

IPhO 2007

Theory problems and solutions · Isfahan, Iran · 3 theory problems with complete solutions

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Problem 1: Dimensional analysis and black holes (Hawking radiation)

In physics, whenever we have an equality relation, both sides of the equation should be of the same type i.e. they must have the same dimensions. For example you cannot have a situation where the quantity on the right-hand side of the equation represents a length and the quantity on the left-hand side represents a time interval. Using this fact, sometimes one can nearly deduce the form of a physical relation without solving the problem analytically. For example if we were asked to find the time it takes for an object to fall from a height of h under the influence of a constant gravitational acceleration g , we could argue that one only needs to build a quantity representing a time interval, using the quantities g and h and the only possible way of doing this is $T = a(h/g)^{1/2}$. Notice that this solution includes an as yet undetermined coefficient a which is *dimensionless* and thus cannot be determined, using this method. This coefficient can be a number such as 1, $1/2$, $\sqrt{3}$, π , or any other real number. This method of deducing physical relations is called *dimensional analysis*. In dimensional analysis the dimensionless coefficients are not important and we do not need to write them. Fortunately in most physical problems these coefficients are of the order of 1 and eliminating them does not change the order of magnitude of the physical quantities. Therefore, by applying the dimensional analysis to the above problem, one obtains $T = (h/g)^{1/2}$.

Generally, the dimensions of a physical quantity are written in terms of the dimensions of four fundamental quantities: M (mass), L (length), T (time), and K (temperature). The dimensions of an arbitrary quantity, x is denoted by $[x]$. As an example, to express the dimensions of velocity v , kinetic energy E_k , and heat capacity C_V we write: $[v] = LT^{-1}$, $[E_k] = ML^2T^{-2}$, $[C_V] = ML^2T^{-2}K^{-1}$.

1 Fundamental Constants and Dimensional Analysis

1.1	Find the dimensions of <i>the fundamental constants</i> , i.e. the Planck's constant, h , the speed of light, c , the universal constant of gravitation, G , and the Boltzmann constant, k_B , in terms of the dimensions of length, mass, time, and temperature.	0.8
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The Stefan-Boltzmann law states that the black body emissive power which is the total energy radiated per unit surface area of a black body in unit time is equal to $\sigma\theta^4$ where σ is the Stefan-Boltzmann's constant and θ is the absolute temperature of the black body.

1.2	Determine the dimensions of the Stefan-Boltzmann's constant in terms of the dimensions of length, mass, time, and temperature.	0.5
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The Stefan-Boltzmann's constant is not a fundamental constant and one can write it in terms of fundamental constants i.e. one can write $\sigma = ah^\alpha c^\beta G^\gamma k_B^\delta$. In this relation a is a dimensionless parameter of the order of 1. As mentioned before, the exact value of a is not significant from our

viewpoint, so we will set it equal to 1.

1.3	Find α , β , γ , and δ using dimensional analysis.	1.0
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2 Physics of Black Holes

In this part of the problem, we would like to find out some properties of black holes using dimensional analysis. According to a certain theorem in physics known as the *no hair theorem*, all the characteristics of the black hole which we are considering in this problem depend only on the mass of the black hole. One characteristic of a black hole is the area of its *event horizon*. Roughly speaking, the event horizon is the boundary of the black hole. Inside this boundary, the gravity is so strong that even light cannot emerge from the region enclosed by the boundary.

We would like to find a relation between the mass of a black hole, m , and the area of its event horizon, A . This area depends on the mass of the black hole, the speed of light, and the universal constant of gravitation. As in 1.3 we shall write $A = G^\alpha c^\beta m^\gamma$.

2.1	Use dimensional analysis to find α , β , and γ .	0.8
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From the result of 2.1 it becomes clear that the area of the event horizon of a black hole increases with its mass. From a classical point of view, nothing comes out of a black hole and therefore in all physical processes the area of the event horizon can only increase. In analogy with the second law of thermodynamics, Bekenstein proposed to assign entropy, S , to a black hole, proportional to the area of its event horizon i.e. $S = \eta A$. This conjecture has been made more plausible using other arguments.

2.2	Use the thermodynamic definition of entropy $dS = dQ/\theta$ to find the dimensions of entropy. dQ is the exchanged heat and θ is the absolute temperature of the system.	0.2
2.3	As in 1.3, express the dimensioned constant η as a function of the fundamental constants h , c , G , and k_B .	1.1

Do not use dimensional analysis for the rest of problem, but you may use the results you have obtained in previous sections.

3 Hawking Radiation

With a semi-quantum mechanical approach, Hawking argued that contrary to the classical point of view, black holes emit radiation similar to the radiation of a black body at a temperature which is called the *Hawking temperature*.

3.1	Use $E = mc^2$, which gives the energy of the black hole in terms of its mass, and the laws of thermodynamics to express the Hawking temperature θ_H of a black hole in terms of its mass and the fundamental constants. Assume that the black hole does no work on its surroundings.	0.8
3.2	The mass of an isolated black hole will thus change because of the Hawking radiation. Use Stefan-Boltzmann's law to find the dependence of this rate of change on the Hawking temperature of the black hole, θ_H and express it in terms of mass of the black hole and the fundamental constants.	0.7
3.3	Find the time t^* , that it takes an isolated black hole of mass m to evaporate completely i.e. to lose all its mass.	1.1

From the viewpoint of thermodynamics, black holes exhibit certain exotic behaviors. For example the heat capacity of a black hole is negative.

3.4	Find the heat capacity of a black hole of mass m .	0.6
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4 Black Holes and the Cosmic Background Radiation

Consider a black hole exposed to the cosmic background radiation. The cosmic background radiation is a black body radiation with a temperature θ_B which fills the entire universe. An object with a total area A will thus receive an energy equal to $\sigma\theta_B^4 \times A$ per unit time. A black hole, therefore, loses energy through Hawking radiation and gains energy from the cosmic background radiation.

