

IPhO 2008

Theory problems and solutions · Hanoi, Vietnam · 3 theory problems with complete solutions

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Problem 1: Water-Powered Rice-Pounding Mortar

A. Introduction

Rice is the main staple food of most people in Vietnam. To make white rice from paddy rice, one needs separate of the husk (a process called "hulling") and separate the bran layer ("milling"). The hilly parts of northern Vietnam are abundant with water streams, and people living there use *water-powered rice-pounding mortar* for bran layer separation. Figure 1 shows one of such mortars, Figure 2 shows how it works.

B. Design and operation

1. Design.

The rice-pounding mortar shown in Figure 1 has the following parts:

The mortar, basically a wooden container for rice.

The lever, which is a tree trunk with one larger end and one smaller end. It can rotate around a horizontal axis. A *pestle* is attached perpendicularly to the lever at the smaller end. The length of the pestle is such that it touches the rice in the mortar when the lever lies horizontally. The larger end of the lever is carved hollow to form a bucket. The shape of the bucket is crucial for the mortar's operation.

2. Modes of operation

The mortar has two modes.

Working mode. In this mode, the mortar goes through an operation cycle illustrated in Figure 2.

The rice-pounding function comes from the work that is transferred from the pestle to the rice during stage f) of Figure 2. If, for some reason, the pestle never touches the rice, we say that the mortar is not working.

Rest mode with the lever lifted up. During stage c) of the operation cycle (Figure 2), as the tilt angle α increases, the amount of water in the bucket decreases. At one particular moment in time, the amount of water is just enough to counterbalance the weight of the lever. Denote the tilting angle at this instant by β . If the lever is kept at angle β and the initial angular velocity is zero, then the lever will remain at this position forever. This is the rest mode with the lever lifted up. The stability of this position depends on the flow rate of water into the bucket, Φ . If Φ exceeds some value Φ_2 , then this rest mode is stable, and the mortar cannot be in the working mode.

In other words, Φ_2 is the minimal flow rate for the mortar not to work.



Figure 1
A water-powered rice-pounding mortar

OPERATION CYCLE OF A WATER-POWERED RICE-POUNDING MORTAR

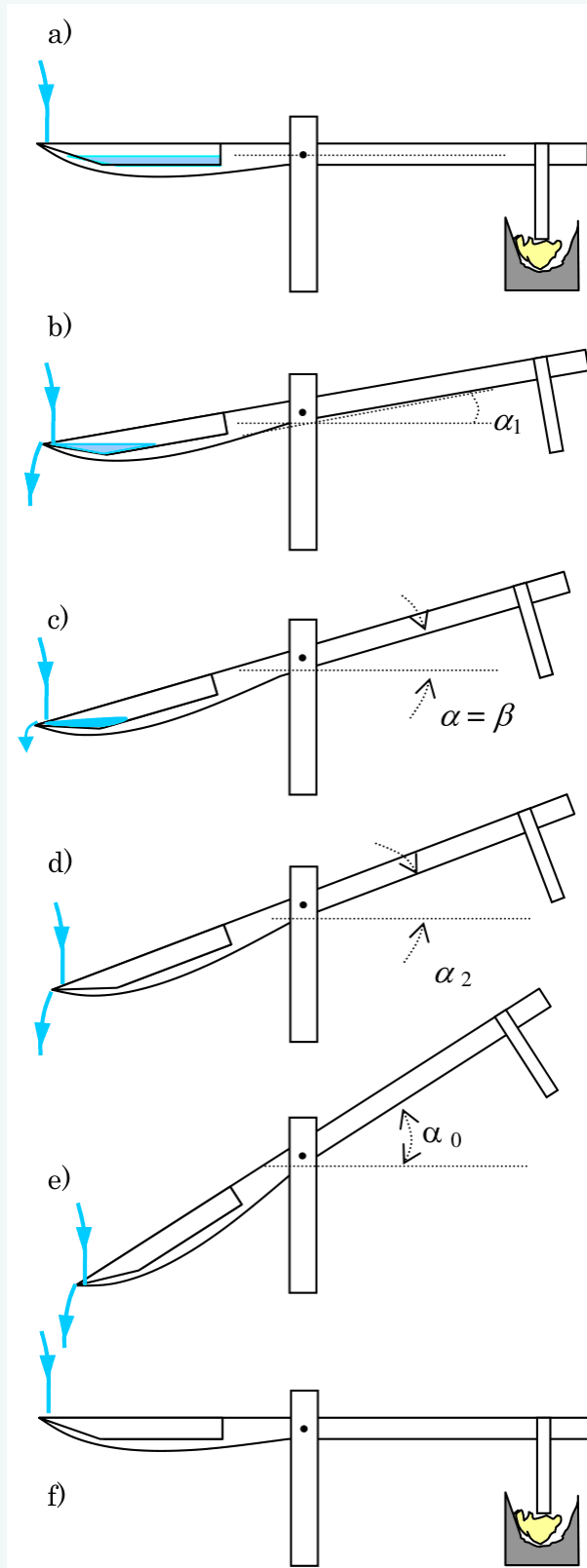


Figure 2

a) At the beginning there is no water in the bucket, the pestle rests on the mortar. Water flows into the bucket with a small rate, but for some time the lever remains in the horizontal position.

b) At some moment the amount of water is enough to lift the lever up. Due to the tilt, water rushes to the farther side of the bucket, tilting the lever more quickly.

Water starts to flow out at $\alpha = \alpha_1$.

c) As the angle α increases, water starts to flow out. At some particular tilt angle, $\alpha = \beta$, the total torque is zero.

d) α continues increasing, water continues to flow out until no water remains in the bucket.

e) α keeps increasing because of inertia. Due to the shape of the bucket, water falls into the bucket but immediately flows out. The inertial motion of the lever continues until α reaches the maximal value α_0 .

f) With no water in the bucket, the weight of the lever pulls it back to the initial horizontal position. The pestle gives the mortar (with rice inside) a pound and a new cycle begins.

C. The problem

Consider a water-powered rice-pounding mortar with the following parameters (Figure 3)

The mass of the lever (including the pestle but without water) is $M = 30 \text{ kg}$,

The center of mass of the lever is G. The lever rotates around the axis T (projected onto the point T on the figure).

The moment of inertia of the lever around T is $I = 12 \text{ kg} \cdot \text{m}^2$.

When there is water in the bucket, the mass of water is denoted as m , the center of mass of the water body is denoted as N.

The tilt angle of the lever with respect to the horizontal axis is α .

The main length measurements of the mortar and the bucket are as in Figure 3.

Neglect friction at the rotation axis and the force due to water falling onto the bucket. In this problem, we make an approximation that the water surface is always horizontal.

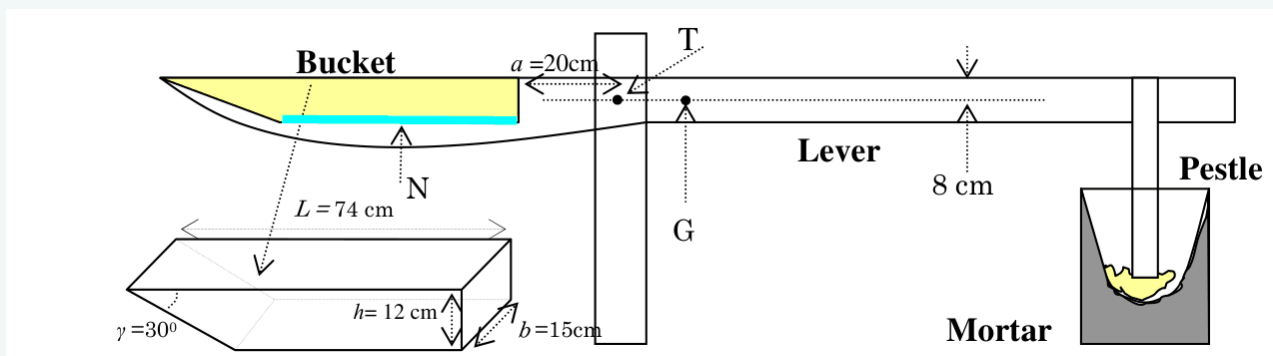


Figure 3 Design and dimensions of the rice-pounding mortar

1. The structure of the mortar

At the beginning, the bucket is empty, and the lever lies horizontally. Then water flows into the bucket until the lever starts rotating. The amount of water in the bucket at this moment is $m = 1.0 \text{ kg}$.

1.1. Determine the distance from the center of mass G of the lever to the rotation axis T. It is known that GT is horizontal when the bucket is empty.

1.2. Water starts flowing out of the bucket when the angle between the lever and the horizontal axis reaches α_1 . The bucket is completely empty when this angle is α_2 .

Determine α_1 and α_2 .

1.3. Let $\mu(\alpha)$ be the total torque (relative to the axis T) which comes from the weight of the lever and the water in the bucket. $\mu(\alpha)$ is zero when $\alpha = \beta$. Determine β and the mass m_1 of water in the bucket at this instant.

2. Parameters of the working mode

Let water flow into the bucket with a flow rate Φ which is constant and small. The amount of water flowing into the bucket when the lever is in motion is negligible. In this part, neglect the change of the moment of inertia during the working cycle.

2.1. Sketch a graph of the torque μ as a function of the angle α , $\mu(\alpha)$, during one operation cycle. Write down explicitly the values of $\mu(\alpha)$ at angle α_1 , α_2 , and $\alpha = 0$.

2.2. From the graph found in section 2.1., discuss and give the geometric interpretation of the value of the total energy W_{total} produced by $\mu(\alpha)$ and the work W_{pounding} that is transferred from the pestle to the rice.

2.3. From the graph representing μ versus α , estimate α_0 and W_{pounding} (assume the kinetic energy of water flowing into the bucket and out of the bucket is negligible.) You may replace curve lines by zigzag lines, if it simplifies the calculation.

3. The rest mode

Let water flow into the bucket with a constant rate Φ , but one cannot neglect the amount of water flowing into the bucket during the motion of the lever.

3.1. Assuming the bucket is always overflowed with water,

3.1.1. Sketch a graph of the torque μ as a function of the angle α in the vicinity of $\alpha = \beta$. To which kind of equilibrium does the position $\alpha = \beta$ of the lever belong?

