = 47$ .
- **2i:**  $\frac{L_{E2}}{L_{M2}} = \frac{I_E}{M_M D_2^2} = 3.6 \times 10^{-3}$ .
- **3a:**  $F_c = \frac{GM_M m}{d_c^2}$ .
- **3b:**  $F_f = \frac{GM_M m}{d_f^2}$ .
- **3c:**  $\tau_c = \frac{GM_M m D_1 r_o \sin \theta}{d_c^3}$ .
- **3d:**  $\tau_f = \frac{GM_M m D_1 r_o \sin \theta}{d_f^3}$ .

- **3e:**  $\tau = \tau_c - \tau_f \simeq \frac{3GM_M m r_o^2}{D_1^3} \sin(2\theta)$ , directed along the Moon's orbital angular momentum.
- **3f:**  $\tau = 4.1 \times 10^{16} \text{ N m}$ .
- **3g:**  $\dot{D}_1 = \frac{2\tau}{M_M} \sqrt{\frac{D_1}{GM_E}} = 3.5 \times 10^{-2} \text{ m yr}^{-1} = 3.5 \text{ cm yr}^{-1}$ .
- **3h:** In one year,  $\Delta\omega_{E1} = -1.6 \times 10^{-14} \text{ s}^{-1}$  and  $\Delta T_1 = +1.9 \times 10^{-5} \text{ s}$ .
- **4a:**  $E = \frac{1}{2} I_E \omega_{E1}^2 - \frac{GM_E M_M}{2D_1}$ .
- **4b:**  $\Delta E \simeq I_E \omega_{E1} \Delta\omega_{E1} + \frac{GM_E M_M}{2D_1^2} \Delta D_1 = -9.1 \times 10^{19} \text{ J}$  in one year.
- **4c:**  $M_{\text{water}} = 4\pi r_o^2 h \rho_{\text{water}} = 2.6 \times 10^{17} \text{ kg}$ .
- **4d:**  $E_{\text{heat, yr}} = 9.2 \times 10^{19} \text{ J}$ , in close agreement with the mechanical energy loss of  $9.1 \times 10^{19} \text{ J}$  per year.

## Solution

All calculations use the assumptions stated in the problem. Numerical results are rounded only at the end. For yearly changes, take

$$t_{\text{yr}} = 365 \times 24 \times 3600 \text{ s}.$$

### 1a. Present angular momentum

The Earth's spin and the Moon's revolution are in the same sense about parallel axes, so their angular momenta add:

$$L_1 = I_E \omega_{E1} + I_{M1} \omega_{M1}.$$

Since the Moon is treated as a point at distance  $D_1$  from the Earth's axis,

$$I_{M1} = M_M D_1^2.$$

### 1b. Final angular momentum

In the final synchronized state, both angular frequencies equal  $\omega_2$ . Thus

$$L_2 = I_E \omega_2 + I_{M2} \omega_2 = (I_E + I_{M2}) \omega_2, \quad I_{M2} = M_M D_2^2.$$

### 1c. Angular momentum conservation

There is no external torque in the stated model. The tidal torques transfer angular momentum internally, giving  $L_1 = L_2$ .

Neglecting only the *final* Earth spin contribution,

$$I_E \omega_{E1} + I_{M1} \omega_{M1} \simeq I_{M2} \omega_2.$$

The present Earth spin must still be included in  $L_1$ .

### 2a. Circular-orbit equation

The gravitational force supplies the Moon's centripetal acceleration:

$$\frac{GM_E M_M}{D_2^2} = M_M \omega_2^2 D_2.$$

Cancelling  $M_M$  gives

$$\omega_2^2 D_2^3 = GM_E.$$

## 2b. Final separation

Within the approximation of part 1c,

$$L_1 = M_M D_2^2 \omega_2.$$

Using  $\omega_2 = \sqrt{GM_E/D_2^3}$ ,

$$L_1 = M_M \sqrt{GM_E D_2}.$$

Squaring and solving,

$$D_2 = \frac{L_1^2}{GM_E M_M^2}.$$

## 2c. Final angular frequency

Substitution of the preceding result into the circular-orbit equation yields

$$\omega_2 = \sqrt{GM_E} \left( \frac{GM_E M_M^2}{L_1^2} \right)^{3/2}.$$

Therefore

$$\omega_2 = \frac{G^2 M_E^2 M_M^3}{L_1^3}.$$

## 2d. Earth's moment of inertia

A thin spherical shell of radius  $r$ , density  $\rho$ , and thickness  $dr$  has

$$dm = 4\pi\rho r^2 dr, \quad dI = \frac{2}{3}r^2 dm = \frac{8\pi}{3}\rho r^4 dr.$$

Integrating over the two density regions,

$$I_E = \frac{8\pi}{3} \left[ \rho_i \int_0^{r_i} r^4 dr + \rho_o \int_{r_i}^{r_o} r^4 dr \right].$$

Thus

$$I_E = \frac{8\pi}{15} [\rho_i r_i^5 + \rho_o (r_o^5 - r_i^5)].$$

## 2e. Numerical moment of inertia

Substituting the given densities and radii,

$$I_E = \frac{8\pi}{15} [(1.3 \times 10^4)(3.5 \times 10^6)^5 + (4.0 \times 10^3) ((6.4 \times 10^6)^5 - (3.5 \times 10^6)^5)] \text{ kg m}^2.$$

Hence

$$I_E = 8.0 \times 10^{37} \text{ kg m}^2.$$

## 2f. Present total angular momentum

Using the given present angular frequencies,

$$L_1 = I_E \omega_{E1} + M_M D_1^2 \omega_{M1}.$$

The two contributions are

$$I_E \omega_{E1} = 5.8 \times 10^{33} \text{ kg m}^2 \text{ s}^{-1},$$

$$M_M D_1^2 \omega_{M1} = 2.8 \times 10^{34} \text{ kg m}^2 \text{ s}^{-1}.$$

Adding them with unrounded intermediate values,

$$L_1 = 3.4 \times 10^{34} \text{ kg m}^2 \text{ s}^{-1}.$$

## 2g. Numerical final separation

Using part 2b and the unrounded value of  $L_1$ ,

$$D_2 = \frac{L_1^2}{GM_E M_M^2} = 5.5 \times 10^8 \text{ m}.$$

Relative to the present separation,

$$\frac{D_2}{D_1} = 1.4.$$

## 2h. Final frequency and day length

From part 2c,

$$\omega_2 = 1.6 \times 10^{-6} \text{ s}^{-1}.$$

Since a rotation period is  $T = 2\pi/\omega$ ,

$$\frac{T_2}{T_1} = \frac{2\pi/\omega_2}{2\pi/\omega_{E1}} = \frac{\omega_{E1}}{\omega_2}.$$

Using unrounded intermediate values gives

$$T_2 = 47T_1,$$

so the final day lasts 47 present days.

## 2i. Check of the neglected final spin

Both final angular momenta contain the same angular frequency:

$$\frac{L_{E2}}{L_{M2}} = \frac{I_E \omega_2}{M_M D_2^2 \omega_2} = \frac{I_E}{M_M D_2^2}.$$

Numerically,

$$\frac{L_{E2}}{L_{M2}} = 3.6 \times 10^{-3}.$$

The final Earth spin angular momentum is only 0.36% of the Moon's orbital angular momentum, justifying its neglect.

### 3a. Force from the closest bulge

Choose the Earth's center as the origin, put the Moon at  $(D_1, 0)$ , and take positive  $y$  along the Moon's instantaneous motion. The closest bulge is at

$$(r_o \cos \theta, r_o \sin \theta).$$

Its distance from the Moon is therefore

$$d_c^2 = (D_1 - r_o \cos \theta)^2 + (r_o \sin \theta)^2 = D_1^2 + r_o^2 - 2D_1 r_o \cos \theta.$$

Newton's gravitational law gives

$$F_c = \frac{GM_M m}{D_1^2 + r_o^2 - 2D_1 r_o \cos \theta}.$$

### 3b. Force from the farthest bulge

The farther bulge is at

$$(-r_o \cos \theta, -r_o \sin \theta).$$

Consequently,

$$d_f^2 = (D_1 + r_o \cos \theta)^2 + (r_o \sin \theta)^2 = D_1^2 + r_o^2 + 2D_1 r_o \cos \theta,$$

and

$$F_f = \frac{GM_M m}{D_1^2 + r_o^2 + 2D_1 r_o \cos \theta}.$$

### 3c. Torque from the closest bulge

Only the force component perpendicular to the Earth-Moon line contributes to the torque about the Earth's axis. For the closer bulge,

$$F_{c,y} = F_c \frac{r_o \sin \theta}{d_c}.$$

Its torque is positive, in the direction of the Moon's orbital angular momentum:

$$\tau_c = D_1 F_{c,y}.$$

Thus its magnitude is

$$\tau_c = \frac{GM_M m D_1 r_o \sin \theta}{(D_1^2 + r_o^2 - 2D_1 r_o \cos \theta)^{3/2}}.$$