4.1	Find the rate of change of a black hole's mass, in terms of the mass of the black hole, the temperature of the cosmic background radiation, and the fundamental constants.	0.8
4.2	At a certain mass, m^* , this rate of change will vanish. Find m^* and express it in terms of θ_B and the fundamental constants.	0.4
4.3	Use your answer to 4.2 to substitute for θ_B in your answer to part 4.1 and express the rate of change of the mass of a black hole in terms of m , m^* , and the fundamental constants.	0.2
4.4	Find the Hawking temperature of a black hole at thermal equilibrium with cosmic background radiation.	0.4
4.5	Is the equilibrium stable or unstable? Why? (Express your answer mathematically)	0.6

Official English text of the problem.

Answers

These results set the undetermined dimensionless coefficients in Parts 1–2 equal to 1, as prescribed. Numerical factors arising from differentiation and integration are retained. Write $\dot{m} = dm/dt$.

- **1.1:** $[h] = ML^2T^{-1}$, $[c] = LT^{-1}$, $[G] = M^{-1}L^3T^{-2}$, $[k_B] = ML^2T^{-2}K^{-1}$.
- **1.2:** $[\sigma] = MT^{-3}K^{-4}$.
- **1.3:** $(\alpha, \beta, \gamma, \delta) = (-3, -2, 0, 4)$, giving $\sigma = \frac{k_B^4}{h^3c^2}$.
- **2.1:** $(\alpha, \beta, \gamma) = (2, -4, 2)$, giving $A = \frac{G^2m^2}{c^4}$.
- **2.2:** $[S] = ML^2T^{-2}K^{-1}$.
- **2.3:** $\eta = \frac{k_Bc^3}{hG}$.
- **3.1:** $\theta_H = \frac{hc^3}{2Gk_Bm}$.
- **3.2:** $\dot{m} = -\frac{\sigma A\theta_H^4}{c^2} = -\frac{k_B^2\theta_H^2}{4hc^2} = -\frac{hc^4}{16G^2m^2}$.
- **3.3:** $t^* = \frac{16G^2m^3}{3hc^4}$.
- **3.4:** $C = -\frac{2Gk_Bm^2}{hc} < 0$.

- **4.1:** $\dot{m} = \frac{G^2 k_B^4 m^2 \theta_B^4}{h^3 c^8} - \frac{hc^4}{16G^2 m^2}.$
- **4.2:** $m^* = \frac{2Gk_B\theta_B}{hc^3}.$
- **4.3:** $\dot{m} = \frac{hc^4}{16G^2 m^2} \left[\left(\frac{m}{m^*} \right)^4 - 1 \right].$
- **4.4:** $\theta_H(m^*) = \theta_B.$
- **4.5: Unstable.** A small mass perturbation satisfies $\delta \dot{m} \simeq \frac{hc^4}{4G^2 (m^*)^3} \delta m$, so its magnitude grows exponentially.

Solution

We use the mass-only black hole model and the black-body emission and absorption laws specified in the problem. All expressions inherit the normalization of the dimensionless coefficients adopted in Parts 1–2. No numerical input values are supplied, so the requested results are symbolic.

1.1

The photon energy relation $E = h\nu$, with $[\nu] = T^{-1}$, gives

$$[h] = \frac{[E]}{[\nu]} = ML^2T^{-1}.$$

Since c is a speed,

$$[c] = LT^{-1}.$$

Newton's gravitational law, $F = Gm_1m_2/r^2$, gives

$$[G] = \frac{[F]L^2}{M^2} = \frac{(MLT^{-2})L^2}{M^2} = M^{-1}L^3T^{-2}.$$

The product $k_B\theta$ has dimensions of energy, so

$$[k_B] = \frac{[E]}{[\theta]} = ML^2T^{-2}K^{-1}.$$

Their SI units are respectively J s, m s^{−1}, m³ kg^{−1} s^{−2}, and J K^{−1}.

1.2

The Stefan–Boltzmann law for total radiated power is

$$P = \sigma A \theta^4.$$

Consequently,

$$[\sigma] = \frac{[P]}{[A][\theta]^4} = \frac{ML^2T^{-3}}{L^2K^4} = \boxed{MT^{-3}K^{-4}}.$$

The corresponding SI unit is W m^{−2} K^{−4}.

1.3

With the prescribed coefficient $a = 1$,

$$\sigma = h^\alpha c^\beta G^\gamma k_B^\delta.$$

The dimensions of the right-hand side are

$$M^{\alpha-\gamma+\delta} L^{2\alpha+\beta+3\gamma+2\delta} T^{-\alpha-\beta-2\gamma-2\delta} K^{-\delta}.$$

Matching these to $MT^{-3}K^{-4}$ gives

$$\begin{aligned}\alpha - \gamma + \delta &= 1, \\ 2\alpha + \beta + 3\gamma + 2\delta &= 0, \\ -\alpha - \beta - 2\gamma - 2\delta &= -3, \\ -\delta &= -4.\end{aligned}$$

Solving,

$$\boxed{\alpha = -3, \quad \beta = -2, \quad \gamma = 0, \quad \delta = 4}.$$

Thus

$$\boxed{\sigma = \frac{k_B^4}{h^3 c^2}}.$$

2.1

For $A = G^\alpha c^\beta m^\gamma$,

$$L^2 = M^{-\alpha+\gamma} L^{3\alpha+\beta} T^{-2\alpha-\beta}.$$

Therefore,

$$-\alpha + \gamma = 0, \quad 3\alpha + \beta = 2, \quad -2\alpha - \beta = 0.$$

The last equation gives $\beta = -2\alpha$; the length equation then gives $\alpha = 2$. Hence

$$\boxed{\alpha = 2, \quad \beta = -4, \quad \gamma = 2},$$

and

$$\boxed{A = \frac{G^2 m^2}{c^4}}.$$

2.2

From the thermodynamic entropy relation,

$$dS = \frac{dQ}{\theta},$$

entropy has dimensions

$$\boxed{[S] = \frac{[Q]}{[\theta]} = ML^2 T^{-2} K^{-1}}.$$

Its SI unit is J K^{-1} .