3.1.2. Find the analytic form of the torque $\mu(\alpha)$ as a function of $\Delta\alpha$ when $\alpha = \beta + \Delta\alpha$, and $\Delta\alpha$ is small.

3.1.3. Write down the equation of motion of the lever, which moves with zero initial velocity from the position $\alpha = \beta + \Delta\alpha$ ($\Delta\alpha$ is small). Show that the motion is, with good accuracy, harmonic oscillation. Compute the period τ .

3.2. At a given Φ , the bucket is overflowed with water at all times only if the lever moves sufficiently slowly. There is an upper limit on the amplitude of harmonic oscillation, which depends on Φ . Determine the minimal value Φ_1 of Φ (in kg/s) so that the lever can make a harmonic oscillator motion with amplitude 1° .

3.3. Assume that Φ is sufficiently large so that during the free motion of the lever when the tilting angle decreases from α_2 to α_1 the bucket is always overflowed with water. However, if Φ is too large the mortar cannot operate. Assuming that the motion of the lever is that of a harmonic oscillator, estimate the minimal flow rate Φ_2 for the rice-pounding mortar to not work.

Official English text of the problem.

Answers

Positive torque denotes increasing α . Numerical results use $\rho = 1000 \text{ kg m}^{-3}$ and $g = 9.8 \text{ m s}^{-2}$.

- 1.1. $GT = 1.57 \text{ cm}$.
- 1.2. $\alpha_1 = 20.62^\circ, \quad \alpha_2 = 30^\circ$.
- 1.3. $\beta = 23.57^\circ, \quad m_1 = 0.605 \text{ kg}$.
- 2.1. On the rising stroke,

$$\mu(0) = 0, \quad \mu(\alpha_1) = +2.69 \text{ N m}, \quad \mu(\alpha_2) = -4.00 \text{ N m}.$$

The empty return branch is $\mu = -4.62 \cos \alpha \text{ N m}$, ending at -4.62 N m . Filling at $\alpha = 0$ connects this value to zero.

- 2.2. $W_{\text{total}} = \oint \mu d\alpha$ is the area enclosed by the cycle. The pounding work is

$$W_{\text{pounding}} = Mg(GT) \sin \alpha_0,$$

equal to W_{total} when the kinetic energy carried away by water is neglected.

- 2.3. $\alpha_0 \approx 37.5^\circ, \quad W_{\text{pounding}} \approx 2.8 \text{ J}$.
- 3.1.1. Stable equilibrium; the torque crosses zero with negative slope at β .
- 3.1.2. With $\Delta\alpha$ in radians,

$$\mu(\beta + \Delta\alpha) \approx -(45.9 \text{ N m rad}^{-1})\Delta\alpha.$$

- 3.1.3. $\alpha(t) = \beta + \Delta\alpha \cos(2\pi t/\tau)$, with $\tau \approx 3.26 \text{ s}$.
- 3.2. $\Phi_1 \approx 0.23 \text{ kg s}^{-1}$.
- 3.3. Using the limiting amplitude $\beta - \alpha_1 \approx 2.95^\circ$,

$$\Phi_2 \approx 0.7 \text{ kg s}^{-1}.$$

The equilibrium is β ; α_1 is the lower turning point used in this estimate.

Solution

Write $m_0 = 1.0 \text{ kg}$ for the water charge that initiates motion, and $r = GT$. From Figure 3,

$$a = 0.20 \text{ m}, \quad L = 0.74 \text{ m}, \quad h = 0.12 \text{ m}, \quad b = 0.15 \text{ m}, \quad \gamma = 30^\circ.$$

The pivot is $d = 0.08 \text{ m}$ below the bucket's upper edge. We use the standard values $\rho = 1000 \text{ kg m}^{-3}$ and $g = 9.8 \text{ m s}^{-2}$, which the problem does not specify.

Useful abbreviations are

$$q = \cot \gamma = \sqrt{3}, \quad R = a + L = 0.94 \text{ m}, \quad C = \frac{\rho b h^2}{2} = 1.08 \text{ kg}.$$

1.1. Position of the lever's centre of mass

The bucket's flat bottom has length

$$\ell = L - h \cot \gamma = 0.532154 \text{ m}.$$

Let η be the initial water depth above this bottom. The longitudinal water cross-section is a trapezoid, so

$$\frac{m_0}{\rho b} = \ell\eta + \frac{1}{2}q\eta^2.$$

Consequently,

$$\eta = \frac{\sqrt{\ell^2 + 2q m_0/(\rho b)} - \ell}{q} = 0.0122822 \text{ m.}$$

Measure u along the bucket from its inner vertical wall toward the outer lip. A horizontal strip at height y above the bottom extends from $u = 0$ to $u = \ell + qy$. Its centroid therefore gives

$$\bar{u} = \frac{\frac{1}{2} \int_0^\eta (\ell + qy)^2 dy}{\int_0^\eta (\ell + qy) dy}.$$

Writing $w = \ell + q\eta$, this becomes

$$\bar{u} = \frac{\ell^2 + \ell w + w^2}{3(\ell + w)} = 0.271430 \text{ m.}$$

At the instant the pestle loses contact with the rice, its supporting reaction vanishes. The two gravitational torques balance:

$$Mgr = m_0 g(a + \bar{u}).$$

Thus

$$r = \frac{m_0}{M}(a + \bar{u}) = 0.0157143 \text{ m} = 1.57 \text{ cm}.$$

Define the corresponding torque scale

$$K = Mgr = 4.6200 \text{ N m.}$$

1.2. Beginning and end of spilling

Use coordinates fixed in the bucket:

- s runs inward along its upper edge from the outer lip;
- z measures depth below that edge.

The sloping end is $z = s \tan \gamma$, the flat bottom is $z = h$, and the pivot has coordinates (R, d) .

When water reaches the outer lip, its horizontal surface passes through $(0, 0)$. In the bucket coordinates its equation is

$$z = s \tan \alpha.$$

The retained water then has a triangular longitudinal cross-section with vertices

$$(0, 0), \quad (hq, h), \quad (h \cot \alpha, h).$$

Its mass is therefore

$$m(\alpha) = \frac{\rho b h^2}{2}(\cot \alpha - \cot \gamma) = C(\cot \alpha - q). \quad (1)$$

For the $m_0 = 1.0$ kg charge, spilling begins when this capacity equals m_0 :

$$\cot \alpha_1 = q + \frac{m_0}{C} = 2.65798.$$

Hence

$$\boxed{\alpha_1 = 20.6176^\circ}.$$

The water disappears when the horizontal free surface becomes parallel to the sloping end:

$$\cot \alpha_2 = \cot \gamma, \quad \boxed{\alpha_2 = \gamma = 30^\circ}.$$

The triangular construction is consistent with the dimensions: at α_1 , its inner endpoint is $h \cot \alpha_1 = 0.319 \text{ m} < L$.

1.3. Balance angle and remaining water mass

The centroid of the overflowing triangular cross-section is

$$s_N = \frac{h}{3}(q + \cot \alpha), \quad z_N = \frac{2h}{3}.$$

The dimensions have the useful property

$$z_N = \frac{2h}{3} = 0.08 \text{ m} = d.$$

Thus TN lies along the lever, and the horizontal moment arm of the water is

$$\left[R - \frac{h}{3}(q + \cot \alpha) \right] \cos \alpha.$$

The total torque while spilling is

$$\boxed{\mu(\alpha) = g \cos \alpha \left\{ C(\cot \alpha - q) \left[R - \frac{h}{3}(\cot \alpha + q) \right] - Mr \right\}.} \quad (2)$$

Set $v = \cot \beta - q$. The equilibrium condition becomes

$$Cv \left[R - \frac{2hq}{3} - \frac{h}{3}v \right] = Mr.$$

The root in the physical range $\alpha_1 < \beta < \alpha_2$ is

$$v = 0.560329.$$

Therefore

$$\boxed{\beta = \operatorname{arccot}(q + v) = 23.5682^\circ},$$

and

$$\boxed{m_1 = Cv = 0.605156 \text{ kg}}.$$

The distance needed later is

$$\ell_\beta = TN = R - \frac{h}{3}(q + \cot \beta) = 0.779023 \text{ m}.$$

2.1. Torque throughout the working cycle

For any water centroid (s_N, z_N) , the water's horizontal moment arm is

$$(R - s_N) \cos \alpha + (d - z_N) \sin \alpha.$$

Hence, before spilling,

$$\mu_{\uparrow}(\alpha) = m_0 g [(R - s_N) \cos \alpha + (d - z_N) \sin \alpha] - K \cos \alpha. \quad (3)$$

The water initially has a trapezoidal cross-section. It becomes triangular when it ceases to touch the inner vertical wall. This occurs at

$$\cot \alpha_c = q + \frac{\rho b \ell^2}{2m_0}, \quad \alpha_c = 2.493^\circ.$$

For $\alpha_c \leq \alpha \leq \alpha_1$, define

$$\delta(\alpha) = \sqrt{\frac{2m_0}{\rho b (\cot \alpha - q)}}.$$

This is the triangle's height measured perpendicular to the bucket bottom. Its centroid is

$$s_N = hq + \frac{\delta}{3} (\cot \alpha - 2q), \quad z_N = h - \frac{\delta}{3}.$$

These expressions, together with (3), determine the rising branch before spilling. Between α_1 and α_2 , use (2).

Once empty, the bucket contributes no torque:

$$\mu_{\text{empty}}(\alpha) = -K \cos \alpha.$$

The graph can be drawn from the following values:

α	Rising-stroke torque (N m)	Empty return torque (N m)
0°	0 at lift-off	-4.620
2.493°	$+0.944$	-4.616
5°	$+1.561$	-4.602
10°	$+2.068$	-4.550
15°	$+2.367$	-4.463
$\alpha_1 = 20.618^\circ$	$+2.687$	-4.324
$\beta = 23.568^\circ$	0	-4.235
$\alpha_2 = 30^\circ$	-4.001	-4.001

The complete cycle has these features:

1. At $\alpha = 0$, filling raises the gravitational torque from $-K$ to zero. This is a vertical segment in the torque-angle plot. The mortar's reaction keeps the lever stationary during this stage.
2. On lifting, the torque rises from zero to its maximum $+2.687$ N m at α_1 .
3. Spilling makes the torque decrease, cross zero at β , and reach -4.001 N m at α_2 .
4. From α_2 to the maximum angle α_0 , and throughout the empty return, the torque follows $-K \cos \alpha$. These outward and return curves coincide for $\alpha \geq \alpha_2$.