### 3d. Torque from the farthest bulge

The farther bulge pulls the Moon in the negative  $y$  direction:

$$F_{f,y} = -F_f \frac{r_o \sin \theta}{d_f}.$$

Its torque opposes the closer bulge's torque. Its magnitude is

$$\tau_f = \frac{GM_M m D_1 r_o \sin \theta}{(D_1^2 + r_o^2 + 2D_1 r_o \cos \theta)^{3/2}}.$$

### 3e. Total torque to lowest nonzero order

Because  $d_c < d_f$ , the closer bulge produces the larger torque. For the geometry shown,

$$\tau = \tau_c - \tau_f = GM_M m D_1 r_o \sin \theta \left( \frac{1}{d_c^3} - \frac{1}{d_f^3} \right).$$

Define  $\epsilon = r_o/D_1 \ll 1$ . Then

$$\frac{1}{d_c^3} = \frac{1}{D_1^3} (1 + \epsilon^2 - 2\epsilon \cos \theta)^{-3/2} \simeq \frac{1}{D_1^3} (1 + 3\epsilon \cos \theta),$$

and

$$\frac{1}{d_f^3} = \frac{1}{D_1^3} (1 + \epsilon^2 + 2\epsilon \cos \theta)^{-3/2} \simeq \frac{1}{D_1^3} (1 - 3\epsilon \cos \theta).$$

The zeroth-order terms cancel, leaving

$$\tau \simeq GM_M m D_1 r_o \sin \theta \frac{6r_o \cos \theta}{D_1^4}.$$

Therefore

$$\tau \simeq \frac{6GM_M m r_o^2}{D_1^3} \sin \theta \cos \theta = \frac{3GM_M m r_o^2}{D_1^3} \sin(2\theta).$$

This torque increases the Moon's orbital angular momentum.

### 3f. Numerical torque

With  $\theta = 3^\circ$ ,

$$\tau \simeq \frac{3(6.7 \times 10^{-11})(7.3 \times 10^{22})(3.6 \times 10^{16})(6.4 \times 10^6)^2}{(3.8 \times 10^8)^3} \sin 6^\circ.$$

Hence

$$\tau = 4.1 \times 10^{16} \text{ N m.}$$

### 3g. Present recession of the Moon

For a circular orbit at separation  $D$ ,

$$\omega_M = \sqrt{\frac{GM_E}{D^3}},$$

so the Moon's angular momentum is

$$L_M = M_M D^2 \omega_M = M_M \sqrt{GM_E D}.$$

Differentiating with respect to time,

$$\tau = \frac{dL_M}{dt} = \frac{M_M}{2} \sqrt{\frac{GM_E}{D}} \dot{D}.$$

Thus, at the present separation,

$$\dot{D}_1 = \frac{2\tau}{M_M} \sqrt{\frac{D_1}{GM_E}}.$$

Numerically,

$$\dot{D}_1 = 1.1 \times 10^{-9} \text{ m s}^{-1}.$$

The increase over one year is

$$\Delta D_1 = \dot{D}_1 t_{\text{yr}},$$

giving

$$\Delta D_1 = 3.5 \times 10^{-2} \text{ m} = 3.5 \text{ cm} \quad \text{in one year.}$$

### 3h. Earth's slowing rotation and increasing day length

The torque on the Earth is opposite to the torque on the Moon. With constant  $I_E$ ,

$$I_E \dot{\omega}_E = -\tau.$$

Therefore

$$\dot{\omega}_{E1} = -\frac{\tau}{I_E} = -5.2 \times 10^{-22} \text{ s}^{-2},$$

and the change in one year is

$$\Delta \omega_{E1} = -\frac{\tau t_{\text{yr}}}{I_E} = -1.6 \times 10^{-14} \text{ s}^{-1}.$$

Since  $T_E = 2\pi/\omega_E$ , a small change satisfies

$$\Delta T_E \simeq -\frac{2\pi}{\omega_E^2} \Delta \omega_E.$$

Consequently,

$$\Delta T_1 = \frac{2\pi \tau t_{\text{yr}}}{I_E \omega_{E1}^2} = 1.9 \times 10^{-5} \text{ s} \quad \text{in one year.}$$

The day lengthens by approximately  $19 \mu\text{s}$  per year.

### 4a. Present mechanical energy

The mechanical energy consists of the Earth's spin kinetic energy, the Moon's orbital kinetic energy, and their gravitational potential energy:

$$E = \frac{1}{2} I_E \omega_{E1}^2 + \frac{1}{2} M_M D_1^2 \omega_{M1}^2 - \frac{GM_E M_M}{D_1}.$$

The circular-orbit relation gives

$$\frac{1}{2} M_M D_1^2 \omega_{M1}^2 = \frac{GM_E M_M}{2D_1}.$$

Hence

$$E = \frac{1}{2} I_E \omega_{E1}^2 - \frac{GM_E M_M}{2D_1}.$$

#### 4b. Annual mechanical energy change

For changes  $\Delta\omega_{E1}$  and  $\Delta D_1$ , the exact difference within this energy model is

$$\Delta E = \frac{1}{2}I_E [(\omega_{E1} + \Delta\omega_{E1})^2 - \omega_{E1}^2] - \frac{GM_E M_M}{2} \left[ \frac{1}{D_1 + \Delta D_1} - \frac{1}{D_1} \right].$$

The annual fractional changes are extremely small, so retaining first-order terms gives

$$\Delta E \simeq I_E \omega_{E1} \Delta\omega_{E1} + \frac{GM_E M_M}{2D_1^2} \Delta D_1.$$

The first term is negative: the Earth loses spin energy. The second is positive: the Moon's orbital energy increases as its orbit expands.

Substituting the annual changes from parts 3g and 3h, retaining their unrounded values,

$$\Delta E = -9.1 \times 10^{19} \text{ J in one year.}$$

The energy balance can also be expressed directly in terms of the torque:

$$\dot{E}_{\text{spin}} = -\tau\omega_{E1}.$$

Using the recession formula,

$$\dot{E}_{\text{orbit}} = \frac{GM_E M_M}{2D_1^2} \dot{D}_1 = \tau \sqrt{\frac{GM_E}{D_1^3}}.$$

Thus

$$\dot{E} = -\tau(\omega_{E1} - \omega_{M1}),$$

where  $\omega_{M1}$  is understood through the circular-orbit relation. Since the Earth spins faster than the Moon revolves, the mechanical energy decreases; the difference becomes heat.

#### 4c. Mass of the water layer

Since the layer is very thin compared with the Earth's radius, its volume is its surface area times its thickness:

$$V \simeq 4\pi r_o^2 h.$$

Therefore

$$M_{\text{water}} = 4\pi r_o^2 h \rho_{\text{water}} = 4\pi (6.4 \times 10^6)^2 (0.5) (10^3) \text{ kg.}$$

Hence

$$M_{\text{water}} = 2.6 \times 10^{17} \text{ kg.}$$

#### 4d. Annual heat dissipation

The stated average rise is  $H = 0.5$  m. The gravitational energy gained by the layer in one rise is

$$E_{\text{rise}} = M_{\text{water}} g H.$$

Ten percent is dissipated as heat during the subsequent descent. With two cycles per day, there are  $2 \times 365$  cycles per year:

$$E_{\text{heat, yr}} = 0.10(2 \times 365) M_{\text{water}} g H.$$

Substitution gives

$$E_{\text{heat,yr}} = 9.2 \times 10^{19} \text{ J.}$$

This agrees closely with the mechanical energy loss

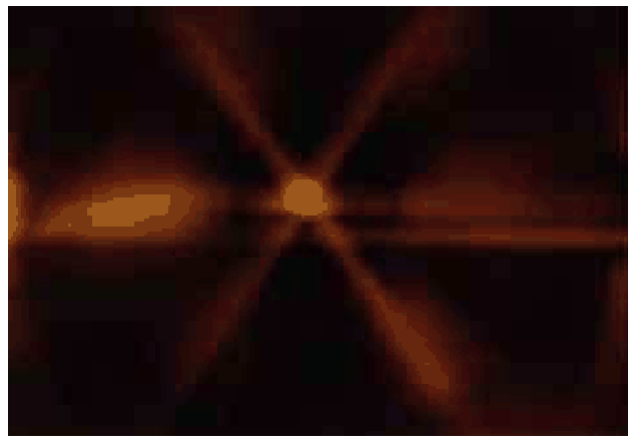
$$|\Delta E| = 9.1 \times 10^{19} \text{ J}$$

found in part 4b. Within the simplified model, tidal dissipation accounts for the lost mechanical energy by converting it into heat.

The solution was checked against the official solution.

## Problem 2: Doppler Laser Cooling and Optical Molasses

The purpose of this problem is to develop a simple theory to understand the so-called “laser cooling” and “optical molasses” phenomena. This refers to the cooling of a beam of neutral atoms, typically alkaline, by counterpropagating laser beams with the same frequency. This is part of the Physics Nobel Prize awarded to S. Chu, P. Phillips and C. Cohen-Tannoudji in 1997.



The image above shows sodium atoms (the bright spot in the center) trapped at the intersection of three orthogonal pairs of opposing laser beams. The trapping region is called “optical molasses” because the dissipative optical force resembles the viscous drag on a body moving through molasses.