2.3

Since $S = \eta A$,

$$[\eta] = \frac{[S]}{[A]} = MT^{-2} K^{-1}.$$

Write

$$\eta = h^p c^q G^r k_B^s.$$

Equating dimensions gives

$$\begin{aligned} p - r + s &= 1, \\ 2p + q + 3r + 2s &= 0, \\ -p - q - 2r - 2s &= -2, \\ -s &= -1. \end{aligned}$$

Thus $s = 1$, $p = r$, and the remaining equations give

$$p = -1, \quad q = 3, \quad r = -1.$$

Therefore,

$$\boxed{\eta = \frac{k_B c^3}{hG}}.$$

Combining this with the horizon area gives the entropy as a function of mass:

$$S(m) = \eta A = \frac{k_B c^3}{hG} \frac{G^2 m^2}{c^4} = \boxed{\frac{k_B G m^2}{hc}}.$$

3.1

For the thermodynamic family of black hole states, the first law with no work term is

$$dE = \theta_H dS.$$

Since

$$E = mc^2, \quad S = \frac{k_B G m^2}{hc},$$

we have

$$\frac{dE}{dm} = c^2, \quad \frac{dS}{dm} = \frac{2k_B G m}{hc}.$$

Consequently,

$$\theta_H = \frac{dE/dm}{dS/dm} = \boxed{\frac{hc^3}{2Gk_B m}}.$$

The temperature decreases as the mass increases.

3.2

The outward radiation power is

$$P_{\text{out}} = \sigma A \theta_H^4.$$

Energy conservation for an isolated black hole requires

$$c^2 \dot{m} = -P_{\text{out}},$$

where the minus sign expresses mass loss. Therefore,

$$\dot{m} = -\frac{\sigma A}{c^2} \theta_H^4 = -\frac{G^2 k_B^4 m^2}{h^3 c^8} \theta_H^4.$$

Substituting the temperature from 3.1,

$$\dot{m} = -\frac{G^2 k_B^4 m^2}{h^3 c^8} \left(\frac{hc^3}{2Gk_B m} \right)^4 = \boxed{-\frac{hc^4}{16G^2 m^2}}.$$

To express the dependence on temperature alone, substitute

$$m = \frac{hc^3}{2Gk_B\theta_H}.$$

This gives

$$\dot{m} = -\frac{k_B^2\theta_H^2}{4hc^2}.$$

Thus the mass-loss rate has magnitude proportional to θ_H^2 . Although the emission per unit area is proportional to θ_H^4 , the horizon area is proportional to m^2 , hence to θ_H^{-2} .

3.3

Let the initial mass be m_i , and denote the instantaneous mass by $\mu(t)$. Define

$$\kappa = \frac{hc^4}{16G^2} > 0.$$

The mass-loss equation is

$$\frac{d\mu}{dt} = -\frac{\kappa}{\mu^2}.$$

Separating variables and integrating from $\mu(0) = m_i$,

$$\int_m^{\mu(t)} \mu'^2 d\mu' = -\kappa \int_0^t dt',$$

so

$$\frac{\mu(t)^3 - m^3}{3} = -\kappa t.$$

Hence

$$\mu(t) = (m^3 - 3\kappa t)^{1/3}.$$

Complete evaporation occurs when $\mu(t^*) = 0$:

$$t^* = \frac{m^3}{3\kappa} = \frac{16G^2m^3}{3hc^4}.$$

This is the lifetime obtained by extending the problem's evaporation model to zero mass.

3.4

Because there is no work term, the heat capacity is

$$C = \frac{dQ}{d\theta_H} = \frac{dE}{d\theta_H}.$$

Using

$$m = \frac{hc^3}{2Gk_B\theta_H},$$

we obtain

$$C = c^2 \frac{dm}{d\theta_H} = -\frac{hc^5}{2Gk_B\theta_H^2}.$$

Substituting $\theta_H = hc^3/(2Gk_Bm)$,

$$C = -\frac{2Gk_Bm^2}{hc}.$$

Equivalently,

$$C = -\frac{mc^2}{\theta_H} = -2S.$$

The heat capacity is negative: absorbing energy increases the mass and lowers the temperature, while losing energy raises the temperature.

4.1

The absorbed and emitted powers are, according to the stated model,

$$P_{\text{in}} = \sigma A \theta_B^4, \quad P_{\text{out}} = \sigma A \theta_H^4.$$

Energy conservation gives

$$c^2 \dot{m} = P_{\text{in}} - P_{\text{out}},$$

so

$$\dot{m} = \frac{\sigma A}{c^2} (\theta_B^4 - \theta_H^4).$$

Substituting the previously derived expressions,

$$\dot{m} = \frac{G^2 k_B^4 m^2 \theta_B^4}{h^3 c^8} - \frac{hc^4}{16G^2 m^2}.$$

The first term represents absorption; the second represents emission.

4.2

At $m = m^*$, the net rate vanishes:

$$\sigma A (\theta_B^4 - \theta_H(m^*)^4) = 0.$$

For positive temperatures, this requires

$$\theta_H(m^*) = \theta_B.$$

Using 3.1,

$$\frac{hc^3}{2Gk_Bm^*} = \theta_B,$$

and therefore

$$m^* = \frac{hc^3}{2Gk_B\theta_B}.$$

4.3

From 4.2,

$$\theta_B = \frac{hc^3}{2Gk_Bm^*}.$$

Substituting into 4.1 gives

$$\begin{aligned}\dot{m} &= \frac{G^2 k_B^4 m^2}{h^3 c^8} \left(\frac{hc^3}{2Gk_B m^*} \right)^4 - \frac{hc^4}{16G^2 m^2} \\ &= \frac{hc^4}{16G^2} \left(\frac{m^2}{(m^*)^4} - \frac{1}{m^2} \right).\end{aligned}$$