In particular, $\mu(0)$ has different values at different stages: zero at lift-off and -4.620 N m with the empty bucket.

2.2. Work and its geometric interpretation

An infinitesimal rotational work is

$$dW = \mu d\alpha,$$

with angles measured in radians.

The total work over a cycle is

$$W_{\text{total}} = \oint \mu d\alpha = \int_0^{\alpha_0} [\mu_{\uparrow}(\alpha) - \mu_{\downarrow}(\alpha)] d\alpha.$$

The branches coincide above α_2 , so

$$W_{\text{total}} = \int_0^{\alpha_2} [\mu_{\uparrow}(\alpha) + K \cos \alpha] d\alpha.$$

Geometrically, this is the area enclosed between the rising branch, the empty return branch, and the filling segment at $\alpha = 0$.

The empty lever starts its return from rest at α_0 . Its gravitational potential-energy loss becomes the kinetic energy delivered at impact:

$$W_{\text{pounding}} = \int_{\alpha_0}^0 (-K \cos \alpha) d\alpha = K \sin \alpha_0.$$

It is the positive area between the empty-return curve and the angle axis, from 0 to α_0 .

If escaping water carries kinetic energy away, that energy reduces the fraction of the cycle's work delivered to the rice. Under the approximation in 2.3 that this energy is negligible,

$$W_{\text{total}} = W_{\text{pounding}}.$$

Equivalently, the positive and negative signed areas on the complete rising stroke cancel, since that stroke starts and ends at rest:

$$\int_0^{\alpha_0} \mu_{\uparrow}(\alpha) d\alpha = 0.$$

2.3. Maximum angle and pounding work

Let

$$E_2 = \int_0^{\alpha_2} \mu_{\uparrow}(\alpha) d\alpha.$$

This is the kinetic energy remaining when the last water leaves.

Connecting the tabulated rising-branch points by straight segments gives

$$E_2 \approx 0.520 \text{ J}.$$

From α_2 to α_0 , the empty lever converts this energy into gravitational potential energy:

$$E_2 = K(\sin \alpha_0 - \sin \alpha_2).$$

Thus the polygonal estimate is

$$\sin \alpha_0 = \frac{1}{2} + \frac{0.520}{4.620},$$

giving

$$\alpha_0 \approx 37.8^\circ, \quad W_{\text{pounding}} = K \sin \alpha_0 \approx 2.83 \text{ J}.$$

Integrating the curved torque branches more accurately gives

$$\int_0^{\alpha_1} \mu_{\uparrow} d\alpha = 0.6796 \text{ J}, \quad \int_{\alpha_1}^{\alpha_2} \mu_{\uparrow} d\alpha = -0.1779 \text{ J}.$$

Hence $E_2 = 0.5016 \text{ J}$, yielding

$$\boxed{\alpha_0 \approx 37.5^\circ, \quad W_{\text{pounding}} \approx 2.8 \text{ J}}.$$

3.1.1. Equilibrium with continuous overflow

Continuous overflow makes the water mass a function of angle given by (1), on both directions of motion.

Near β , increasing α reduces the water mass and produces a negative torque. Decreasing α increases the retained water mass and produces a positive torque.

Thus the graph passes through $(\beta, 0)$ with negative slope, and

$$\boxed{\alpha = \beta \text{ is a stable equilibrium.}}$$

In the stated idealization, small disturbances produce undamped oscillations.

3.1.2. Linearized torque

Write (2) as

$$\mu(\alpha) = g \cos \alpha F(\cot \alpha),$$

where

$$F(c) = C(c - q) \left[R - \frac{h}{3}(c + q) \right] - Mr.$$

At equilibrium $F(\cot \beta) = 0$. Also,

$$\frac{dF}{dc} = C \left(R - \frac{2h}{3}c \right), \quad \frac{d(\cot \alpha)}{d\alpha} = -\csc^2 \alpha.$$

Therefore

$$\mu'(\beta) = -\frac{gC \cos \beta}{\sin^2 \beta} \left(R - \frac{2h}{3} \cot \beta \right).$$

Define

$$\boxed{\kappa = \frac{gC \cos \beta}{\sin^2 \beta} \left(R - \frac{2h}{3} \cot \beta \right) = 45.91 \text{ N m rad}^{-1}}.$$

For a small angular displacement in radians,

$$\boxed{\mu(\beta + \Delta\alpha) = -\kappa \Delta\alpha + O((\Delta\alpha)^2)}.$$

3.1.3. Equation of motion and period

To first order, evaluate the water's inertia at equilibrium. Approximating its small contribution by a mass at its centroid gives

$$J_\beta \approx I + m_1 \ell_\beta^2 = 12 + 0.6052(0.7790)^2 \approx 12.37 \text{ kg m}^2.$$

The water's moment about its own centroid is only about 0.0026 kg m^2 , so including its spatial extent has negligible effect on the quoted period.

Changes of inertia multiply small accelerations or velocities and contribute beyond the linear approximation. Thus, putting $x = \alpha - \beta$,

$$J_\beta \ddot{x} + \kappa x = 0.$$

This is the harmonic-oscillator equation, with

$$\omega = \sqrt{\frac{\kappa}{J_\beta}} \approx 1.927 \text{ s}^{-1}.$$

For $x(0) = \Delta\alpha$ and $\dot{x}(0) = 0$,

$$\alpha(t) = \beta + \Delta\alpha \cos(\omega t).$$

The period is

$$\tau = 2\pi \sqrt{\frac{J_\beta}{\kappa}} \approx 3.26 \text{ s}.$$

3.2. Flow required for amplitude 1°

The overflowing bucket's capacity obeys

$$\frac{dm}{d\alpha} = -C \csc^2 \alpha.$$

During the downward part of the oscillation, this capacity increases. To maintain overflow, the incoming mass flow must satisfy

$$\Phi \geq \frac{dm}{dt} = -C \csc^2 \alpha \dot{\alpha}.$$

For small oscillations of amplitude A ,

$$\alpha = \beta + A \cos \omega t.$$

To first order,

$$\frac{dm}{dt} = \frac{C}{\sin^2 \beta} A \omega \sin \omega t.$$

Its maximum occurs at the downward passage through equilibrium. Therefore

$$\Phi_{\min}(A) = \frac{C}{\sin^2 \beta} \omega A. \quad (4)$$

Numerically,

$$\frac{C}{\sin^2 \beta} = 6.755 \text{ kg rad}^{-1}.$$

For $A = 1^\circ = \pi/180$ radians,

$$\Phi_1 = 6.755(1.927) \frac{\pi}{180} = 0.227 \text{ kg s}^{-1}.$$

Hence

$$\Phi_1 \approx 0.23 \text{ kg s}^{-1}.$$

3.3. Estimated flow threshold for cessation of pounding

Use the marginal raised oscillation whose lower turning point is α_1 . Its centre is the equilibrium angle β , so its amplitude is

$$A_2 = \beta - \alpha_1 = 2.9506^\circ = 0.051497 \text{ rad}.$$

This limiting oscillation remains above the horizontal position, so the pestle does not reach the rice.

Applying the same overflow condition (4),

$$\Phi_2 \approx \frac{C}{\sin^2 \beta} \omega (\beta - \alpha_1).$$

Equivalently,

$$\Phi_2 \approx \Phi_1 \frac{\beta - \alpha_1}{1^\circ}.$$

Thus

$$\Phi_2 \approx 0.227(2.9506) = 0.670 \text{ kg s}^{-1}.$$

The finite-amplitude correction can be checked while retaining a harmonic trajectory. On its descending branch,

$$-\dot{\alpha} = \omega \sqrt{A_2^2 - (\alpha - \beta)^2},$$

so the exact capacity-growth rate is

$$\frac{dm}{dt} = \frac{C\omega}{\sin^2 \alpha} \sqrt{A_2^2 - (\alpha - \beta)^2}.$$

Its maximum is approximately 0.69 kg s^{-1} , consistent with the estimate

$$\Phi_2 \approx 0.7 \text{ kg s}^{-1}.$$

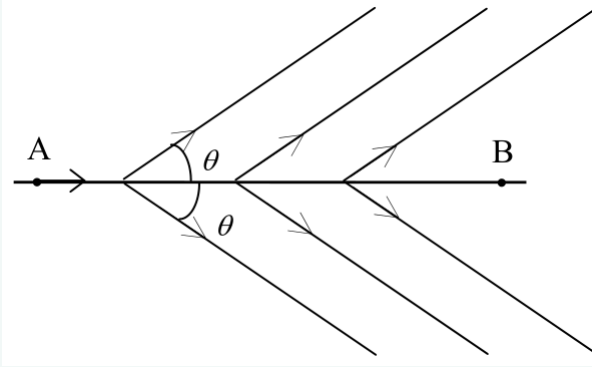
The solution was checked against the official solution.

Problem 2: Cherenkov Light and Ring Imaging Counter

Light propagates in vacuum with the speed c . There is no particle which moves with a speed higher than c . However, it is possible that in a transparent medium a particle moves with a speed v higher than the speed of the light in the same medium $\frac{c}{n}$, where n is the refraction index of the medium. Experiment (Cherenkov, 1934) and theory (Tamm and Frank, 1937) showed that a charged particle, moving with a speed v in a transparent medium with refractive index n such that $v > \frac{c}{n}$, radiates light, called *Cherenkov light*, in directions forming with the trajectory an angle

$$\theta = \arccos \frac{1}{\beta n} \quad (1)$$

where $\beta = \frac{v}{c}$.



1. To establish this fact, consider a particle moving at constant velocity $v > \frac{c}{n}$ on a straight line. It passes A at time 0 and B at time t_1 . As the problem is symmetric with respect to rotations around AB, it is sufficient to consider light rays in a plane containing AB.

At any point C between A and B, the particle emits a spherical light wave, which propagates with velocity $\frac{c}{n}$. We define the wave front at a given time t as the envelope of all these spheres at this time.

1.1. Determine the wave front at time t_1 and draw its intersection with a plane containing the trajectory of the particle.

1.2. Express the angle φ between this intersection and the trajectory of the particle in terms of n and β .

2. Let us consider a beam of particles moving with velocity $v > \frac{c}{n}$, such that the angle θ is small, along a straight line IS. The beam crosses a concave spherical mirror of focal length f and center C, at point S. SC makes with SI a small angle α (see the figure in the Answer Sheet). The particle beam creates a ring image in the focal plane of the mirror.