In this problem you will analyze the basic phenomenon of the interaction between a photon incident on an atom and the basis of the dissipative mechanism in one dimension.

### PART I: BASICS OF LASER COOLING

Consider an atom of mass  $m$  moving in the  $+x$  direction with velocity  $v$ . For simplicity, we shall consider the problem to be one-dimensional, namely, we shall ignore the  $y$  and  $z$  directions (see figure 1). The atom has two internal energy levels. The energy of the lowest state is considered to be zero and the energy of the excited state to be  $\hbar\omega_0$ , where  $\hbar = h/2\pi$ . The atom is initially in the lowest state. A laser beam with frequency  $\omega_L$  in the laboratory is directed in the  $-x$  direction and it is incident on the atom. Quantum mechanically the laser is composed of a large number of photons, each with energy  $\hbar\omega_L$  and momentum  $-\hbar q$ . A photon can be absorbed by the atom and later spontaneously emitted; this emission can occur with equal probabilities along the  $+x$  and  $-x$  directions. Since the atom moves at non-relativistic speeds,  $v/c \ll 1$  (with  $c$  the speed of

light) keep terms up to first order in this quantity only. Consider also  $\hbar q/mv \ll 1$ , namely, that the momentum of the atom is much larger than the momentum of a single photon. In writing your answers, keep only corrections linear in either of the above quantities.

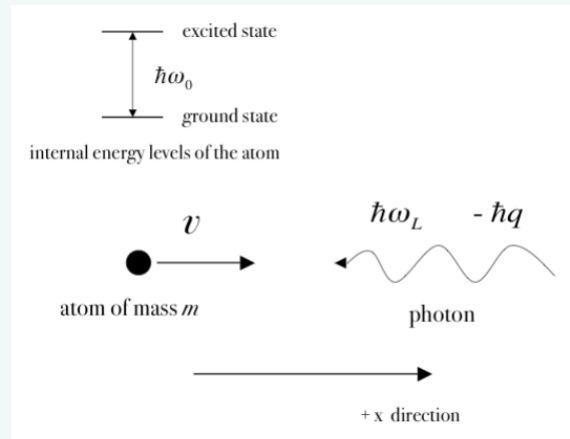


Fig.1 Sketch of an atom of mass  $m$  with velocity  $v$  in the  $+x$  direction, colliding with a photon with energy  $\hbar\omega_L$  and momentum  $-\hbar q$ . The atom has two internal states with energy difference  $\hbar\omega_0$ .

Assume that the laser frequency  $\omega_L$  is tuned such that, as seen by the moving atom, it is in resonance with the internal transition of the atom. Answer the following questions:

### 1. Absorption.

|    |                                                                                                         |     |
|----|---------------------------------------------------------------------------------------------------------|-----|
| 1a | Write down the resonance condition for the absorption of the photon.                                    | 0.2 |
| 1b | Write down the momentum $p_{at}$ of the atom after absorption, as seen in the laboratory.               | 0.2 |
| 1c | Write down the total energy $\varepsilon_{at}$ of the atom after absorption, as seen in the laboratory. | 0.2 |

### 2. Spontaneous emission of a photon in the $-x$ direction.

At some time after the absorption of the incident photon, the atom may emit a photon in the  $-x$  direction.

|    |                                                                                                                                                |     |
|----|------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 2a | Write down the energy of the emitted photon, $\varepsilon_{ph}$ , after the emission process in the $-x$ direction, as seen in the laboratory. | 0.2 |
| 2b | Write down the momentum of the emitted photon $p_{ph}$ , after the emission process in the $-x$ direction, as seen in the laboratory.          | 0.2 |
| 2c | Write down the momentum of the atom $p_{at}$ , after the emission process in the $-x$ direction, as seen in the laboratory.                    | 0.2 |
| 2d | Write down the total energy of the atom $\varepsilon_{at}$ , after the emission process in the $-x$ direction, as seen in the laboratory.      | 0.2 |

### 3. Spontaneous emission of a photon in the $+x$ direction.

At some time after the absorption of the incident photon, the atom may instead emit a photon in the  $+x$  direction.

|    |                                                                                                                                                |     |
|----|------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 3a | Write down the energy of the emitted photon, $\varepsilon_{ph}$ , after the emission process in the $+x$ direction, as seen in the laboratory. | 0.2 |
| 3b | Write down the momentum of the emitted photon $p_{ph}$ , after the emission process in the $+x$ direction, as seen in the laboratory.          | 0.2 |

|    |                                                                                                                                           |     |
|----|-------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 3c | Write down the momentum of the atom $p_{at}$ , after the emission process in the $+x$ direction, as seen in the laboratory.               | 0.2 |
| 3d | Write down the total energy of the atom $\varepsilon_{at}$ , after the emission process in the $+x$ direction, as seen in the laboratory. | 0.2 |

#### 4. Average emission after the absorption.

The spontaneous emission of a photon in the  $-x$  or in the  $+x$  directions occurs with the same probability. Taking this into account, answer the following questions.

|    |                                                                                                      |     |
|----|------------------------------------------------------------------------------------------------------|-----|
| 4a | Write down the average energy of an emitted photon, $\varepsilon_{ph}$ , after the emission process. | 0.2 |
| 4b | Write down the average momentum of an emitted photon $p_{ph}$ , after the emission process.          | 0.2 |
| 4c | Write down the average total energy of the atom $\varepsilon_{at}$ , after the emission process.     | 0.2 |
| 4d | Write down the average momentum of the atom $p_{at}$ , after the emission process.                   | 0.2 |

#### 5. Energy and momentum transfer.

Assuming a complete one-photon absorption-emission process only, as described above, there is a net average momentum and energy transfer between the laser radiation and the atom.

|    |                                                                                                                               |     |
|----|-------------------------------------------------------------------------------------------------------------------------------|-----|
| 5a | Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process. | 0.2 |
| 5b | Write down the average momentum change $\Delta p$ of the atom after a complete one-photon absorption-emission process.        | 0.2 |

#### 6. Energy and momentum transfer by a laser beam along the $+x$ direction.

Consider now that a laser beam of frequency  $\omega'_L$  is incident on the atom along the  $+x$  direction, while the atom moves also in the  $+x$  direction with velocity  $v$ . Assuming a resonance condition between the internal transition of the atom and the laser beam, as seen by the atom, answer the following questions:

|    |                                                                                                                               |     |
|----|-------------------------------------------------------------------------------------------------------------------------------|-----|
| 6a | Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process. | 0.3 |
| 6b | Write down the average momentum change $\Delta p$ of the atom after a complete one-photon absorption-emission process.        | 0.3 |

### PART II: DISSIPATION AND THE FUNDAMENTALS OF OPTICAL MOLASSES

Nature, however, imposes an inherent uncertainty in quantum processes. Thus, the fact that the atom can spontaneously emit a photon in a *finite time* after absorption, gives as a result that the resonance condition does not have to be obeyed *exactly* as in the discussion above. That is, the frequency of the laser beams  $\omega_L$  and  $\omega'_L$  may have any value and the absorption-emission process can still occur. These will happen with different (quantum) probabilities and, as one should expect, the maximum probability is found at the exact resonance condition. On the average, the time elapsed between a single process of absorption and emission is called the lifetime of the excited energy level of the atom and it is denoted by  $\Gamma^{-1}$ .

Consider a collection of  $N$  atoms *at rest* in the laboratory frame of reference, and a laser beam of frequency  $\omega_L$  incident on them. The atoms absorb and emit continuously such that there is, on average,  $N_{exc}$  atoms in the excited state (and therefore,  $N - N_{exc}$  atoms in the ground state). A

quantum mechanical calculation yields the following result:

$$N_{exc} = N \frac{\Omega_R^2}{(\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2}$$

where  $\omega_0$  is the resonance frequency of the atomic transition and  $\Omega_R$  is the so-called Rabi frequency;  $\Omega_R^2$  is proportional to the *intensity* of the laser beam. As mentioned above, you can see that this number is different from zero even if the resonance frequency  $\omega_0$  is different from the frequency of the laser beam  $\omega_L$ . An alternative way of expressing the previous result is that the number of absorption-emission processes per unit of time is  $N_{exc}\Gamma$ .

Consider the physical situation depicted in Figure 2, in which two counter propagating laser beams with the *same* but *arbitrary* frequency  $\omega_L$  are incident on a gas of  $N$  atoms that move in the  $+x$  direction with velocity  $v$ .

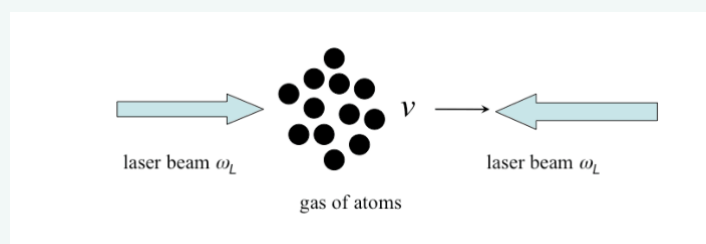


Figure 2. Two counter propagating laser beams with the *same* but *arbitrary* frequency  $\omega_L$  are incident on a gas of  $N$  atoms that move in the  $+x$  direction with velocity  $v$ .