Equivalently,

$$\boxed{\dot{m} = \frac{hc^4}{16G^2 m^2} \left[\left(\frac{m}{m^*} \right)^4 - 1 \right]}.$$

4.4

At thermal equilibrium, absorbed and emitted powers are equal, so

$$\boxed{\theta_H(m^*) = \frac{hc^3}{2Gk_B m^*} = \theta_B}.$$

4.5

Hold the background temperature fixed, so m^* is fixed. Write

$$\dot{m} = f(m) = \kappa \left(\frac{m^2}{(m^*)^4} - \frac{1}{m^2} \right), \quad \kappa = \frac{hc^4}{16G^2}.$$

Differentiating,

$$f'(m) = \kappa \left(\frac{2m}{(m^*)^4} + \frac{2}{m^3} \right).$$

At equilibrium,

$$f'(m^*) = \frac{4\kappa}{(m^*)^3} = \frac{hc^4}{4G^2 (m^*)^3} > 0.$$

For a small perturbation $m = m^* + \delta m$, linearization therefore gives

$$\delta \dot{m} \simeq \frac{hc^4}{4G^2 (m^*)^3} \delta m,$$

with solution

$$\delta m(t) \simeq \delta m(0) \exp \left[\frac{hc^4 t}{4G^2 (m^*)^3} \right].$$

The perturbation grows, so

$$\boxed{\text{the equilibrium is unstable}}.$$

The signs in 4.3 show the same result directly:

$$m > m^* \implies \dot{m} > 0, \quad m < m^* \implies \dot{m} < 0.$$

A slightly heavier black hole is colder than the background and gains still more mass. A slightly lighter one is hotter and loses still more mass.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Capacitive accelerometer for a car air bag

In this problem we deal with a simplified model of accelerometers designed to activate the safety air bags of automobiles during a collision. We would like to build an electromechanical system in such a way that when the acceleration exceeds a certain limit, one of the electrical parameters of the system such as the voltage at a certain point of the circuit will exceed a threshold and the air bag will be activated as a result.

Note: Ignore gravity in this problem.

1 Consider a capacitor with parallel plates as in Figure 1. The area of each plate in the capacitor is A and the distance between the two plates is d . The distance between the two plates is much smaller than the dimensions of the plates. One of these plates is in contact with a wall through a spring with a spring constant k , and the other plate is fixed. When the distance between the plates is d the spring is neither compressed nor stretched, in other words no force is exerted on the spring in this state. Assume that the permittivity of the air between the plates is that of free vacuum ϵ_0 . The capacitance corresponding to this distance between the plates of the capacitor is $C_0 = \epsilon_0 A/d$. We put charges $+Q$ and $-Q$ on the plates and let the system achieve mechanical equilibrium.

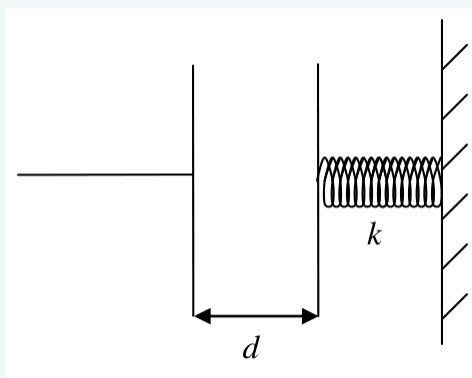


Figure 1

Part	Question	Points
1.1	Calculate the electrical force, F_E , exerted by the plates on each other.	0.8
1.2	Let x be the displacement of the plate connected to the spring. Find x .	0.6
1.3	In this state, what is the electrical potential difference V between the plates of the capacitor in terms of Q , A , d , k ?	0.4
1.4	Let C be the capacitance of the capacitor, defined as the ratio of charge to potential difference. Find C/C_0 as a function of Q , A , d and k .	0.3
1.5	What is the total energy, U , stored in the system in terms of Q , A , d and k ?	0.6

Figure 2, shows a mass M which is attached to a conducting plate with negligible mass and also to two springs having identical spring constants k . The conducting plate can move back and forth in the space between two fixed conducting plates. All these plates are similar and have the same area A . Thus these three plates constitute two capacitors. As shown in Figure 2, the fixed plates are connected to the given potentials V and $-V$, and the middle plate is connected through a two-state switch to the ground. The wire connected to the movable plate does not disturb the

motion of the plate and the three plates will always remain parallel. When the whole complex is not being accelerated, the distance from each fixed plate to the movable plate is d which is much smaller than the dimensions of the plates. The thickness of the movable plate can be ignored.