Explain why with the help of a sketch illustrating this fact. Give the position of the center O and the radius r of the ring image.

This set up is used in *ring imaging Cherenkov counters* (RICH) and the medium which the particle traverses is called the *radiator*.

Note: in all questions of the present problem, terms of second order and higher in α and θ will be neglected.

3. A beam of particles of known momentum $p = 10.0 \text{ GeV}/c$ consists of three types of particles: protons, kaons and pions, with rest mass $M_p = 0.94 \text{ GeV}/c^2$, $M_\kappa = 0.50 \text{ GeV}/c^2$ and $M_\pi = 0.14 \text{ GeV}/c^2$, respectively. Remember that pc and Mc^2 have the dimension of an energy, and 1 eV is the energy acquired by an electron after being accelerated by a voltage 1 V, and $1 \text{ GeV} = 10^9 \text{ eV}$, $1 \text{ MeV} = 10^6 \text{ eV}$.

The particle beam traverses an air medium (the radiator) under the pressure P . The refraction index of air depends on the air pressure P according to the relation $n = 1 + aP$ where $a = 2.7 \times 10^{-4} \text{ atm}^{-1}$

3.1. Calculate for each of the three particle types the minimal value $P_{\min}$ of the air pressure such that they emit Cherenkov light.

3.2. Calculate the pressure $P_{\frac{1}{2}}$ such that the ring image of kaons has a radius equal to one half of that corresponding to pions. Calculate the values of θ_{κ} and θ_{π} in this case.

Is it possible to observe the ring image of protons under this pressure?

4. Assume now that the beam is not perfectly monochromatic: the particles momenta are distributed over an interval centered at $10 \text{ GeV}/c$ having a half width at half height Δp . This makes the ring image broaden, correspondingly θ distribution has a half width at half height $\Delta\theta$. The pressure of the radiator is $P_{\frac{1}{2}}$ determined in 3.2.

4.1. Calculate $\frac{\Delta\theta_{\kappa}}{\Delta p}$ and $\frac{\Delta\theta_{\pi}}{\Delta p}$, the values taken by $\frac{\Delta\theta}{\Delta p}$ in the pions and kaons cases.

4.2. When the separation between the two ring images, $\theta_{\pi} - \theta_{\kappa}$, is greater than 10 times the half-width sum $\Delta\theta = \Delta\theta_{\kappa} + \Delta\theta_{\pi}$, that is $\theta_{\pi} - \theta_{\kappa} > 10\Delta\theta$, it is possible to distinguish well the two ring images. Calculate the maximal value of Δp such that the two ring images can still be well distinguished.

5. Cherenkov first discovered the effect bearing his name when he was observing a bottle of water located near a radioactive source. He saw that the water in the bottle emitted light.

5.1. Find out the minimal kinetic energy $T_{\min}$ of a particle with a rest mass M moving in water, such that it emits Cherenkov light. The index of refraction of water is $n = 1.33$.

5.2. The radioactive source used by Cherenkov emits either α particles (i.e. helium nuclei) having a rest mass $M_{\alpha} = 3.8 \text{ GeV}/c^2$, or β particles (i.e. electrons) having a rest mass $M_e = 0.51 \text{ MeV}/c^2$. Calculate the numerical values of $T_{\min}$ for α particles and β particles.

Knowing that the kinetic energy of particles emitted by radioactive sources never exceeds a few MeV, find out which particles give rise to the radiation observed by Cherenkov.

6. In the previous sections of the problem, the dependence of the Cherenkov effect on wavelength λ has been ignored. We now take into account the fact that the Cherenkov radiation of a particle has a broad continuous spectrum including the visible range (wavelengths from $0.4 \mu\text{m}$ to $0.8 \mu\text{m}$). We know also that the index of refraction n of the radiator decreases linearly by 2% of $n - 1$ when λ increases over this range.

6.1. Consider a beam of pions with definite momentum of $10.0 \text{ GeV}/c$ moving in air at pressure 6 atm. Find out the angular difference $\delta\theta$ associated with the two ends of the visible range.

6.2. On this basis, study qualitatively the effect of the dispersion on the ring image of pions with momentum distributed over an interval centered at $p = 10 \text{ GeV}/c$ and having a half width at half height $\Delta p = 0.3 \text{ GeV}/c$.

6.2.1. Calculate the broadening due to dispersion (varying refraction index) and that due to achromaticity of the beam (varying momentum).

6.2.2. Describe how the color of the ring changes when going from its inner to outer edges by checking the appropriate boxes in the Answer Sheet.

Official English text of the problem.

Answers

- **1.1.** The common-tangent wave front is a right circular cone with vertex B and axis AB, opening backward along the trajectory. Its axial section consists of two tangent segments meeting at B. For emission beginning at A, the full outer boundary also includes the exposed rear cap of the first spherical wave.

- **1.2.**

$$\varphi = \arcsin \frac{1}{n\beta}, \quad \theta = \frac{\pi}{2} - \varphi.$$

- **2.** Let F be the midpoint of SC. The center O lies in the focal plane through F, in the plane ISC, on the opposite side of SC from I:

$$FO \simeq f\alpha, \quad CO \parallel IS, \quad r \simeq f\theta.$$

The reference ray along IS reflects at S along SO.

- **3.1.**

$$P_{\min} = \frac{\sqrt{1 + (Mc/p)^2} - 1}{a}.$$

The threshold pressures are

$$P_{\min,p} \simeq 16.3 \text{ atm}, \quad P_{\min,\kappa} \simeq 4.63 \text{ atm}, \quad P_{\min,\pi} \simeq 0.363 \text{ atm}.$$

Nonzero emission requires pressure above the threshold.

- **3.2.**

$$P_{\frac{1}{2}} \simeq 6.05 \text{ atm}, \quad \theta_{\kappa} \simeq 0.0277 \text{ rad}, \quad \theta_{\pi} \simeq 0.0554 \text{ rad}.$$

Protons produce no ring at this pressure.

- **4.1.**

$$\frac{\Delta\theta_{\kappa}}{\Delta p} \simeq 9.0 \times 10^{-3} \frac{\text{rad}}{\text{GeV}/c}, \quad \frac{\Delta\theta_{\pi}}{\Delta p} \simeq 3.5 \times 10^{-4} \frac{\text{rad}}{\text{GeV}/c}.$$

- **4.2.** The limiting half-width is

$$\Delta p_{\lim} \simeq 0.296 \text{ GeV}/c.$$

The stated strict separation criterion requires $\Delta p < \Delta p_{\lim}$.

- **5.1.**

$$T_{\min} = Mc^2 \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right) = 0.5168 Mc^2 \quad (n = 1.33).$$

- **5.2.**

$$T_{\min,\alpha} \simeq 2.0 \text{ GeV}, \quad T_{\min,e} \simeq 0.26 \text{ MeV}.$$

The observed radiation is produced by the β particles, namely electrons.

- 6.1.

$$\delta\theta = \theta(0.4 \mu\text{m}) - \theta(0.8 \mu\text{m}) \simeq 5.9 \times 10^{-4} \text{ rad} = 0.59 \text{ mrad}.$$

- 6.2.1. Dispersion gives a full visible-spectrum angular span of approximately 0.59 mrad. Momentum spread gives

$$\Delta\theta_p \simeq 0.11 \text{ mrad (half-width)}, \quad 2\Delta\theta_p \simeq 0.21 \text{ mrad (FWHM)}.$$

Corresponding radial widths are obtained by multiplying these angular widths by f .

- 6.2.2. The inner edge is red and the outer edge is blue/violet. Dominant wavelengths decrease outward; momentum spread blurs and mixes neighboring colors.

Solution

Angles are measured in radians. The mirror geometry is treated to first order in the small angles. The relativistic momentum–speed and Cherenkov relations are retained when evaluating the numerical results.

1.1. Wave-front construction

Write the light speed in the medium as

$$u = \frac{c}{n}.$$

Suppose the particle passes C at time t_C . At time t_1 , the wave emitted there has radius

$$R_C = u(t_1 - t_C),$$

while the particle has moved from C to B through the distance

$$CB = v(t_1 - t_C).$$

Consequently,

$$\frac{R_C}{CB} = \frac{u}{v} = \frac{1}{n\beta},$$

independent of C.

In any plane containing AB, the spherical waves appear as circles. Their radii are proportional to their centers' distances from B. They therefore have the same two tangent lines through B. Rotating these tangents about AB produces a right circular cone with its vertex at B, opening backward.

The required sketch is constructed as follows:

1. Draw AB horizontally, with B to the right and $AB = vt_1$.
2. Draw the largest wave circle centered at A, with radius ut_1 .
3. Draw smaller wave circles centered at intermediate points C, with radii $u(t_1 - t_C)$.
4. Draw the upper and lower common tangents from B to these circles. Mark a tangency point T on the circle centered at A and draw $AT \perp BT$.

For the finite emission interval A–B, the conical portion extends from B to the tangency circle on the first sphere. If the whole outer boundary of the disturbed region is shown, its rear portion is the exposed spherical cap centered at A.

1.2. Wave-front angle and Cherenkov angle

In the right triangle ATB,

$$AT = \frac{c}{n}t_1, \quad AB = vt_1.$$

The acute angle between the tangent BT and the trajectory line is φ , so

$$\sin \varphi = \frac{AT}{AB} = \frac{c}{nv} = \frac{1}{n\beta}.$$

Thus

$$\varphi = \arcsin \frac{1}{n\beta}.$$

The light rays are normal to the wave front. Therefore the angle θ between a ray and the forward particle direction satisfies

$$\theta + \varphi = \frac{\pi}{2},$$

and hence

$$\cos \theta = \sin \varphi = \frac{1}{n\beta}.$$

This establishes equation (1).

2. Formation, center and radius of the ring

Here C denotes the mirror's center of curvature. Since the mirror is spherical,

$$SC = 2f.$$

Let F be the midpoint of SC; the paraxial focal plane is perpendicular to SC at F.

First consider a reference light ray traveling along the particle-beam direction IS and striking the mirror at S. The normal to the mirror at S is SC. By the law of reflection, the reflected ray SO makes the same angle α with SC as SI does, on the opposite side.