### 7. Force on the atomic beam by the lasers.

|    |                                                                                                                                       |     |
|----|---------------------------------------------------------------------------------------------------------------------------------------|-----|
| 7a | With the information found so far, find the force that the lasers exert on the atomic beam. You should assume that $mv \gg \hbar q$ . | 1.5 |
|----|---------------------------------------------------------------------------------------------------------------------------------------|-----|

### 8. Low velocity limit.

Assume now that the velocity of the atoms is small enough such that you can expand the force up to first order in  $v$ .

|    |                                                                         |     |
|----|-------------------------------------------------------------------------|-----|
| 8a | Find an expression for the force found in Question (7a), in this limit. | 1.5 |
|----|-------------------------------------------------------------------------|-----|

Using this result, you can find the conditions for speeding up, slowing down, or no effect at all on the atoms by the laser radiation.

|    |                                                                                                                                                            |      |
|----|------------------------------------------------------------------------------------------------------------------------------------------------------------|------|
| 8b | Write down the condition to obtain a positive force (speeding up the atoms).                                                                               | 0.25 |
| 8c | Write down the condition to obtain a zero force.                                                                                                           | 0.25 |
| 8d | Write down the condition to obtain a negative force (slowing down the atoms).                                                                              | 0.25 |
| 8e | Consider now that the atoms are moving with a velocity $-v$ (in the $-x$ direction). Write down the condition to obtain a slowing down force on the atoms. | 0.25 |

### 9. Optical molasses.

In the case of a negative force, one obtains a frictional dissipative force. Assume that initially, at  $t = 0$ , the gas of atoms has velocity  $v_0$ .

|    |                                                                                                                                                                     |     |
|----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 9a | In the limit of low velocities, find the velocity of the atoms after the laser beams have been on for a time $\tau$ .                                               | 1.5 |
| 9b | Assume now that the gas of atoms is in thermal equilibrium at a temperature $T_0$ . Find the temperature $T$ after the laser beams have been on for a time $\tau$ . | 0.5 |

This model does not allow you to go to arbitrarily low temperatures.

Official English text of the problem.

## Answers

Define

$$\beta = \frac{v}{c}, \quad q = \frac{\omega_L}{c}, \quad q' = \frac{\omega'_L}{c}, \quad K = \frac{1}{2}mv^2.$$

For Part II, define

$$\Delta = \omega_0 - \omega_L, \quad D = \Delta^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2, \quad \alpha = \frac{4\hbar q^2 \Gamma \Omega_R^2 \Delta}{D^2}.$$

The two beams are taken to have the same  $\Omega_R$ .

Photon quantities below retain their first Doppler correction. Atomic quantities retain the leading recoil correction, omitting products of  $v/c$  and  $\hbar q/(mv)$ . The small momentum corrections before this truncation are displayed in the solution. Atomic energies omit the constant rest energy.

- **1a:**  $\omega_L(1 + \beta) = \omega_0$ , equivalently  $\omega_L + qv = \omega_0$ .
- **1b:**  $p_{at} = mv - \hbar q$ .
- **1c:**  $\varepsilon_{at} = K + \hbar\omega_L = K + \hbar\omega_0 - \hbar qv$ .
- **2a:**  $\varepsilon_{ph} = \hbar\omega_0(1 - \beta) \simeq \hbar\omega_L$ .
- **2b:**  $p_{ph} = -\frac{\hbar\omega_0}{c}(1 - \beta) \simeq -\hbar q$ .
- **2c:**  $p_{at} = mv$ .
- **2d:**  $\varepsilon_{at} = K$ .
- **3a:**  $\varepsilon_{ph} = \hbar\omega_0(1 + \beta) \simeq \hbar\omega_L(1 + 2\beta)$ .
- **3b:**  $p_{ph} = \frac{\hbar\omega_0}{c}(1 + \beta) \simeq \hbar q(1 + 2\beta)$ .
- **3c:**  $p_{at} \simeq mv - 2\hbar q$ .
- **3d:**  $\varepsilon_{at} \simeq K - 2\hbar qv$ .
- **4a:**  $\langle \varepsilon_{ph} \rangle = \hbar\omega_0$ .
- **4b:**  $\langle p_{ph} \rangle = \frac{\hbar\omega_0 v}{c^2} \simeq \hbar q\beta$ .
- **4c:**  $\langle \varepsilon_{at} \rangle \simeq K - \hbar qv$ .
- **4d:**  $\langle p_{at} \rangle \simeq mv - \hbar q$ .
- **5a:**  $\Delta\varepsilon = -\frac{\hbar\omega_0 v}{c} \simeq -\hbar qv$ .
- **5b:**  $\Delta p \simeq -\hbar q$ .
- **6a:**  $\Delta\varepsilon = +\frac{\hbar\omega_0 v}{c} \simeq +\hbar q'v$ .
- **6b:**  $\Delta p \simeq +\hbar q'$ .

- **7a:**

$$F(v) = N\hbar q \Gamma \Omega_R^2 \left[ \frac{1}{(\Delta + qv)^2 + \Gamma^2/4 + 2\Omega_R^2} - \frac{1}{(\Delta - qv)^2 + \Gamma^2/4 + 2\Omega_R^2} \right].$$

- **8a:**

$$F(v) \simeq -N\alpha v = -\frac{4N\hbar q^2 \Gamma \Omega_R^2 (\omega_0 - \omega_L)}{[(\omega_0 - \omega_L)^2 + \Gamma^2/4 + 2\Omega_R^2]^2} v.$$

- **8b:**  $\omega_L > \omega_0$ .
- **8c:**  $\omega_L = \omega_0$ .
- **8d:**  $\omega_L < \omega_0$ .
- **8e:**  $\omega_L < \omega_0$ ; the force then points in  $+x$ .
- **9a:**  $v(\tau) = v_0 e^{-\alpha\tau/m}$ , with  $\alpha > 0$ .
- **9b:**  $T(\tau) = T_0 e^{-2\alpha\tau/m}$ .

## Solution

The photon energy–momentum relation gives

$$\hbar\omega_L = c\hbar q, \quad q = \frac{\omega_L}{c}.$$

Write the two small parameters as

$$\beta = \frac{v}{c} \ll 1, \quad \eta = \frac{\hbar q}{mv} \ll 1.$$

The recoil energy  $(\hbar q)^2/(2m)$  is quadratic in  $\eta$  relative to the initial atomic kinetic energy, so it is omitted. The change in atomic velocity following absorption is  $-\hbar q/m$ ; its contribution to a Doppler factor is

$$\frac{\hbar q}{mc} = \eta\beta,$$

which is also omitted.

Thus, when calculating emitted photon frequencies, the emitting atom can be assigned velocity  $v$ . Momentum corrections of order  $\hbar q\beta = mv\eta\beta$  are shown where useful, but omitted from the leading atomic recoil results.

### 1a. Resonance condition

For a photon propagating in  $-x$ , the frequency measured by an atom moving in  $+x$  is, to first order,

$$\omega_{\text{atom}} = \omega_L - \mathbf{k} \cdot \mathbf{v} = \omega_L + qv.$$

Resonance requires this frequency to equal  $\omega_0$ :

$$\boxed{\omega_L + qv = \omega_0}, \quad \boxed{\omega_L \left(1 + \frac{v}{c}\right) = \omega_0}.$$

This also follows from conservation of energy:

$$\frac{1}{2}mv^2 + \hbar\omega_L = \frac{(mv - \hbar q)^2}{2m} + \hbar\omega_0.$$

Discarding the recoil energy gives the same condition.

### 1b. Atomic momentum after absorption

Conservation of momentum gives

$$p_{at} = mv - \hbar q.$$

### 1c. Atomic energy after absorption

The atom acquires the incident photon's energy:

$$\varepsilon_{at} = \frac{1}{2}mv^2 + \hbar\omega_L.$$

Equivalently, adding its kinetic and internal energies,

$$\varepsilon_{at} = \frac{(mv - \hbar q)^2}{2m} + \hbar\omega_0 \simeq \frac{1}{2}mv^2 - \hbar qv + \hbar\omega_0.$$

The resonance condition makes these expressions equal to the retained order.

For the emission questions, let  $s = +1$  denote a photon emitted in  $+x$ , and  $s = -1$  a photon emitted in  $-x$ . Neglecting recoil in the transition frequency, its energy in the emitting atom's rest frame is  $\hbar\omega_0$ .