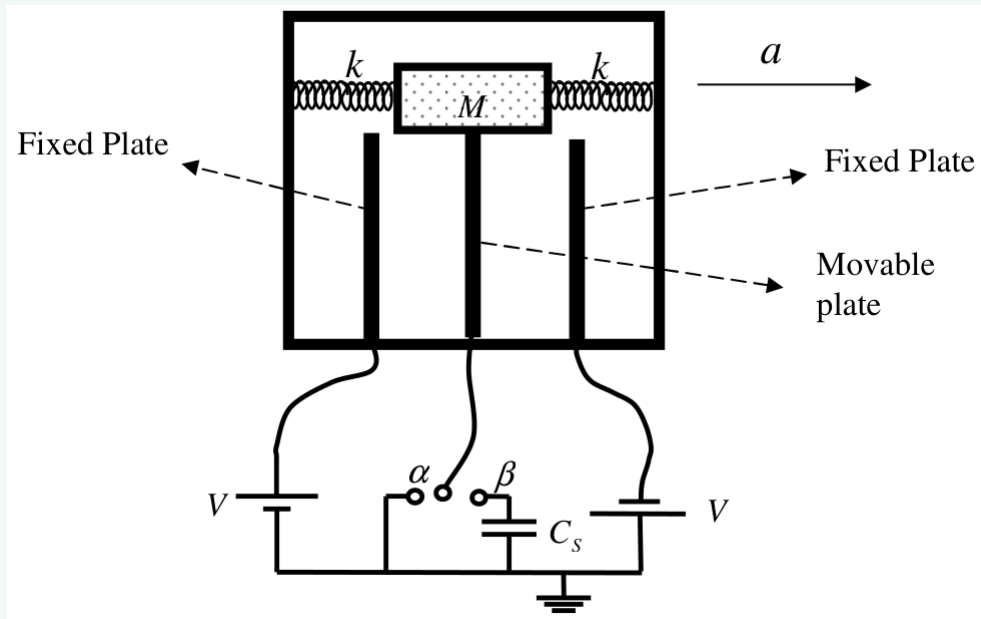


Figure 2

The switch can be in either one of the two states α and β . Assume that the capacitor complex is being accelerated along with the automobile, and the acceleration is constant. Assume that during this constant acceleration the spring does not oscillate and all components of this complex capacitor are in their equilibrium positions, i.e. they do not move with respect to each other, and hence with respect to the automobile.

Due to the acceleration, the movable plate will be displaced a certain amount x from the middle of the two fixed plates.

2 Consider the case where the switch is in state α i.e. the movable plate is connected to the ground through a wire, then

Part	Question	Points
2.1	Find the charge on each capacitor as a function of x .	0.4
2.2	Find the net electrical force on the movable plate, F_E , as a function of x .	0.4
2.3	Assume $d \gg x$ and terms of order x^2 can be ignored compared to terms of order d^2 . Simplify the answer to the previous part.	0.2
2.4	Write the total force on the movable plate (the sum of the electrical and the spring forces) as $-k_{\text{eff}}x$ and give the form of k_{eff} .	0.7
2.5	Express the constant acceleration a as a function of x .	0.4

3 Now assume that the switch is in state β i.e. the movable plate is connected to the ground through a capacitor, the capacitance of which is C_S (there is no initial charge on the capacitors). If the movable plate is displaced by an amount x from its central position,

Part	Question	Points
3.1	Find V_S the electrical potential difference across the capacitor C_S as a function of x .	1.5

Part	Question	Points
3.2	Again assume that $d \gg x$ and ignore terms of order x^2 compared to terms of order d^2 . Simplify your answer to the previous part.	0.2

4 We would like to adjust the parameters in the problem such that the air bag will not be activated in normal braking but opens fast enough during a collision to prevent the driver's head from colliding with the windshield or the steering wheel. As you have seen in Part 2, the force exerted on the movable plate by the springs and the electrical charges can be represented as that of a spring with an effective spring constant k_{eff} . The whole capacitor complex is similar to a *mass and spring system* of mass M and spring constant k_{eff} under the influence of a constant acceleration a , which in this problem is the acceleration of the automobile.

Note: In this part of the problem, the assumption that the mass and spring are in equilibrium under a constant acceleration and hence are fixed relative to the automobile, no longer holds.

Ignore friction and consider the following numerical values for the parameters of the problem:

$$d = 1.0 \text{ cm}, \quad A = 2.5 \times 10^{-2} \text{ m}^2, \quad k = 4.2 \times 10^3 \text{ N/m}, \\ \epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/(\text{N m}^2), \quad V = 12 \text{ V}, \quad M = 0.15 \text{ kg}.$$

Part	Question	Points
4.1	Using this data, find the ratio of the electrical force you calculated in section 2.3 to the force of the springs and show that one can ignore the electrical forces compared to the spring forces.	0.6

Although we did not calculate the electrical forces for the case when the switch is in the state β , it can be shown that in this situation, quite similarly, the electrical forces are as small and can be ignored.

Part	Question	Points
4.2	If the automobile while traveling with a constant velocity, suddenly brakes with a constant acceleration a , what is the maximum displacement of the movable plate? Give your answer in parameter.	0.6

Assume that the switch is in state β and the system has been designed such that when the electrical voltage across the capacitor reaches $V_S = 0.15 \text{ V}$, the air bag is activated. We would like the air bag not to be activated during normal braking when the automobile's acceleration is less than the acceleration of gravity $g = 9.8 \text{ m/s}^2$, but be activated otherwise.

Part	Question	Points
4.3	How much should C_S be for this purpose?	0.6

We would like to find out if the air bag will be activated fast enough to prevent the driver's head from hitting the windshield or the steering wheel. Assume that as a result of collision, the automobile experiences a deceleration equal to g but the driver's head keeps moving at a constant speed.

Part	Question	Points
4.4	By estimating the distance between the driver's head and the steering wheel, find the time t_1 it takes before the driver's head hits the steering wheel.	0.8

Part	Question	Points
4.5	Find the time t_2 before the air bag is activated and compare it to t_1 . Is the air bag activated in time? Assume that airbag opens instantaneously.	0.9

Official English text of the problem.

Answers

Take $x > 0$ towards the left fixed plate. In parts 2–4, this is the plate at $+V$; take the automobile's acceleration $a > 0$ in the opposite direction, as drawn. Force components are positive in the direction of positive x , and V_S is the movable plate's potential relative to ground. Write $C_0 = \varepsilon_0 A/d$.