Its intersection O with the focal plane consequently satisfies

$$FO = f \tan \alpha \simeq f\alpha, \quad SO \simeq f.$$

Because $CF = SF = f$, the same construction gives

$$CO \parallel IS.$$

Now fix an azimuth around the particle trajectory. All Cherenkov rays emitted at that azimuth have the same direction, even when emitted at different points of the trajectory. They form a parallel bundle, which the mirror focuses to one point in its focal plane.

Changing the azimuth changes the ray direction around a cone of half-angle θ . A paraxial mirror converts an angular displacement into a focal-plane displacement equal to f times that angle. Thus the image points have coordinates relative to O of the form

$$X = f\theta \cos \psi, \quad Y = f\theta \sin \psi,$$

where ψ is the azimuth. Therefore

$$X^2 + Y^2 = f^2\theta^2,$$

which is a circle:

$$r = f\theta.$$

For the sketch, draw S at the right, C to its left, and F halfway between them. Put I above SC. The reference ray reflects at S toward O below SC. Draw the focal plane through F and mark the two ring intersections in the plane ISC at signed distances

$$-f\alpha + f\theta, \quad -f\alpha - f\theta$$

from F, taking the side toward I as positive. These points are separated by $2f\theta$ and have midpoint O. The corresponding points from all azimuths form the ring.

3.1. Threshold pressures

Relativistic energy and momentum obey

$$E^2 = p^2 c^2 + M^2 c^4, \quad \beta = \frac{pc}{E}.$$

It is convenient to define

$$q = \frac{1}{\beta} = \sqrt{1 + \left(\frac{Mc}{p}\right)^2}.$$

The Cherenkov condition is

$$n\beta > 1.$$

At the threshold,

$$n_{\min} = q.$$

Using $n = 1 + aP$,

$$P_{\min} = \frac{q - 1}{a} = \frac{\sqrt{1 + (Mc/p)^2} - 1}{a}.$$

With the supplied masses and $p = 10.0 \text{ GeV}/c$:

Particle	β	$P_{\min}$
Proton	0.995611	16.3 atm
Kaon	0.998752	4.63 atm
Pion	0.999902	0.363 atm

At equality the Cherenkov angle is zero; a nonzero ring requires pressure above the threshold.

3.2. Pressure for a kaon ring of half the pion radius

Since $r = f\theta$, the required radius ratio gives

$$\theta_{\pi} = 2\theta_{\kappa}$$

to the stated optical accuracy.

For each species,

$$\cos \theta_i = \frac{q_i}{n}, \quad q_i = \sqrt{1 + \left(\frac{M_i c}{p}\right)^2}.$$

Using the double-angle identity,

$$\frac{q_\pi}{n} = \cos(2\theta_\kappa) = 2\left(\frac{q_\kappa}{n}\right)^2 - 1.$$

Thus

$$n^2 + q_\pi n - 2q_\kappa^2 = 0.$$

The positive root is

$$n_* = \frac{\sqrt{q_\pi^2 + 8q_\kappa^2} - q_\pi}{2}.$$

Numerically,

$$q_\kappa = \sqrt{1 + 0.050^2}, \quad q_\pi = \sqrt{1 + 0.014^2},$$

giving

$$n_* \simeq 1.00163306.$$

Therefore

$$P_{\frac{1}{2}} = \frac{n_* - 1}{a} \simeq 6.05 \text{ atm.}$$

The corresponding angles are

$$\theta_\kappa = \arccos \frac{q_\kappa}{n_*} \simeq 0.027685 \text{ rad},$$

$$\theta_\pi = \arccos \frac{q_\pi}{n_*} \simeq 0.055371 \text{ rad}.$$

Hence

$$\theta_\kappa \simeq 27.7 \text{ mrad}, \quad \theta_\pi \simeq 55.4 \text{ mrad}.$$

The proton threshold is 16.3 atm, substantially above this pressure. Thus

$$\text{No proton ring is produced.}$$

4.1. Angular broadening from momentum spread

At fixed pressure, n is constant. Differentiate

$$\cos \theta = \frac{q}{n}, \quad q = \sqrt{1 + \frac{M^2 c^2}{p^2}}.$$

Since

$$\frac{dq}{dp} = -\frac{M^2 c^2}{p^3 q},$$

we obtain

$$\frac{d\theta}{dp} = \frac{M^2 c^2}{np^3 q \sin \theta}.$$

Using $q = 1/\beta$ and $q/n = \cos \theta$, this can also be written

$$\frac{d\theta}{dp} = \frac{1 - \beta^2}{p} \cot \theta.$$

For a narrow momentum distribution, its transformation is locally linear, so

$$\Delta\theta \simeq \left| \frac{d\theta}{dp} \right| \Delta p.$$

The derivative is positive: increasing momentum increases the Cherenkov angle.

Evaluating at $p = 10.0 \text{ GeV}/c$ and the pressure from 3.2 gives

$$\begin{aligned} \left. \frac{d\theta}{dp} \right|_{\kappa} &\simeq 9.005 \times 10^{-3} \frac{\text{rad}}{\text{GeV}/c}, \\ \left. \frac{d\theta}{dp} \right|_{\pi} &\simeq 3.535 \times 10^{-4} \frac{\text{rad}}{\text{GeV}/c}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\Delta\theta_{\kappa}}{\Delta p} &\simeq 9.0 \times 10^{-3} \frac{\text{rad}}{\text{GeV}/c}, \\ \frac{\Delta\theta_{\pi}}{\Delta p} &\simeq 3.5 \times 10^{-4} \frac{\text{rad}}{\text{GeV}/c}. \end{aligned}$$

4.2. Largest permitted momentum half-width

The required separation condition is

$$\theta_{\pi} - \theta_{\kappa} > 10 (\Delta\theta_{\kappa} + \Delta\theta_{\pi}).$$

Substituting the linear broadening relations gives

$$\Delta p < \frac{\theta_{\pi} - \theta_{\kappa}}{10 [(d\theta/dp)_{\kappa} + (d\theta/dp)_{\pi}]}.$$

Using the unrounded values,

$$\Delta p_{\text{lim}} \simeq \frac{0.027685}{10(0.0090052 + 0.00035355)} \text{ GeV}/c.$$

Thus

$$\Delta p_{\text{lim}} \simeq 0.296 \text{ GeV}/c.$$

This is approximately 3.0% of the central momentum. Since the problem states a strict inequality, the well-distinguished regime is

$$\Delta p < 0.296 \text{ GeV}/c.$$

5.1. Kinetic-energy threshold in water

The relativistic kinetic energy is

$$T = (\gamma - 1)Mc^2, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

At the Cherenkov threshold,

$$\beta_{\text{min}} = \frac{1}{n}.$$

Therefore

$$\gamma_{\min} = \frac{1}{\sqrt{1 - 1/n^2}} = \frac{n}{\sqrt{n^2 - 1}},$$

and

$$T_{\min} = Mc^2 \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right).$$

For water,

$$n = 1.33, \quad \gamma_{\min} \simeq 1.51676,$$

so

$$T_{\min} \simeq 0.5168 Mc^2.$$

Nonzero radiation requires kinetic energy above this threshold.

5.2. Alpha particles and electrons

For an alpha particle,

$$T_{\min,\alpha} = 0.51676(3.8 \text{ GeV}) \simeq 1.96 \text{ GeV} \simeq 2.0 \text{ GeV}.$$

For an electron,

$$T_{\min,e} = 0.51676(0.51 \text{ MeV}) \simeq 0.264 \text{ MeV} \simeq 0.26 \text{ MeV}.$$

An alpha particle with kinetic energy of only a few MeV is far below its threshold of approximately 1960 MeV. Electrons with energies of a few MeV can readily exceed their threshold. Consequently,

The observed Cherenkov radiation is produced by electrons.

6.1. Angular difference across the visible spectrum

At $P = 6 \text{ atm}$,

$$n = 1 + aP = 1 + (2.7 \times 10^{-4})(6) = 1.00162.$$

The magnitude of the refractive-index decrease across the visible range is

$$\delta n = 0.02(n - 1) = 3.24 \times 10^{-5}.$$

For $10.0 \text{ GeV}/c$ pions, the representative Cherenkov angle is

$$\theta_0 = \arccos \left(\frac{\sqrt{1 + (0.14/10.0)^2}}{1.00162} \right) \simeq 0.05513 \text{ rad}.$$

At fixed particle speed, differentiating

$$\cos \theta = \frac{1}{\beta n}$$

with respect to n gives

$$-\sin \theta \frac{d\theta}{dn} = -\frac{1}{\beta n^2}.$$

Hence

$$\frac{d\theta}{dn} = \frac{\cot \theta}{n}.$$

To first order in the index change,

$$\delta\theta \simeq \frac{\cot \theta_0}{n} \delta n \simeq 5.86 \times 10^{-4} \text{ rad.}$$

Thus

$$\delta\theta \simeq 5.9 \times 10^{-4} \text{ rad} = 0.59 \text{ mrad.}$$

The shorter wavelength has the larger index and therefore the larger angle:

$$\theta(0.4 \mu\text{m}) > \theta(0.8 \mu\text{m}).$$

6.2.1. Dispersion broadening and momentum broadening

The dispersion broadening is the full angular span between the two visible endpoints:

$$w_{\text{disp}} = \delta\theta \simeq 0.59 \text{ mrad.}$$

The corresponding radial span is

$$w_{r,\text{disp}} = f\delta\theta \simeq 5.9 \times 10^{-4} f.$$

For the momentum spread, use the derivative from 4.1, now evaluated at 6 atm:

$$\left. \frac{d\theta}{dp} \right|_{\pi, 6 \text{ atm}} = \frac{1 - \beta_\pi^2}{p} \cot \theta_0 \simeq 3.55 \times 10^{-4} \frac{\text{rad}}{\text{GeV}/c}.$$

With $\Delta p = 0.3 \text{ GeV}/c$,

$$\Delta\theta_p \simeq (3.55 \times 10^{-4})(0.3) = 1.07 \times 10^{-4} \text{ rad.}$$

Therefore

$$\Delta\theta_p \simeq 0.11 \text{ mrad}$$

is the **half-width at half height**, while

$$w_{p,\text{FWHM}} = 2\Delta\theta_p \simeq 0.21 \text{ mrad}$$

is the **full width at half maximum**.