The first-order Doppler transformation gives

$$\varepsilon_{ph}^{(s)} = \hbar\omega_0(1 + s\beta), \quad p_{ph}^{(s)} = s \frac{\hbar\omega_0}{c}(1 + s\beta).$$

### 2a. Photon energy for emission in $-x$

Taking  $s = -1$ ,

$$\varepsilon_{ph}^{(-)} = \hbar\omega_0(1 - \beta).$$

Since

$$\omega_L = \frac{\omega_0}{1 + \beta} \simeq \omega_0(1 - \beta),$$

this is

$$\varepsilon_{ph}^{(-)} \simeq \hbar\omega_L.$$

### 2b. Photon momentum for emission in $-x$

Using  $p_{ph} = -\varepsilon_{ph}/c$ ,

$$p_{ph}^{(-)} = -\frac{\hbar\omega_0}{c}(1 - \beta) \simeq -\hbar q.$$

### 2c. Atomic momentum after emission in $-x$

Conservation of momentum over the complete process gives

$$p_{at}^{(-)} = mv - \hbar q - p_{ph}^{(-)} \simeq mv - \hbar q + \hbar q.$$

Therefore

$$p_{at}^{(-)} = mv.$$

### 2d. Atomic energy after emission in $-x$

The emitted photon carries away the energy supplied by the absorbed photon:

$$\varepsilon_{at}^{(-)} = K + \hbar\omega_L - \varepsilon_{ph}^{(-)} \simeq K.$$

Thus

$$\varepsilon_{at}^{(-)} = \frac{1}{2}mv^2.$$

### 3a. Photon energy for emission in $+x$

For  $s = +1$ ,

$$\varepsilon_{ph}^{(+)} = \hbar\omega_0(1 + \beta).$$

Using  $\omega_0 = \omega_L(1 + \beta)$ ,

$$\varepsilon_{ph}^{(+)} \simeq \hbar\omega_L(1 + 2\beta).$$

### 3b. Photon momentum for emission in $+x$

The momentum is positive:

$$p_{ph}^{(+)} = \frac{\hbar\omega_0}{c}(1 + \beta) \simeq \hbar q(1 + 2\beta).$$

### 3c. Atomic momentum after emission in $+x$

Momentum conservation, retaining the photon's Doppler correction temporarily, gives

$$p_{at}^{(+)} = mv - \hbar q - \hbar q(1 + 2\beta) = mv - 2\hbar q - 2\hbar q\beta.$$

The last term is of relative order  $\eta\beta$  in the atomic momentum. With that mixed correction omitted,

$$p_{at}^{(+)} \simeq mv - 2\hbar q.$$

### 3d. Atomic energy after emission in $+x$

The atom returns to its ground state. Conservation of energy gives

$$\begin{aligned} \varepsilon_{at}^{(+)} &= K + \hbar\omega_L - \hbar\omega_0(1 + \beta) \\ &\simeq K + \hbar\omega_0(1 - \beta) - \hbar\omega_0(1 + \beta) \\ &= K - 2\hbar\omega_0\beta. \end{aligned}$$

At the retained atomic accuracy,

$$\varepsilon_{at}^{(+)} \simeq \frac{1}{2}mv^2 - 2\hbar qv.$$

This also follows by expanding

$$\frac{(mv - 2\hbar q)^2}{2m} = K - 2\hbar qv + \frac{2\hbar^2 q^2}{m}$$

and discarding the quadratic recoil term.

#### 4a. Average emitted photon energy

The two directions have equal probability:

$$\begin{aligned}\langle \varepsilon_{ph} \rangle &= \frac{1}{2} \hbar \omega_0 (1 - \beta) + \frac{1}{2} \hbar \omega_0 (1 + \beta) \\ &= \boxed{\hbar \omega_0}.\end{aligned}$$

#### 4b. Average emitted photon momentum

Averaging the signed momenta,

$$\begin{aligned}\langle p_{ph} \rangle &= \frac{1}{2} \left[ -\frac{\hbar \omega_0}{c} (1 - \beta) + \frac{\hbar \omega_0}{c} (1 + \beta) \right] \\ &= \boxed{\frac{\hbar \omega_0 v}{c^2}} \simeq \boxed{\hbar q \beta}.\end{aligned}$$

Although the directions are equally probable, the two photon energies differ in the laboratory. Their mean laboratory momentum is consequently small and positive.

Relative to the atomic momentum, this mean is of order

$$\frac{\langle p_{ph} \rangle}{mv} \simeq \eta \beta.$$

It is therefore omitted in the leading atomic momentum changes below.

#### 4c. Average atomic energy

Using the results of 2d and 3d,

$$\langle \varepsilon_{at} \rangle = \frac{1}{2} K + \frac{1}{2} (K - 2\hbar q v).$$

Hence

$$\boxed{\langle \varepsilon_{at} \rangle \simeq K - \hbar q v}.$$

Equivalently,

$$\langle \varepsilon_{at} \rangle = K + \hbar \omega_L - \hbar \omega_0.$$

#### 4d. Average atomic momentum

Before discarding the mixed correction, momentum conservation gives

$$\langle p_{at} \rangle = mv - \hbar q - \langle p_{ph} \rangle = mv - \hbar q - \frac{\hbar \omega_0 v}{c^2}.$$

Thus, to the requested leading atomic recoil order,

$$\boxed{\langle p_{at} \rangle \simeq mv - \hbar q}.$$

#### 5a. Average atomic energy change

Subtracting the initial energy,

$$\Delta \varepsilon = \langle \varepsilon_{at} \rangle - K = \hbar (\omega_L - \omega_0).$$

The resonance condition yields

$$\boxed{\Delta \varepsilon = -\frac{\hbar \omega_0 v}{c} \simeq -\hbar q v}.$$

The emitted photon has, on average, more energy than the absorbed photon. That difference comes from the atom's kinetic energy.

### 5b. Average atomic momentum change

Subtracting the initial momentum from 4d gives

$$\Delta p \simeq -\hbar q.$$

The smaller correction before truncation is

$$-\langle p_{ph} \rangle = -\frac{\hbar \omega_0 v}{c^2}.$$

At leading recoil order, the energy and momentum changes obey

$$\Delta \varepsilon \simeq v \Delta p.$$

### 6a. Energy change for a laser propagating in $+x$

The atom now moves away from the incident light. The resonance condition is

$$\omega'_L \left(1 - \frac{v}{c}\right) = \omega_0,$$

so

$$\omega'_L \simeq \omega_0(1 + \beta).$$

The mean emitted photon energy remains  $\hbar \omega_0$ . Therefore

$$\Delta \varepsilon = \hbar(\omega'_L - \omega_0) = \boxed{\frac{\hbar \omega_0 v}{c}} \simeq \boxed{\hbar q' v}, \quad q' = \frac{\omega'_L}{c}.$$

The atom gains kinetic energy.

### 6b. Momentum change for a laser propagating in $+x$

The absorbed photon supplies momentum  $+\hbar q'$ . Consequently,

$$\Delta p = \hbar q' - \frac{\hbar \omega_0 v}{c^2} \simeq \boxed{+\hbar q'}.$$

The omitted second term is again a mixed Doppler–recoil correction in the atomic momentum.

### 7a. Force from the two laser beams

Use the supplied scattering law separately for the two beams, taking their intensities, and hence their Rabi frequencies, to be equal.

Let  $+$  label the beam propagating in  $+x$ , and  $-$  the beam propagating in  $-x$ . The frequencies seen by the moving atoms are

$$\omega_{\text{atom},+} = \omega_L - qv, \quad \omega_{\text{atom},-} = \omega_L + qv.$$

Thus their detunings from the atomic transition are

$$\omega_0 - \omega_{\text{atom},+} = \Delta + qv, \quad \omega_0 - \omega_{\text{atom},-} = \Delta - qv.$$

The total numbers of scattering events per unit time are

$$R_+(v) = \frac{N\Gamma\Omega_R^2}{(\Delta + qv)^2 + \Gamma^2/4 + 2\Omega_R^2},$$

$$R_{-}(v) = \frac{N\Gamma\Omega_R^2}{(\Delta - qv)^2 + \Gamma^2/4 + 2\Omega_R^2}.$$

At leading recoil order, each event transfers  $+\hbar q$  or  $-\hbar q$ , respectively. Therefore

$$F(v) = \hbar q R_{+}(v) - \hbar q R_{-}(v),$$

giving

$$F(v) = N\hbar q \Gamma \Omega_R^2 \left[ \frac{1}{(\Delta + qv)^2 + \Gamma^2/4 + 2\Omega_R^2} - \frac{1}{(\Delta - qv)^2 + \Gamma^2/4 + 2\Omega_R^2} \right].$$

This is the force on all  $N$  atoms. The force on one atom is  $F/N$ .

The Doppler shift  $qv$  is retained in the scattering probabilities: it can be significant compared with the transition linewidth even when  $v/c$  is very small.

### 8a. Low-velocity expansion

Put

$$D = \Delta^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2.$$

For sufficiently small  $qv$ ,

$$\frac{1}{(\Delta \pm qv)^2 + \Gamma^2/4 + 2\Omega_R^2} \simeq \frac{1}{D} \mp \frac{2\Delta qv}{D^2}.$$

For example,  $|qv| \ll \sqrt{D}$  is a sufficient small-velocity condition.

Subtracting the two expansions,

$$F(v) \simeq N\hbar q \Gamma \Omega_R^2 \left( -\frac{4\Delta qv}{D^2} \right).$$

Hence

$$F(v) \simeq -N\alpha v, \quad \alpha = \frac{4\hbar q^2 \Gamma \Omega_R^2 \Delta}{D^2}.$$

The coefficient  $\alpha$  has units of mass per time, so  $\alpha v$  has units of force.