- **1.1:** $F_E = \frac{Q^2}{2\varepsilon_0 A}$, attractive.
- **1.2:** $x = \frac{Q^2}{2k\varepsilon_0 A}$, towards the fixed plate.
- **1.3:** $V = \frac{Q}{\varepsilon_0 A} \left(d - \frac{Q^2}{2k\varepsilon_0 A} \right)$.
- **1.4:** $\frac{C}{C_0} = \left(1 - \frac{Q^2}{2k\varepsilon_0 A d} \right)^{-1}$.
- **1.5:** $U = \frac{Q^2 d}{2\varepsilon_0 A} - \frac{Q^4}{8k\varepsilon_0^2 A^2}$.
- **2.1:** Charge magnitudes: $Q_L = \frac{\varepsilon_0 A V}{d - x}$, $Q_R = \frac{\varepsilon_0 A V}{d + x}$. The corresponding charges on the movable plate are $-Q_L$ and $+Q_R$.
- **2.2:** $F_E = \frac{\varepsilon_0 A V^2}{2} \left[\frac{1}{(d - x)^2} - \frac{1}{(d + x)^2} \right] = \frac{2\varepsilon_0 A V^2 dx}{(d^2 - x^2)^2}$.
- **2.3:** $F_E \simeq \frac{2\varepsilon_0 A V^2}{d^3} x$.
- **2.4:** $F_{\text{tot}} = -k_{\text{eff}} x$, $k_{\text{eff}} = 2k - \frac{2\varepsilon_0 A V^2}{d^3}$.
- **2.5:** $a = \frac{k_{\text{eff}}}{M} x = \frac{2}{M} \left(k - \frac{\varepsilon_0 A V^2}{d^3} \right) x$, in the small-displacement approximation.
- **3.1:** $V_S = \frac{2\varepsilon_0 A V x}{2\varepsilon_0 A d + C_S(d^2 - x^2)}$.
- **3.2:** $V_S \simeq \frac{2C_0}{2C_0 + C_S} \frac{V x}{d}$.
- **4.1:** $\frac{|F_E|}{|F_{\text{springs}}|} = \frac{\varepsilon_0 A V^2}{k d^3} = 7.6 \times 10^{-9}$. Thus $k_{\text{eff}} \simeq 2k$.
- **4.2:** $x_{\text{max}} = \frac{2Ma}{k_{\text{eff}}} \simeq \frac{Ma}{k}$.
- **4.3:** $C_S \simeq 8.0 \times 10^{-11} \text{ F} = 80 \text{ pF}$.
- **4.4:** $t_1 = \sqrt{\frac{2L}{g}}$. Estimating the initial head-to-wheel distance as $L = 0.50 \text{ m}$ gives $t_1 \simeq 0.32 \text{ s}$.

- **4.5:** $t_2 = \pi \sqrt{\frac{M}{2k}} \simeq 1.3 \times 10^{-2} \text{ s} = 13 \text{ ms}$. Since $t_2 \ll t_1$, the air bag activates in time under the stated assumptions.

Solution

We neglect fringing fields throughout, as justified by the small plate separations compared with the plate dimensions.

1.1

The surface charge density is $\sigma = Q/A$. A uniformly charged plate produces a field of magnitude $\sigma/(2\epsilon_0)$ at the other plate. A plate experiences force from the other plate's field, so the attractive force has magnitude

$$F_E = Q \frac{\sigma}{2\epsilon_0} = \frac{Q^2}{2\epsilon_0 A}.$$

At fixed charge, this force is independent of the plate separation within the parallel-plate approximation.

1.2

Attraction moves the movable plate towards the fixed plate and stretches the spring by x . Mechanical equilibrium requires

$$kx = F_E.$$

Consequently,

$$x = \frac{Q^2}{2k\epsilon_0 A}.$$

The separated-plate solution requires $x < d$, or

$$Q^2 < 2k\epsilon_0 Ad.$$

1.3

The electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}.$$

Their equilibrium separation is $d - x$, hence

$$V = E(d - x).$$

Substituting the displacement gives

$$V = \frac{Q}{\epsilon_0 A} \left(d - \frac{Q^2}{2k\epsilon_0 A} \right).$$

1.4

By the given definition,

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d - x}.$$

Since $C_0 = \varepsilon_0 A/d$,

$$\frac{C}{C_0} = \frac{d}{d-x} = \left(1 - \frac{Q^2}{2k\varepsilon_0 A d}\right)^{-1}.$$

1.5

At equilibrium, the stored energy consists of the capacitor's electric-field energy and the spring's elastic energy:

$$U = \frac{Q^2}{2C} + \frac{1}{2}kx^2 = \frac{Q^2(d-x)}{2\varepsilon_0 A} + \frac{1}{2}kx^2.$$

Using $x = Q^2/(2k\varepsilon_0 A)$,

$$\begin{aligned} U &= \frac{Q^2 d}{2\varepsilon_0 A} - \frac{Q^4}{4k\varepsilon_0^2 A^2} + \frac{Q^4}{8k\varepsilon_0^2 A^2} \\ &= \boxed{\frac{Q^2 d}{2\varepsilon_0 A} - \frac{Q^4}{8k\varepsilon_0^2 A^2}}. \end{aligned}$$

2.1

With x positive towards the left plate, the left and right gaps are $d-x$ and $d+x$. Their capacitances are therefore

$$C_L = \frac{\varepsilon_0 A}{d-x}, \quad C_R = \frac{\varepsilon_0 A}{d+x}.$$

In switch position α , the movable plate is grounded. Each capacitor has voltage magnitude V , giving

$$\boxed{Q_L = C_L V = \frac{\varepsilon_0 A V}{d-x}, \quad Q_R = C_R V = \frac{\varepsilon_0 A V}{d+x}}.$$

The left fixed plate carries $+Q_L$, and the facing surface of the movable plate carries $-Q_L$. The right fixed plate carries $-Q_R$, and the facing surface of the movable plate carries $+Q_R$.

2.2

Each fixed plate attracts the movable plate. The left attraction acts in the positive- x direction and the right attraction in the negative- x direction. Using the force from 1.1,

$$F_E = \frac{Q_L^2 - Q_R^2}{2\varepsilon_0 A}.$$

Thus

$$F_E = \frac{\varepsilon_0 A V^2}{2} \left[\frac{1}{(d-x)^2} - \frac{1}{(d+x)^2} \right].$$

Combining the fractions gives

$$\boxed{F_E = \frac{2\varepsilon_0 A V^2 dx}{(d^2 - x^2)^2}}.$$

The force has the same sign as x : electrical attraction tends to increase the displacement towards the nearer plate.