The corresponding radial half-width and FWHM are

$$\Delta r_p \simeq 1.1 \times 10^{-4} f, \quad w_{r,p,\text{FWHM}} \simeq 2.1 \times 10^{-4} f.$$

The visible-spectrum span from dispersion is about 2.8 times the momentum-induced FWHM. Dispersion therefore supplies the larger broadening scale. A unique combined FWHM cannot be calculated without specifying the momentum profile and the spectral weighting.

6.2.2. Color ordering across the ring

Because

$$\frac{dn}{d\lambda} < 0, \quad \frac{d\theta}{dn} = \frac{\cot \theta}{n} > 0,$$

we have

$$\frac{d\theta}{d\lambda} < 0.$$

Since $r = f\theta$,

$$\frac{dr}{d\lambda} < 0.$$

Longer wavelengths therefore produce smaller radii. The correct color choice is:

- **Inner edge:** red, corresponding to the longer wavelengths.
- **Moving outward:** progressively shorter dominant wavelengths, with neighboring colors overlapping.
- **Outer edge:** blue/violet, corresponding to the shortest wavelengths.

The momentum spread blurs this ordering, but its characteristic width is smaller than the chromatic span, so the red-inner, violet-outer pattern remains.

The solution was checked against the official solution.

Problem 3: Change of Air Temperature with Altitude, Atmospheric Stability and Air Pollution

Vertical motion of air governs many atmospheric processes, such as the formation of clouds and precipitation and the dispersal of air pollutants. If the atmosphere is *stable*, vertical motion is restricted and air pollutants tend to be accumulated around the emission site rather than dispersed and diluted. Meanwhile, in an *unstable* atmosphere, vertical motion of air encourages the vertical dispersal of air pollutants. Therefore, the pollutants' concentrations depend not only on the strength of emission sources but also on the *stability* of the atmosphere.

We shall determine the atmospheric stability by using the concept of *air parcel* in meteorology and compare the temperature of the air parcel rising or sinking adiabatically in the atmosphere to that of the surrounding air. We will see that in many cases an air parcel containing air pollutants and rising from the ground will come to rest at a certain altitude, called a *mixing height*. The greater the mixing height, the lower the air pollutant concentration. We will evaluate the mixing height and the concentration of carbon monoxide emitted by motorbikes in the Hanoi metropolitan area for a morning rush hour scenario, in which the vertical mixing is restricted due to a temperature inversion (air temperature increases with altitude) at elevations above 119 m.

Let us consider the air as an ideal diatomic gas, with molar mass $\mu = 29 \text{ g/mol}$.

Quasi equilibrium adiabatic transformation obey the equation $pV^\gamma = \text{const}$, where $\gamma = \frac{c_p}{c_v}$ is the ratio between isobaric and isochoric heat capacities of the gas.

The student may use the following data if necessary:

The universal gas constant is $R = 8.31 \text{ J/(mol} \cdot \text{K)}$.

The atmospheric pressure on ground is $p_0 = 101.3 \text{ kPa}$

The acceleration due to gravity is constant, $g = 9.81 \text{ m/s}^2$

The molar isobaric heat capacity is $c_p = \frac{7}{2}R$ for air.

The molar isochoric heat capacity is $c_V = \frac{5}{2}R$ for air.

Mathematical hints

a.

$$\int \frac{dx}{A+Bx} = \frac{1}{B} \int \frac{d(A+Bx)}{A+Bx} = \frac{1}{B} \ln(A+Bx)$$

b. The solution of the differential equation $\frac{dx}{dt} + Ax = B$ (with A and B constant) is

$$x(t) = x_1(t) + \frac{B}{A}$$

where $x_1(t)$ is the solution of the differential equation $\frac{dx}{dt} + Ax = 0$.

c.

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

1. Change of pressure with altitude.

1.1. Assume that the temperature of the atmosphere is uniform and equal to T_0 . Write down the expression giving the atmospheric pressure p as a function of the altitude z .

1.2. Assume that the temperature of the atmosphere varies with the altitude according to the relation

$$T(z) = T(0) - \Lambda z$$

where Λ is a constant, called the *temperature lapse rate* of the atmosphere (the vertical gradient of temperature is $-\Lambda$).

1.2.1. Write down the expression giving the atmospheric pressure p as a function of the altitude z .

1.2.2. A process called free convection occurs when the air density increases with altitude. At which values of Λ does the free convection occur?

2. Change of the temperature of an air parcel in vertical motion

Consider an air parcel moving upward and downward in the atmosphere. An air parcel is a body of air of sufficient dimension, several meters across, to be treated as an independent thermodynamical entity, yet small enough for its temperature to be considered uniform. The vertical motion of an air parcel can be treated as a quasi adiabatic process, i.e. the exchange of heat with the surrounding air is negligible. If the air parcel rises in the atmosphere, it expands and cools. Conversely, if it moves downward, the increasing outside pressure will compress the air inside the parcel and its temperature will increase.

As the size of the parcel is not large, the atmospheric pressure at different points on the parcel boundary can be considered to have the same value $p(z)$, with z - the altitude of the parcel center. The temperature in the parcel is uniform and equals to $T_{\text{parcel}}(z)$, which is generally different from the temperature of the surrounding air $T(z)$. In parts 2.1 and 2.2, we do not make any assumption about the form of $T(z)$.

2.1. The change of the parcel temperature T_{parcel} with altitude is defined by

$$\frac{dT_{\text{parcel}}}{dz} = -G.$$

Derive the expression of $G(T, T_{\text{parcel}})$.

2.2. Consider a special atmospheric condition in which at any altitude z the temperature T of the atmosphere equals to that of the parcel T_{parcel} , $T(z) = T_{\text{parcel}}(z)$. We use Γ to denote the value of G when $T = T_{\text{parcel}}$, that is

$$\Gamma = -\frac{dT_{\text{parcel}}}{dz}$$

(with $T = T_{\text{parcel}}$). Γ is called *dry adiabatic lapse rate*.

2.2.1. Derive the expression of Γ

2.2.2. Calculate the numerical value of Γ .

2.2.3. Derive the expression of the atmospheric temperature $T(z)$ as a function of the altitude.

2.3. Assume that the atmospheric temperature depends on altitude according to the relation $T(z) = T(0) - \Lambda z$, where Λ is a constant. Find the dependence of the parcel temperature $T_{\text{parcel}}(z)$ on altitude z .

2.4. Write down the approximate expression of $T_{\text{parcel}}(z)$ when $|\Lambda z| \ll T(0)$ and $T(0) \approx T_{\text{parcel}}(0)$.

3. The atmospheric stability.

In this part, we assume that T changes linearly with altitude.

3.1. Consider an air parcel initially in equilibrium with its surrounding air at altitude z_0 , i.e. it has the same temperature $T(z_0)$ as that of the surrounding air. If the parcel is moved slightly up and down (e.g. by atmospheric turbulence), one of the three following cases may occur:

- The air parcel finds its way back to the original altitude z_0 , the equilibrium of the parcel is stable. The atmosphere is said to be stable.
- The parcel keeps moving in the original direction, the equilibrium of the parcel is unstable. The atmosphere is unstable.
- The air parcel remains at its new position, the equilibrium of the parcel is indifferent. The atmosphere is said to be neutral.

What is the condition on Λ for the atmosphere to be stable, unstable or neutral?

3.2. A parcel has its temperature on ground $T_{\text{parcel}}(0)$ higher than the temperature $T(0)$ of the surrounding air. The buoyancy force will make the parcel rise. Derive the expression for the maximal altitude the parcel can reach in the case of a stable atmosphere in terms of Λ and Γ .

4. The mixing height

4.1. Table 1 shows air temperatures recorded by a radio sounding balloon at 7: 00 am on a November day in Hanoi. The change of temperature with altitude can be approximately described by the formula $T(z) = T(0) - \Lambda z$ with different lapse rates Λ in the three layers $0 < z < 96 \text{ m}$, $96 \text{ m} < z < 119 \text{ m}$ and $119 \text{ m} < z < 215 \text{ m}$.

Consider an air parcel with temperature $T_{\text{parcel}}(0) = 22^\circ\text{C}$ ascending from ground. On the basis of the data given in Table 1 and using the above linear approximation, calculate the temperature of the parcel at the altitudes of 96 m and 119 m.

4.2. Determine the maximal elevation H the parcel can reach, and the temperature $T_{\text{parcel}}(H)$ of the parcel.

H is called the mixing height. Air pollutants emitted from ground can mix with the air in the atmosphere (e.g. by wind, turbulence and dispersion) and become diluted within this layer.

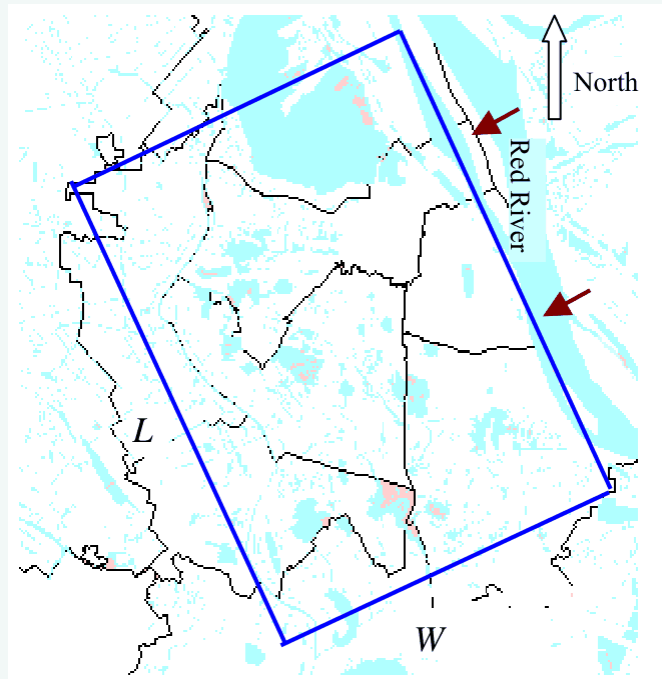
Table 1

Data recorded by a radio sounding balloon at 7:00 am on a November day in Hanoi.