### 8b. Positive force

For  $v > 0$ , all factors in  $\alpha$  except  $\Delta$  are positive. A positive force requires

$$\Delta < 0.$$

Therefore

$$\omega_L > \omega_0.$$

This is blue detuning.

### 8c. Zero force

For nonzero intensity and a moving atom, the detuning condition is

$$\Delta = 0,$$

or

$$\omega_L = \omega_0.$$

The two scattering rates are then equal. Independently, symmetry also gives zero force at  $v = 0$ .

### 8d. Negative force

For  $v > 0$ , a negative force requires  $\alpha > 0$ , hence

$$\omega_L < \omega_0.$$

This is red detuning. The counterpropagating beam is Doppler shifted closer to resonance and is scattered more strongly.

### 8e. Slowing atoms moving in $-x$

For velocity  $-v$ , with  $v > 0$ ,

$$F(-v) = +N\alpha v.$$

Slowing requires a force in  $+x$ , so again  $\alpha > 0$ . Thus

$$\omega_L < \omega_0.$$

Red detuning produces a force opposing motion in either direction.

### 9a. Velocity after time $\tau$

For red detuning,  $\alpha > 0$ . The gas has total mass  $Nm$ , so Newton's second law gives

$$Nm \frac{dv}{dt} = -N\alpha v.$$

The factor  $N$  cancels:

$$\frac{dv}{dt} = -\frac{\alpha}{m}v.$$

Integrating with  $v(0) = v_0$ ,

$$v(\tau) = v_0 e^{-\alpha\tau/m}.$$

Explicitly,

$$v(\tau) = v_0 \exp \left[ -\frac{4\hbar q^2 \Gamma \Omega_R^2 (\omega_0 - \omega_L) \tau}{m [(\omega_0 - \omega_L)^2 + \Gamma^2/4 + 2\Omega_R^2]^2} \right].$$

The velocity decay time is  $m/\alpha$ .

### 9b. Temperature after time $\tau$

In this deterministic friction model, each atomic velocity is multiplied by

$$e^{-\alpha\tau/m}.$$

Therefore the variance of the velocities is multiplied by

$$e^{-2\alpha\tau/m}.$$

For one-dimensional thermal motion,

$$m \langle (v - \langle v \rangle)^2 \rangle = k_B T.$$

Consequently,

$$T(\tau) = T_0 e^{-2\alpha\tau/m}.$$

Explicitly,

$$T(\tau) = T_0 \exp \left[ -\frac{8\hbar q^2 \Gamma \Omega_R^2 (\omega_0 - \omega_L) \tau}{m [(\omega_0 - \omega_L)^2 + \Gamma^2/4 + 2\Omega_R^2]^2} \right].$$

This exponential cannot be extrapolated to arbitrarily low temperatures. The calculation omitted quadratic recoil contributions and the fluctuations caused by random scattering events. In particular, the assumption  $mv \gg \hbar q$ , applied to a typical thermal speed, requires

$$k_B T \gg \frac{(\hbar q)^2}{m}.$$

As the atoms cool, the terms omitted from this model become relevant.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

### Problem 3: Why Are Stars So Large?

The stars are spheres of hot gas. Most of them shine because they are fusing hydrogen into helium in their central parts. In this problem we use concepts of both classical and quantum mechanics, as well as of electrostatics and thermodynamics, to understand why stars have to be big enough to achieve this fusion process and also derive what would be the mass and radius of the smallest star that can fuse hydrogen.

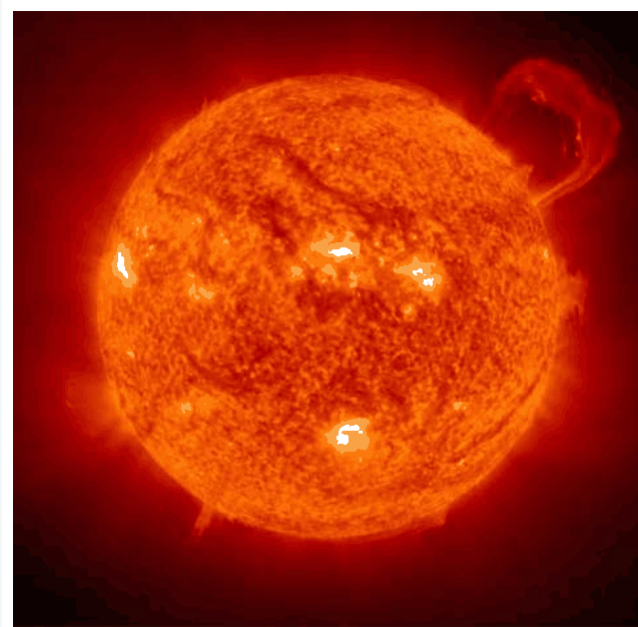


Figure 1. Our Sun, as most stars, shines as a result of thermonuclear fusion of hydrogen into helium in its central parts.

#### USEFUL CONSTANTS

Gravitational constant =  $G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$  (misprint in the official paper:  $\text{s}^2$  instead of  $\text{s}^{-2}$ )

### USEFUL CONSTANTS

Boltzmann's constant =  $k = 1.4 \times 10^{-23} \text{ J K}^{-1}$

Planck's constant =  $h = 6.6 \times 10^{-34} \text{ m}^2 \text{ kg s}^{-1}$

Mass of the proton =  $m_p = 1.7 \times 10^{-27} \text{ kg}$

Mass of the electron =  $m_e = 9.1 \times 10^{-31} \text{ kg}$

Unit of electric charge =  $q = 1.6 \times 10^{-19} \text{ C}$

Electric constant (vacuum permittivity) =  $\epsilon_0 = 8.9 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$

Radius of the Sun =  $R_S = 7.0 \times 10^8 \text{ m}$

Mass of the Sun =  $M_S = 2.0 \times 10^{30} \text{ kg}$

#### 1. A classical estimate of the temperature at the center of the stars.

Assume that the gas that forms the star is pure ionized hydrogen (electrons and protons in equal amounts), and that it behaves like an ideal gas. From the classical point of view, to fuse two protons, they need to get as close as  $10^{-15} \text{ m}$  for the short range strong nuclear force, which is attractive, to become dominant. However, to bring them together they have to overcome first the repulsive action of Coulomb's force. Assume classically that the two protons (taken to be point sources) are moving in an antiparallel way, each with velocity  $v_{rms}$ , the root-mean-square (rms) velocity of the protons, in a one-dimensional frontal collision.

|    |                                                                                                                                                                                                                                     |     |
|----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 1a | What has to be the temperature of the gas, $T_c$ , so that the distance of closest approach of the protons, $d_c$ , equals $10^{-15} \text{ m}$ ? Give this and all numerical values in this problem up to two significant figures. | 1.5 |
|----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|

#### 2. Finding that the previous temperature estimate is wrong.

To check if the previous temperature estimate is reasonable, one needs an independent way of estimating the central temperature of a star. The structure of the stars is very complicated, but we can gain significant understanding making some assumptions. Stars are in equilibrium, that is, they do not expand or contract because the inward force of gravity is balanced by the outward force of pressure (see Figure 2). For a slab of gas the equation of hydrostatic equilibrium at a given distance  $r$  from the center of the star, is given by

$$\frac{\Delta P}{\Delta r} = -\frac{GM_r \rho_r}{r^2},$$

where  $P$  is the pressure of the gas,  $G$  the gravitational constant,  $M_r$  the mass of the star within a sphere of radius  $r$ , and  $\rho_r$  is the density of the gas in the slab.

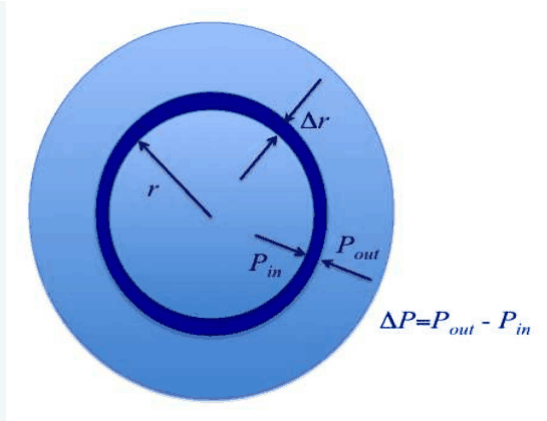


Figure 2. The stars are in hydrostatic equilibrium, with the pressure difference balancing gravity.

An order of magnitude estimate of the central temperature of the star can be obtained with values of the parameters at the center and at the surface of the star, making the following approximations:

$$\Delta P \approx P_o - P_c,$$

where  $P_c$  and  $P_o$  are the pressures at the center and surface of the star, respectively. Since  $P_c \gg P_o$ , we can assume that

$$\Delta P \approx -P_c.$$

Within the same approximation, we can write

$$\Delta r \approx R,$$

where  $R$  is the total radius of the star, and

$$M_r \approx M_R = M,$$

with  $M$  the total mass of the star.