2.3

Rewrite the exact result as

$$F_E = \frac{2\varepsilon_0 AV^2}{d^3} x \left(1 - \frac{x^2}{d^2}\right)^{-2}.$$

For $|x| \ll d$, neglecting x^2/d^2 gives

$$F_E \simeq \frac{2\varepsilon_0 AV^2}{d^3} x.$$

2.4

Both springs exert restoring forces. One is compressed and the other stretched, and their total force is

$$F_{\text{springs}} = -2kx.$$

Adding the linearized electrical force,

$$F_{\text{tot}} = -2kx + \frac{2\varepsilon_0 AV^2}{d^3} x = -k_{\text{eff}} x.$$

Therefore,

$$k_{\text{eff}} = 2k - \frac{2\varepsilon_0 AV^2}{d^3}.$$

The electrical interaction reduces the restoring stiffness. A stable small-displacement response requires $k_{\text{eff}} > 0$.

2.5

The automobile accelerates opposite to positive x . In the automobile frame, the inertial force on the mass is therefore $+Ma$. Relative equilibrium requires

$$-k_{\text{eff}} x + Ma = 0.$$

Hence

$$a = \frac{k_{\text{eff}} x}{M} = \frac{2}{M} \left(k - \frac{\varepsilon_0 AV^2}{d^3} \right) x.$$

This expression uses the small-displacement approximation introduced in 2.3.

3.1

In position β , the movable plate, its connecting wire, and the connected plate of C_S form an isolated conductor. Its initially zero total charge remains zero.

Let its potential relative to ground be V_S . The charges on its three capacitor surfaces are

$$q_L = C_L(V_S - V), \quad q_R = C_R(V_S + V), \quad q_S = C_S V_S.$$

Charge conservation gives

$$C_L(V_S - V) + C_R(V_S + V) + C_S V_S = 0.$$

Therefore,

$$V_S = \frac{V(C_L - C_R)}{C_L + C_R + C_S}.$$

Now

$$C_L - C_R = \frac{2\varepsilon_0 Ax}{d^2 - x^2}, \quad C_L + C_R = \frac{2\varepsilon_0 Ad}{d^2 - x^2}.$$

Substitution yields

$$V_S = \frac{2\varepsilon_0 AVx}{2\varepsilon_0 Ad + C_S(d^2 - x^2)}.$$

In particular, $V_S = 0$ at the central position and $V_S > 0$ for displacement towards the $+V$ plate.

3.2

Neglect x^2 compared with d^2 in the denominator:

$$V_S \simeq \frac{2\varepsilon_0 AVx}{2\varepsilon_0 Ad + C_S d^2}.$$

Using $C_0 = \varepsilon_0 A/d$,

$$V_S \simeq \frac{2C_0}{2C_0 + C_S} \frac{Vx}{d}.$$

Thus the measured voltage is proportional to displacement in this approximation.

4.1

The electrical and spring forces oppose each other. Their magnitude ratio is

$$\frac{|F_E|}{|F_{\text{springs}}|} = \frac{(2\varepsilon_0 AV^2/d^3)|x|}{2k|x|} = \frac{\varepsilon_0 AV^2}{kd^3}.$$

With $d = 1.0 \times 10^{-2}$ m,

$$\frac{|F_E|}{|F_{\text{springs}}|} = \frac{(8.85 \times 10^{-12})(2.5 \times 10^{-2})(12)^2}{(4.2 \times 10^3)(1.0 \times 10^{-2})^3} = \boxed{7.6 \times 10^{-9}}.$$

The electrical contribution is negligible, so

$$k_{\text{eff}} \simeq 2k = 8.4 \times 10^3 \text{ N/m}.$$

As permitted by the statement, we also neglect electrical forces in position β .

4.2

Before braking, the mass is at the centre and at rest relative to the automobile:

$$x(0) = 0, \quad \dot{x}(0) = 0.$$

After the sudden onset of constant acceleration, its equation of motion in the automobile frame is

$$M\ddot{x} + k_{\text{eff}}x = Ma.$$

Define

$$\omega = \sqrt{\frac{k_{\text{eff}}}{M}}, \quad x_{\text{eq}} = \frac{Ma}{k_{\text{eff}}}.$$

The solution satisfying the initial conditions is

$$x(t) = x_{\text{eq}}(1 - \cos \omega t).$$

The first maximum occurs at $\omega t = \pi$, giving

$$x_{\max} = 2x_{\text{eq}} = \frac{2Ma}{k_{\text{eff}}} \simeq \frac{Ma}{k}.$$

The maximum displacement is twice the static equilibrium displacement because the undamped mass overshoots its new equilibrium position.

4.3

Let the trigger voltage be

$$V_{\text{th}} = 0.15 \text{ V}.$$

To put the activation boundary at $a = g$, the maximum voltage during a constant acceleration g must equal V_{th} .

The corresponding maximum displacement is

$$x_g = \frac{Mg}{k} = \frac{(0.15)(9.8)}{4.2 \times 10^3} = 3.5 \times 10^{-4} \text{ m}.$$

Thus

$$\frac{x_g}{d} = 0.035,$$

which supports the small-displacement approximation.

From 3.2,

$$V_{\text{th}} = \frac{2C_0}{2C_0 + C_S} \frac{Vx_g}{d}.$$

Solving for the required capacitance,

$$C_S = 2C_0 \left(\frac{Vx_g}{dV_{\text{th}}} - 1 \right) = 2C_0 \left(\frac{VMg}{kdV_{\text{th}}} - 1 \right).$$

Here,

$$C_0 = \frac{\varepsilon_0 A}{d} = 2.2125 \times 10^{-11} \text{ F} = 22.125 \text{ pF},$$

and

$$\frac{Vx_g}{d} = (12)(0.035) = 0.42 \text{ V}.$$

Consequently,

$$C_S = 2(22.125 \text{ pF}) \left(\frac{0.42}{0.15} - 1 \right) = 79.65 \text{ pF}.$$

To the precision of the data,

$$C_S \simeq 80 \text{ pF} = 8.0 \times 10^{-11} \text{ F}.$$

For this constant-acceleration braking model, the maximum voltage is proportional to a , so accelerations below g do not reach the threshold, while those at or above g do.