Altitude, m	Temperature, °C
5	21.5
60	20.6
64	20.5
69	20.5
75	20.4
81	20.3
90	20.2
96	20.1
102	20.1
109	20.1
113	20.1
119	20.1
128	20.2
136	20.3
145	20.4
153	20.5
159	20.6
168	20.8
178	21.0
189	21.5
202	21.8
215	22.0
225	22.1
234	22.2
246	22.3
257	22.3

5. Estimation of carbon monoxide (CO) pollution during a morning motorbike rush hour in Hanoi.

Hanoi metropolitan area can be approximated by a rectangle with base dimensions L and W as shown in the figure, with one side taken along the south-west bank of the Red River.



It is estimated that during the morning rush hour, from 7:00 am to 8:00 am, there are 8×10^5 motorbikes on the road, each running on average 5 km and emitting 12 g of CO per kilometer. The amount of CO pollutant is approximately considered as emitted uniformly in time, at a constant rate M during the rush hour. At the same time, the clean north-east wind blows perpendicularly to the Red River (i.e. perpendicularly to the sides L of the rectangle) with velocity u , passes the city with the same velocity, and carries a part of the CO-polluted air out of the city atmosphere.

Also, we use the following rough approximate model:

- The CO spreads quickly throughout the entire volume of the mixing layer above the Hanoi metropolitan area, so that the concentration $C(t)$ of CO at time t can be assumed to be constant throughout that rectangular box of dimensions L , W and H .
- The upwind air entering the box is clean and no pollution is assumed to be lost from the box through the sides parallel to the wind.
- Before 7:00 am, the CO concentration in the atmosphere is negligible.

5.1. Derive the differential equation determining the CO pollutant concentration $C(t)$ as a function of time.

5.2. Write down the solution of that equation for $C(t)$.

5.3. Calculate the numerical value of the concentration $C(t)$ at 8:00 a.m.

Given $L = 15$ km, $W = 8$ km, $u = 1$ m/s.

Official English text of the problem.

Answers

Write $T_0 = T(0)$, $T_p = T_{\text{parcel}}$, and $T_{p0} = T_p(0)$. Temperature ratios use kelvin. In Part 5, t is the elapsed time since 7:00 am.

- 1.1.

$$p(z) = p_0 \exp\left(-\frac{\mu g z}{RT_0}\right)$$

- 1.2.1. For $\Lambda \neq 0$ and $T_0 - \Lambda z > 0$,

$$p(z) = p_0 \left(1 - \frac{\Lambda z}{T_0}\right)^{\mu g / (R\Lambda)}$$

The limit $\Lambda = 0$ gives the result of 1.1.

- 1.2.2. Using the density-increase criterion specified in this part,

$$\Lambda > \frac{\mu g}{R} = 3.42 \times 10^{-2} \text{ K m}^{-1} = 34.2 \text{ K km}^{-1}$$

- 2.1.

$$G(T, T_p) = \frac{\mu g}{c_p} \frac{T_p}{T} = \frac{\gamma - 1}{\gamma} \frac{\mu g}{R} \frac{T_p}{T}$$

- 2.2.1.

$$\Gamma = \frac{\mu g}{c_p} = \frac{2\mu g}{7R}$$

- 2.2.2.

$$\Gamma = 9.78 \times 10^{-3} \text{ K m}^{-1} = 9.78 \text{ K km}^{-1}$$

- 2.2.3.

$$T(z) = T_0 - \Gamma z$$

- 2.3.

$$T_p(z) = \begin{cases} T_{p0} \left(1 - \frac{\Lambda z}{T_0}\right)^{\Gamma/\Lambda}, & \Lambda \neq 0, \\ T_{p0} \exp\left(-\frac{\Gamma z}{T_0}\right), & \Lambda = 0. \end{cases}$$

- 2.4. To first order in height,

$$T_p(z) \simeq T_{p0} - \Gamma \frac{T_{p0}}{T_0} z \simeq T_{p0} - \Gamma z$$

- 3.1.

$$\begin{array}{l} \Lambda < \Gamma : \text{stable,} \\ \Lambda > \Gamma : \text{unstable,} \\ \Lambda = \Gamma : \text{neutral.} \end{array}$$

- 3.2. In the intended mixing-height approximation, neglecting inertial overshoot,

$$z_{\max} \simeq \frac{T_{p0} - T_0}{\Gamma - \Lambda}, \quad \Lambda < \Gamma.$$

The corresponding expression without linearizing the parcel temperature is given in the solution.

- 4.1.

$$T_p(96 \text{ m}) \simeq 21.1^\circ\text{C}, \quad T_p(119 \text{ m}) \simeq 20.8^\circ\text{C}$$

- 4.2.

$$H \simeq 144 \text{ m}, \quad T_p(H) \simeq 20.6^\circ\text{C}$$

- 5.1.

$$\frac{dC}{dt} + \frac{u}{W}C = \frac{M}{LWH}, \quad C(0) = 0$$

- 5.2. During the rush hour,

$$C(t) = \frac{M}{uLH} \left(1 - e^{-ut/W} \right)$$

- 5.3.

$$C(8:00 \text{ am}) \simeq 2.2 \text{ mg m}^{-3}$$

Keeping intermediate digits gives 2.24 mg m^{-3} .

Solution

1.1. Isothermal atmosphere

Hydrostatic equilibrium of a horizontal atmospheric layer requires

$$\frac{dp}{dz} = -\rho g.$$

For ideal air, the mass density is

$$\rho = \frac{\mu p}{RT}.$$

Thus, when $T = T_0$,

$$\frac{dp}{p} = -\frac{\mu g}{RT_0} dz.$$

Integrating from the ground, where $p = p_0$, to altitude z ,

$$\ln \frac{p(z)}{p_0} = -\frac{\mu g z}{RT_0}.$$

Therefore,

$$\boxed{p(z) = p_0 e^{-\mu g z / (RT_0)}}.$$

1.2.1. Atmosphere with a constant temperature lapse rate

With $T(z) = T_0 - \Lambda z$, hydrostatic equilibrium becomes

$$\frac{dp}{p} = -\frac{\mu g}{R} \frac{dz}{T_0 - \Lambda z}.$$

For $\Lambda \neq 0$, integration gives

$$\ln \frac{p(z)}{p_0} = \frac{\mu g}{R\Lambda} \ln \frac{T_0 - \Lambda z}{T_0}.$$

Hence

$$\boxed{p(z) = p_0 \left(1 - \frac{\Lambda z}{T_0}\right)^{\mu g / (R\Lambda)}}.$$

This formula applies where the absolute temperature remains positive.

As $\Lambda \rightarrow 0$,

$$\frac{\mu g}{R\Lambda} \ln \left(1 - \frac{\Lambda z}{T_0}\right) \rightarrow -\frac{\mu g z}{RT_0},$$

recovering the isothermal result.

1.2.2. Density increasing with height

Using $\rho = \mu p / (RT)$,

$$\frac{d \ln \rho}{dz} = \frac{d \ln p}{dz} - \frac{d \ln T}{dz} = -\frac{\mu g}{RT} + \frac{\Lambda}{T}.$$

Therefore,

$$\frac{d\rho}{dz} = \frac{\rho}{T} \left(\Lambda - \frac{\mu g}{R} \right).$$

Since $\rho, T > 0$, density increases upward precisely when

$$\boxed{\Lambda > \frac{\mu g}{R}}.$$

Numerically,

$$\frac{\mu g}{R} = \frac{0.029(9.81)}{8.31} = 0.0342347 \text{ K m}^{-1},$$

so the requested condition is

$$\boxed{\Lambda > 34.2 \text{ K km}^{-1}}.$$

This is the density-inversion criterion stated in this question. The adiabatic parcel-instability criterion in Part 3 is less restrictive.

2.1. Temperature change of an adiabatic parcel

The parcel contains a fixed amount of gas, remains in pressure equilibrium with its surroundings, and exchanges negligible heat with them. Its adiabatic and ideal-gas relations are

$$pV^\gamma = \text{const}, \quad pV = nRT_p.$$

Eliminating V ,

$$T_p^\gamma p^{1-\gamma} = \text{const},$$

or

$$T_p p^{-(\gamma-1)/\gamma} = \text{const}.$$

Taking logarithmic derivatives,

$$\frac{1}{T_p} \frac{dT_p}{dz} = \frac{\gamma-1}{\gamma} \frac{1}{p} \frac{dp}{dz}.$$

The pressure gradient is set by the surrounding atmosphere, whose temperature is T :

$$\frac{1}{p} \frac{dp}{dz} = -\frac{\mu g}{RT}.$$

Consequently,

$$\frac{dT_p}{dz} = -\frac{\gamma-1}{\gamma} \frac{\mu g}{R} \frac{T_p}{T}.$$

Because $c_p - c_V = R$ and $\gamma = c_p/c_V$,

$$\frac{\gamma-1}{\gamma} = \frac{R}{c_p}.$$

Thus

$$\boxed{G(T, T_p) = \frac{\mu g}{c_p} \frac{T_p}{T}}.$$

The factor T_p/T matters: the parcel's cooling rate is not exactly constant when its temperature differs from that of its surroundings.

2.2.1. Dry adiabatic lapse rate

When $T_p = T$, the temperature ratio in the preceding expression equals one:

$$\boxed{\Gamma = \frac{\mu g}{c_p}}.$$

For the specified diatomic gas,

$$c_p = \frac{7}{2}R,$$

so

$$\boxed{\Gamma = \frac{2\mu g}{7R}}.$$

2.2.2. Numerical value

Substitution gives

$$\Gamma = \frac{2(0.029)(9.81)}{7(8.31)} = 0.00978133 \text{ K m}^{-1}.$$

Therefore,

$$\boxed{\Gamma = 9.78 \text{ K km}^{-1}}.$$

2.2.3. Atmospheric temperature profile

Under the special condition $T(z) = T_p(z)$,

$$\frac{dT}{dz} = -\Gamma.$$

Since Γ is constant,

$$\boxed{T(z) = T(0) - \Gamma z}.$$

2.3. Parcel temperature in an atmosphere with constant Λ

From Part 2.1,

$$\frac{dT_p}{dz} = -\Gamma \frac{T_p}{T_0 - \Lambda z}.$$

Separating variables,

$$\frac{dT_p}{T_p} = -\Gamma \frac{dz}{T_0 - \Lambda z}.$$

Integrating,

$$\ln \frac{T_p(z)}{T_{p0}} = \frac{\Gamma}{\Lambda} \ln \left(1 - \frac{\Lambda z}{T_0} \right).$$

Hence, for $\Lambda \neq 0$,

$$\boxed{T_p(z) = T_{p0} \left(1 - \frac{\Lambda z}{T_0} \right)^{\Gamma/\Lambda}}.$$

If $\Lambda = 0$, the surrounding atmosphere is isothermal. Direct integration instead yields

$$\boxed{T_p(z) = T_{p0} e^{-\Gamma z/T_0}}.$$

For a layer beginning at altitude z_i , the same result can be written

$$T_p(z) = T_p(z_i) \left[\frac{T(z)}{T(z_i)} \right]^{\Gamma/\Lambda},$$

which allows successive layers to be joined continuously.