The density may be approximated by its value at the center,

$$\rho_r \approx \rho_c.$$

You can assume that the pressure is that of an ideal gas.

|    |                                                                                                                                                     |     |
|----|-----------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 2a | Find an equation for the temperature at the center of the star, $T_c$ , in terms of the radius and mass of the star and of physical constants only. | 0.5 |
|----|-----------------------------------------------------------------------------------------------------------------------------------------------------|-----|

We can use now the following prediction of this model as a criterion for its validity:

|    |                                                                                                                                 |     |
|----|---------------------------------------------------------------------------------------------------------------------------------|-----|
| 2b | Using the equation found in (2a) write down the ratio $M/R$ expected for a star in terms of physical constants and $T_c$ only.  | 0.5 |
| 2c | Use the value of $T_c$ derived in section (1a) and find the numerical value of the ratio $M/R$ expected for a star.             | 0.5 |
| 2d | Now, calculate the ratio $M(\text{Sun})/R(\text{Sun})$ , and verify that this value is much smaller than the one found in (2c). | 0.5 |

### 3. A quantum mechanical estimate of the temperature at the center of the stars

The large discrepancy found in (2d) suggests that the classical estimate for  $T_c$  obtained in (1a) is not correct. The solution to this discrepancy is found when we consider quantum mechanical effects, that tell us that the protons behave as waves and that a single proton is smeared on a size

of the order of  $\lambda_p$ , the de Broglie wavelength. This implies that if  $d_c$ , the distance of closest approach of the protons is of the order of  $\lambda_p$ , the protons in a quantum mechanical sense overlap and can fuse.

|    |                                                                                                                                                                                                                              |     |
|----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 3a | Assuming that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition that allows fusion, for a proton with velocity $v_{rms}$ , find an equation for $T_c$ in terms of physical constants only.                                  | 1.0 |
| 3b | Evaluate numerically the value of $T_c$ obtained in (3a).                                                                                                                                                                    | 0.5 |
| 3c | Use the value of $T_c$ derived in (3b) to find the numerical value of the ratio $M/R$ expected for a star, using the formula derived in (2b). Verify that this value is quite similar to the ratio $M(Sun)/R(Sun)$ observed. | 0.5 |

Indeed, stars in the so-called *main sequence* (fusing hydrogen) approximately do follow this ratio for a large range of masses.

#### 4. The mass/radius ratio of the stars.

The previous agreement suggests that the quantum mechanical approach for estimating the temperature at the center of the Sun is correct.

|    |                                                                                                                                                                                                                                     |     |
|----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 4a | Use the previous results to demonstrate that for any star fusing hydrogen, the ratio of mass $M$ to radius $R$ is the same and depends only on physical constants. Find the equation for the ratio $M/R$ for stars fusing hydrogen. | 0.5 |
|----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|

#### 5. The mass and radius of the smallest star.

The result found in (4a) suggests that there could be stars of any mass as long as such a relationship is fulfilled; however, this is not true.

The gas inside normal stars fusing hydrogen is known to behave approximately as an ideal gas. This means that  $d_e$ , the typical *separation between electrons* is on the average larger than  $\lambda_e$ , their typical de Broglie wavelength. If closer, the electrons would be in a so-called degenerate state and the stars would behave differently. Note the distinction in the ways we treat protons and electrons inside the star. For protons, their de Broglie waves should overlap closely as they collide in order to fuse, whereas for electrons their de Broglie waves should not overlap in order to remain as an ideal gas.

The density in the stars increases with decreasing radius. Nevertheless, for this order-of-magnitude estimate assume they are of uniform density. You may further use that  $m_p \gg m_e$ .

|    |                                                                                                                                                                                                                               |     |
|----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 5a | Find an equation for $n_e$ , the average electron number density inside the star.                                                                                                                                             | 0.5 |
| 5b | Find an equation for $d_e$ , the typical separation between electrons inside the star.                                                                                                                                        | 0.5 |
| 5c | Use the $d_e \geq \frac{\lambda_e}{2^{1/2}}$ condition to write down an equation for the radius of the smallest normal star possible. Take the temperature at the center of the star as typical for all the stellar interior. | 1.5 |
| 5d | Find the numerical value of the radius of the smallest normal star possible, both in meters and in units of solar radius.                                                                                                     | 0.5 |
| 5e | Find the numerical value of the mass of the smallest normal star possible, both in kg and in units of solar masses.                                                                                                           | 0.5 |

## 6. Fusing helium nuclei in older stars.

As stars get older they will have fused most of the hydrogen in their cores into helium (He), so they are forced to start fusing helium into heavier elements in order to continue shining. A helium nucleus has two protons and two neutrons, so it has twice the charge and approximately four times the mass of a proton. We saw before that  $d_c = \frac{\lambda_p}{2^{1/2}}$  is the condition for the protons to fuse.

|    |                                                                                                                                                                       |     |
|----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 6a | Set the equivalent condition for helium nuclei and find $v_{rms}(He)$ , the rms velocity of the helium nuclei and $T(He)$ , the temperature needed for helium fusion. | 0.5 |
|----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----|

Official English text of the problem.

## Answers

All results use the approximations prescribed in the problem. Numerical results are rounded to two significant figures, with unrounded values retained in intermediate calculations.

- 1a.  $T_c = \frac{q^2}{12\pi\epsilon_0 k d_c} = 5.4 \times 10^9 \text{ K}.$
- 2a.  $T_c \simeq \frac{GMm_p}{2kR}.$
- 2b.  $\frac{M}{R} \simeq \frac{2kT_c}{Gm_p}.$
- 2c.  $\frac{M}{R} = 1.3 \times 10^{24} \text{ kg m}^{-1}.$
- 2d.  $\frac{M_S}{R_S} = 2.9 \times 10^{21} \text{ kg m}^{-1}.$  The classical prediction is  $4.7 \times 10^2$  times larger.
- 3a.  $T_c = \frac{m_p q^4}{24\pi^2 \epsilon_0^2 k h^2}.$
- 3b.  $T_c = 9.7 \times 10^6 \text{ K}.$
- 3c.  $\frac{M}{R} = 2.4 \times 10^{21} \text{ kg m}^{-1} \simeq 0.84 \frac{M_S}{R_S}.$
- 4a.  $\frac{M}{R} = \frac{q^4}{12\pi^2 \epsilon_0^2 G h^2}.$
- 5a.  $n_e = \frac{3M}{4\pi m_p R^3}.$
- 5b.  $d_e \simeq n_e^{-1/3} = \left( \frac{4\pi m_p R^3}{3M} \right)^{1/3}.$
- 5c.  $R_{\min} = \left( \frac{\epsilon_0 h^4}{2Gq^2 m_p^{5/2} m_e^{3/2}} \right)^{1/2}.$
- 5d.  $R_{\min} = 6.9 \times 10^7 \text{ m} = 0.099 R_S.$
- 5e.  $M_{\min} = 1.7 \times 10^{29} \text{ kg} = 0.083 M_S.$

- **6a.** The condition is  $d_c(He) = \frac{\lambda_{He}}{\sqrt{2}} = \frac{h}{4\sqrt{2} m_p v_{rms}(He)}$ , giving

$$v_{rms}(He) = \frac{\sqrt{2} q^2}{\pi \epsilon_0 h} = 2.0 \times 10^6 \text{ m s}^{-1}, \quad T(He) = \frac{8 m_p q^4}{3 \pi^2 \epsilon_0^2 k h^2} = 6.2 \times 10^8 \text{ K}.$$

## Solution

### 1a. Classical temperature required for fusion

For a nonrelativistic ideal gas, equipartition gives the proton rms speed through

$$\frac{1}{2} m_p v_{rms}^2 = \frac{3}{2} k T_c.$$

The collision is head-on, but  $v_{rms}$  is the rms speed in the three-dimensional gas.