2.4. Linear approximation

Expanding the result of Part 2.3 about $z = 0$,

$$T_p(z) = T_{p0} \left[1 - \frac{\Gamma z}{T_0} + \frac{\Gamma(\Gamma - \Lambda)z^2}{2T_0^2} + \dots \right].$$

To first order,

—

$$T_p(z) \simeq T_{p0} - \Gamma \frac{T_{p0}}{T_0} z.$$

With $T_{p0} \approx T_0$,

$$\boxed{T_p(z) \simeq T_{p0} - \Gamma z}.$$

Strictly, this is a small-height expansion. If Λ is very small, the stated condition $|\Lambda z| \ll T_0$ alone does not ensure linearity; one also needs $\Gamma|z|/T_0 \ll 1$. Both temperature changes are small compared with the absolute temperature in Part 4.

3.1. Stability of a displaced parcel

Let the parcel initially be at z_0 , with

$$T_p(z_0) = T(z_0) = T_*.$$

Displace it upward by a small signed distance ξ . At its initial equilibrium, its adiabatic temperature gradient is $-\Gamma$, so

$$T_p(z_0 + \xi) = T_* - \Gamma\xi + O(\xi^2).$$

The environmental temperature is

$$T(z_0 + \xi) = T_* - \Lambda\xi.$$

The parcel's temperature excess is therefore

$$T_p - T = (\Lambda - \Gamma)\xi + O(\xi^2).$$

At equal pressure, density is inversely proportional to absolute temperature. The parcel's upward acceleration due to buoyancy and gravity is

$$a_z = g \left(\frac{\rho}{\rho_p} - 1 \right) = g \left(\frac{T_p}{T} - 1 \right) \simeq \frac{g}{T_*} (\Lambda - \Gamma)\xi.$$

Thus:

- If $\Lambda < \Gamma$, acceleration opposes the displacement: the atmosphere is **stable**.
- If $\Lambda > \Gamma$, acceleration reinforces the displacement: the atmosphere is **unstable**.
- If $\Lambda = \Gamma$, the parcel remains at the same temperature and density as its surroundings: the atmosphere is **neutral**.

Hence

$$\boxed{\text{stable: } \Lambda < \Gamma, \quad \text{unstable: } \Lambda > \Gamma, \quad \text{neutral: } \Lambda = \Gamma.}$$

3.2. Height reached by an initially warmer parcel

In the linear approximation,

$$T_p(z) \simeq T_{p0} - \Gamma z, \quad T(z) = T_0 - \Lambda z.$$

Therefore,

$$T_p(z) - T(z) = (T_{p0} - T_0) - (\Gamma - \Lambda)z.$$

In a stable atmosphere, $\Gamma - \Lambda > 0$, so the parcel's initial temperature excess decreases with height. The neutral-buoyancy level is obtained by setting this difference to zero:

$$z_{\max} \simeq \frac{T_{p0} - T_0}{\Gamma - \Lambda}.$$

Here the notation $z_{\max}$ follows the problem's intended mixing-height model, which neglects inertial overshoot.

For completeness, the neutral-buoyancy height can also be found without the approximation of Part 2.4. Setting the exact parcel temperature equal to the atmospheric temperature gives

$$T_{p0} \left(1 - \frac{\Lambda z}{T_0}\right)^{\Gamma/\Lambda} = T_0 \left(1 - \frac{\Lambda z}{T_0}\right).$$

Thus, for $\Lambda \neq 0$,

$$z_n = \frac{T_0}{\Lambda} \left[1 - \left(\frac{T_0}{T_{p0}} \right)^{\Lambda/(\Gamma-\Lambda)} \right].$$

For $\Lambda = 0$, its limit is

$$z_n = \frac{T_0}{\Gamma} \ln \frac{T_{p0}}{T_0}.$$

Both reduce to the stated linear result when $T_{p0} - T_0 \ll T_0$.

A dynamical qualification is necessary: zero buoyancy means zero acceleration, not necessarily zero velocity. A freely moving parcel can overshoot this level. For example, in the same linear approximation, an undamped parcel released from rest obeys

$$\ddot{z} \simeq \frac{g}{T_0} [(T_{p0} - T_0) - (\Gamma - \Lambda)z],$$

and its first turning point is $2z_n$. The following parts use the intended neutral-buoyancy mixing height.

4.1. Parcel temperatures at 96 m and 119 m

Use straight lines through the measured endpoints of each specified layer. Their lapse rates are

$$\Lambda_1 = \frac{21.5 - 20.1}{96 - 5} = 0.0153846 \text{ K m}^{-1},$$

$$\Lambda_2 = 0,$$

and

$$\Lambda_3 = \frac{20.1 - 22.0}{215 - 119} = -0.0197917 \text{ K m}^{-1}.$$

The first line extrapolates to a ground temperature

$$T(0) \simeq 21.5 + 5\Lambda_1 = 21.5769^\circ\text{C},$$

close to the parcel's initial 22°C . All relevant temperature changes are small compared with approximately 294 K, so Part 2.4 applies.

Since a kelvin temperature difference equals a Celsius temperature difference,

$$T_p(z) \simeq 22^\circ\text{C} - \Gamma z.$$

Consequently,

$$T_p(96) \simeq 22 - 0.00978133(96) = 21.06099^\circ\text{C},$$

and

$$T_p(119) \simeq 22 - 0.00978133(119) = 20.83602^\circ\text{C}.$$

To the precision of the temperature data,

$$\boxed{T_p(96 \text{ m}) \simeq 21.1^\circ\text{C}}, \quad \boxed{T_p(119 \text{ m}) \simeq 20.8^\circ\text{C}}.$$

As a check, applying the exact formula of Part 2.3 separately to the first layer and the isothermal second layer gives 21.059°C and 20.833°C , respectively. These round to the same answers.

4.2. Mixing height

At both 96 m and 119 m, the surrounding temperature is 20.1°C . The parcel is warmer at both heights, so it continues into the inversion layer above 119 m.

Write the positive temperature gradient there as

$$k = -\Lambda_3 = \frac{22.0 - 20.1}{215 - 119} = 0.0197917 \text{ K m}^{-1}.$$

The environmental temperature in that layer is

$$T(z) = 20.1^\circ\text{C} + k(z - 119 \text{ m}).$$

At the mixing height, the parcel and atmosphere have equal temperatures:

$$22 - \Gamma H = 20.1 + k(H - 119),$$

with H in metres and the displayed temperatures in degrees Celsius. Solving,

$$H = \frac{22 - 20.1 + 119k}{\Gamma + k} = \frac{4.2552083}{0.029572997} = 143.888 \text{ m}.$$

This lies within the assumed inversion layer, as required.

The parcel temperature there is

$$T_p(H) \simeq 22 - \Gamma H = 20.5926^\circ\text{C}.$$

Therefore,

$$\boxed{H \simeq 144 \text{ m}, \quad T_p(H) \simeq 20.6^\circ\text{C}}.$$

Using the exact parcel-temperature evolution through all three linear atmospheric layers gives $H = 143.78 \text{ m}$, confirming that the linear parcel approximation is adequate.

5.1. Conservation equation for CO

The uniformly mixed air volume is

$$V = LWH.$$

If $C(t)$ is the CO mass concentration, the total CO mass inside the box is

$$m_{\text{CO}} = LWH C.$$

The wind crosses a face of width L and height H , so its volume flow rate is

$$Q = uLH.$$

Because incoming air is clean, the incoming CO flux is zero. The outgoing CO mass rate is

$$QC = uLH C.$$

Conservation of CO therefore gives

$$\frac{d}{dt}(LWH C) = M - uLH C.$$

The box dimensions are constant, hence

$$\boxed{\frac{dC}{dt} + \frac{u}{W}C = \frac{M}{LWH}}, \quad \boxed{C(0) = 0}.$$

5.2. Concentration as a function of time

A constant particular solution is

$$C_{ss} = \frac{M}{uLH}.$$

The homogeneous solution is proportional to $e^{-ut/W}$, so

$$C(t) = \frac{M}{uLH} + Ae^{-ut/W}.$$

Using $C(0) = 0$,

$$A = -\frac{M}{uLH}.$$

Thus

$$\boxed{C(t) = \frac{M}{uLH} \left(1 - e^{-ut/W}\right)}, \quad 0 \leq t \leq 3600 \text{ s.}$$

The ventilation timescale is W/u . In the limit of no wind, this expression correctly reduces to $C(t) = Mt/(LWH)$.

5.3. CO concentration at 8:00 am

The total CO emitted during the hour is

$$m_{\text{emitted}} = (8 \times 10^5)(5 \text{ km})(12 \text{ g km}^{-1}) = 4.8 \times 10^7 \text{ g} = 4.8 \times 10^4 \text{ kg}.$$

Therefore,

$$M = \frac{4.8 \times 10^4}{3600} = 13.3333 \text{ kg s}^{-1}.$$

Using

$$L = 15000 \text{ m}, \quad W = 8000 \text{ m}, \quad u = 1 \text{ m s}^{-1}, \quad H \simeq 143.888 \text{ m},$$

the dimensionless elapsed time is

$$\frac{ut}{W} = \frac{1(3600)}{8000} = 0.45.$$

Hence

$$C(3600) = \frac{13.3333}{(1)(15000)(143.888)} (1 - e^{-0.45}) \text{ kg m}^{-3},$$

giving

$$C(3600) = 2.2386 \times 10^{-6} \text{ kg m}^{-3} = 2.2386 \text{ mg m}^{-3}.$$

With precision appropriate to the rough input data,

$$C(8:00 \text{ am}) \simeq 2.2 \text{ mg m}^{-3}.$$

The solution was checked against the official solution.