The two protons have equal and opposite momenta, so their center of mass is stationary. At closest approach, both momentarily stop. Taking their initial electrostatic potential energy to be negligible, conservation of energy gives

$$2 \left( \frac{1}{2} m_p v_{rms}^2 \right) = \frac{q^2}{4 \pi \epsilon_0 d_c}.$$

Thus

$$3 k T_c = \frac{q^2}{4 \pi \epsilon_0 d_c}, \quad \boxed{T_c = \frac{q^2}{12 \pi \epsilon_0 k d_c}}.$$

For  $d_c = 10^{-15} \text{ m}$ ,

$$T_c = \frac{(1.6 \times 10^{-19})^2}{12 \pi (8.9 \times 10^{-12}) (1.4 \times 10^{-23}) (10^{-15})},$$

so

$$\boxed{T_c = 5.4 \times 10^9 \text{ K}}.$$

### 2a. Central temperature from hydrostatic equilibrium

Using the prescribed approximations, including a characteristic radial scale  $r \simeq R$ ,

$$-\frac{P_c}{R} \simeq -\frac{GM \rho_c}{R^2},$$

hence

$$P_c \simeq \frac{GM \rho_c}{R}.$$

In fully ionized hydrogen, the proton and electron number densities are equal. Neglecting electron mass in the mass density,

$$n_{p,c} = n_{e,c} \simeq \frac{\rho_c}{m_p}.$$

Both species contribute to the ideal-gas pressure:

$$P_c = (n_{p,c} + n_{e,c})kT_c = \frac{2\rho_c kT_c}{m_p}.$$

Equating the pressure expressions and cancelling  $\rho_c$ ,

$$T_c \simeq \frac{GMm_p}{2kR}.$$

## 2b. Mass-to-radius ratio

Rearranging the result of (2a),

$$\frac{M}{R} \simeq \frac{2kT_c}{Gm_p}.$$

## 2c. Ratio predicted using the classical fusion temperature

Substitution of the temperature from (1a) gives

$$\frac{M}{R} = \frac{2k}{Gm_p} \frac{q^2}{12\pi\epsilon_0 k d_c} = \frac{q^2}{6\pi\epsilon_0 Gm_p d_c}.$$

Numerically,

$$\frac{M}{R} = 1.3 \times 10^{24} \text{ kg m}^{-1}.$$

## 2d. Comparison with the Sun

The supplied solar mass and radius give

$$\frac{M_S}{R_S} = \frac{2.0 \times 10^{30}}{7.0 \times 10^8} = \boxed{2.9 \times 10^{21} \text{ kg m}^{-1}}.$$

Using unrounded values, the comparison is

$$\frac{(M/R)_{\text{classical}}}{M_S/R_S} = \boxed{4.7 \times 10^2}.$$

The Sun's ratio is therefore much smaller than the classical prediction. Within the hydrostatic model, the classical fusion temperature is far too high.

## 3a. Quantum estimate of the fusion temperature

For a proton moving at  $v_{rms}$ , the de Broglie wavelength is

$$\lambda_p = \frac{h}{m_p v_{rms}}.$$

The collision-energy relation from (1a) gives

$$d_c = \frac{q^2}{4\pi\epsilon_0 m_p v_{rms}^2}.$$

Imposing the problem's fusion condition,

$$d_c = \frac{\lambda_p}{\sqrt{2}},$$

we obtain

$$\frac{q^2}{4\pi\epsilon_0 m_p v_{rms}^2} = \frac{h}{\sqrt{2} m_p v_{rms}},$$

and consequently

$$v_{rms} = \frac{\sqrt{2} q^2}{4\pi\epsilon_0 h}.$$

Using  $m_p v_{rms}^2 = 3kT_c$ ,

$$T_c = \frac{m_p}{3k} \left( \frac{\sqrt{2} q^2}{4\pi\epsilon_0 h} \right)^2.$$

Therefore,

$$T_c = \frac{m_p q^4}{24\pi^2 \epsilon_0^2 k h^2}.$$

### 3b. Numerical quantum temperature

Substituting the given constants,

$$T_c = \frac{(1.7 \times 10^{-27})(1.6 \times 10^{-19})^4}{24\pi^2 (8.9 \times 10^{-12})^2 (1.4 \times 10^{-23})(6.6 \times 10^{-34})^2},$$

which gives

$$T_c = 9.7 \times 10^6 \text{ K}.$$

The corresponding proton rms speed is  $4.9 \times 10^5 \text{ m s}^{-1}$ .

### 3c. Mass-to-radius ratio using the quantum temperature

Using (2b) with the temperature from (3b),

$$\frac{M}{R} = \frac{2kT_c}{Gm_p} = 2.4 \times 10^{21} \text{ kg m}^{-1}.$$

The ratio to the solar value is

$$\frac{M/R}{M_S/R_S} = 0.84.$$

Thus the prediction is about 16% below the solar value, a close agreement for the order-of-magnitude approximations used.

### 4a. Common mass-to-radius ratio

Within this model, hydrogen fusion has the same characteristic temperature in every star. Combining (2b) and (3a),

$$\frac{M}{R} = \frac{2k}{Gm_p} \frac{m_p q^4}{24\pi^2 \epsilon_0^2 k h^2}.$$

The factors  $m_p$  and  $k$  cancel:

$$\frac{M}{R} = \frac{q^4}{12\pi^2 \epsilon_0^2 G h^2}.$$

This depends only on physical constants. For the following parts, denote this common ratio by

$$A \equiv \frac{q^4}{12\pi^2\epsilon_0^2 G h^2}, \quad M = AR.$$

### 5a. Mean electron number density

Each hydrogen proton supplies one electron. Neglecting electron mass,

$$N_e = N_p \simeq \frac{M}{m_p}.$$

For a uniform spherical star of volume  $V = 4\pi R^3/3$ ,

$$n_e = \frac{N_e}{V} = \frac{3M}{4\pi m_p R^3}.$$

### 5b. Typical electron separation

The typical volume per electron is  $1/n_e$ . Taking its cube root as the characteristic spacing,

$$d_e \simeq n_e^{-1/3} = \left( \frac{4\pi m_p R^3}{3M} \right)^{1/3}.$$

### 5c. Radius of the smallest normal star

Equipartition for electrons gives

$$m_e v_{e,rms}^2 = 3kT_c.$$

Their characteristic de Broglie wavelength is therefore

$$\lambda_e = \frac{h}{m_e v_{e,rms}} = \frac{h}{\sqrt{3m_e kT_c}}.$$

The prescribed condition for avoiding electron degeneracy becomes

$$d_e \geq \frac{\lambda_e}{\sqrt{2}} = \frac{h}{\sqrt{6m_e kT_c}}.$$

Using  $M = AR$  in the spacing formula,

$$d_e^3 = \frac{4\pi m_p R^2}{3A}.$$

Thus, along the model's mass-radius relation, the spacing decreases as  $R^{2/3}$ . The minimum permitted radius occurs at equality:

$$\frac{4\pi m_p R_{\min}^2}{3A} = \left( \frac{h}{\sqrt{6m_e kT_c}} \right)^3.$$

Hence

$$R_{\min}^2 = \frac{3Ah^3}{4\pi m_p (6m_e kT_c)^{3/2}}.$$

First substitute  $A = 2kT_c/(Gm_p)$ :

$$R_{\min}^2 = \frac{h^3}{4\pi G m_p^2 m_e^{3/2} \sqrt{6kT_c}}.$$

From (3a),

$$\sqrt{6kT_c} = \frac{q^2 \sqrt{m_p}}{2\pi\epsilon_0 h}.$$

Therefore,

$$R_{\min}^2 = \frac{\epsilon_0 h^4}{2Gq^2 m_p^{5/2} m_e^{3/2}},$$

or

$$R_{\min} = \left( \frac{\epsilon_0 h^4}{2Gq^2 m_p^{5/2} m_e^{3/2}} \right)^{1/2}.$$

For a smaller radius along  $M = AR$ , the electron waves would overlap according to the stated criterion, and the assumed ideal-gas description would cease to apply.

#### 5d. Numerical minimum radius

Substitution gives

$$R_{\min} = 6.9 \times 10^7 \text{ m}.$$

Dividing the unrounded result by the supplied solar radius,

$$\frac{R_{\min}}{R_s} = 0.099.$$

Thus the smallest normal star in this model has a radius of approximately one-tenth of the solar radius.

#### 5e. Numerical minimum mass

Because  $M = AR$  with fixed positive  $A$ , the minimum radius also gives the minimum mass:

$$M_{\min} = AR_{\min} = \frac{q^4 R_{\min}}{12\pi^2 \epsilon_0^2 G h^2}.$$

Numerically,

$$M_{\min} = 1.7 \times 10^{29} \text{ kg}, \quad \frac{M_{\min}}{M_s} = 0.083.$$

#### 6a. Helium fusion

A helium nucleus has mass  $m_{He} \simeq 4m_p$  and charge  $Q_{He} = 2q$ . Its de Broglie wavelength is

$$\lambda_{He} = \frac{h}{4m_p v_{rms}(He)}.$$

The corresponding fusion condition is

$$d_c(He) = \frac{\lambda_{He}}{\sqrt{2}} = \frac{h}{4\sqrt{2} m_p v_{rms}(He)}.$$

For two helium nuclei approaching with equal and opposite rms velocities, energy conservation gives

$$4m_p v_{rms}(He)^2 = \frac{(2q)^2}{4\pi\epsilon_0 d_c(He)}.$$

Substituting the fusion condition and simplifying,

$$v_{rms}(He) = \frac{\sqrt{2} q^2}{\pi\epsilon_0 h}.$$

This is four times the proton speed obtained in (3a), so

$$v_{rms}(He) = 2.0 \times 10^6 \text{ m s}^{-1}.$$

Equipartition now gives

$$T(He) = \frac{4m_p v_{rms}(He)^2}{3k}.$$

Since the mass is four times greater and the speed is four times greater,

$$T(He) = 4(4^2)T_c = 64T_c,$$

where  $T_c$  is the hydrogen fusion temperature from (3b). Therefore,

$$T(He) = \frac{8m_p q^4}{3\pi^2 \epsilon_0^2 k h^2} = 6.2 \times 10^8 \text{ K}.$$

---

The solution was checked against the official solution.