For comparison, retaining the exact geometry from 3.1 gives

$$C_{S,\text{exact}} = \frac{2\varepsilon_0 A (V x_g / V_{\text{th}} - d)}{d^2 - x_g^2} = 79.75 \text{ pF},$$

which has the same rounded value.

4.4

Let L be the initial distance between the driver's head and the steering wheel along the direction of motion.

Initially they have the same velocity. The head continues at constant velocity, while the steering wheel decelerates with the automobile at magnitude g . Their relative acceleration therefore has magnitude g , and the distance closed after time t is

$$s_{\text{rel}} = \frac{1}{2}gt^2.$$

Impact occurs when this equals L :

$$t_1 = \sqrt{\frac{2L}{g}}.$$

Taking a reasonable estimate $L = 0.50 \text{ m}$,

$$t_1 = \sqrt{\frac{2(0.50)}{9.8}} = 0.319 \text{ s} \simeq \boxed{0.32 \text{ s}}.$$

4.5

At $a = g$, the capacitor was chosen so that the trigger voltage is reached at the maximum displacement x_g . The motion is

$$x(t) = \frac{Mg}{2k}(1 - \cos \omega t), \quad \omega = \sqrt{\frac{2k}{M}}.$$

Since $x_g = Mg/k$, the first trigger occurs when

$$1 - \cos \omega t_2 = 2,$$

or

$$\omega t_2 = \pi.$$

Therefore,

$$t_2 = \pi \sqrt{\frac{M}{2k}}.$$

Numerically,

$$t_2 = \pi \sqrt{\frac{0.15}{8.4 \times 10^3}} = 0.01328 \text{ s} \simeq \boxed{13 \text{ ms}}.$$

Comparing with the estimated head-impact time,

$$\frac{t_2}{t_1} \simeq 0.042.$$

The head closes only

$$\frac{1}{2}gt_2^2 \simeq 8.6 \times 10^{-4} \text{ m} = 0.86 \text{ mm}$$

before activation. Thus **the air bag activates well before the head reaches the steering wheel**, assuming instantaneous opening as stipulated.

The solution was checked against the official solution.

Problem 3: Binary stars

Two stars rotating around their center of mass form a binary star system. Almost half of the stars in our galaxy are binary star systems. It is not easy to realize the binary nature of most of these star systems from Earth, since the distance between the two stars is much less than their distance from us and thus the stars cannot be resolved with telescopes. Therefore, we have to use either photometry or spectrometry to observe the variations in the intensity or the spectrum of a particular star to find out whether it is a binary system or not.

Photometry of Binary Stars

If we are exactly on the plane of motion of the two stars, then one star will occult (pass in front of) the other star at certain times and the intensity of the whole system will vary with time from our observation point. These binary systems are called eclipsing binaries.

1 Assume that two stars are moving on circular orbits around their common center of mass with a constant angular speed ω and we are exactly on the plane of motion of the binary system. Also assume that the surface temperatures of the stars are T_1 and T_2 ($T_1 > T_2$), and the corresponding radii are R_1 and R_2 ($R_1 > R_2$), respectively. The total intensity of light, measured on Earth, is plotted in Figure 1 as a function of time. Careful measurements indicate that the intensities of the incident light from the stars corresponding to the minima are respectively 90 and 63 percent of the maximum intensity, I_0 , received from both stars ($I_0 = 4.8 \times 10^{-9} \text{ W/m}^2$). The vertical axis in Figure 1 shows the ratio I/I_0 and the horizontal axis is marked in days.

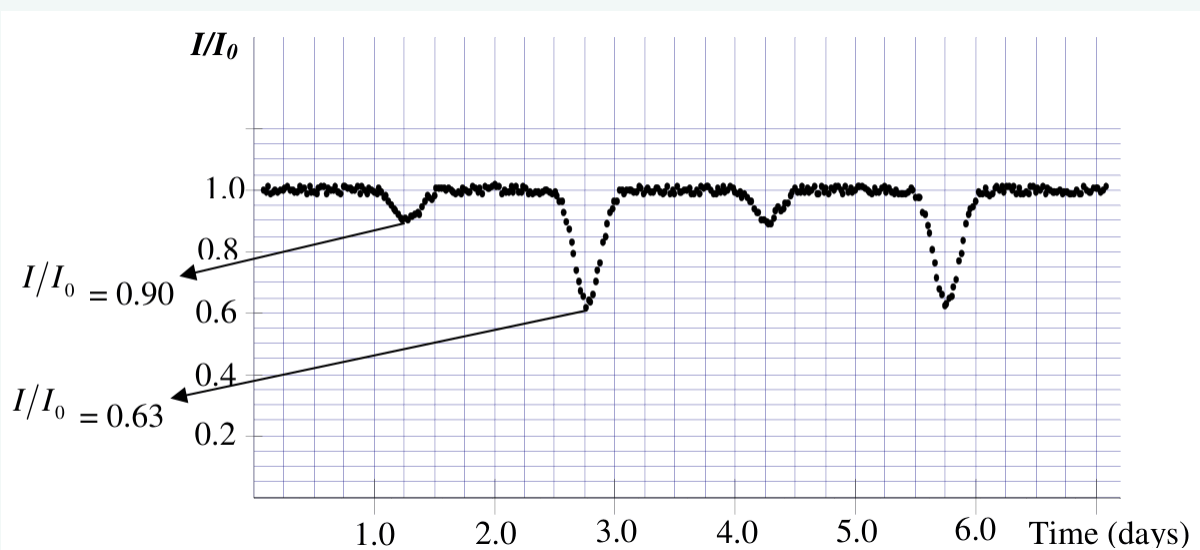


Figure 1. The relative intensity received from the binary star system as a function of time. The vertical axis has been scaled by $I_0 = 4.8 \times 10^{-9} \text{ W/m}^2$. Time is given in days.

1.1 — 0.8 points

Find the period of the orbital motion. Give your answer in **seconds** up to two significant digits.
What is the angular frequency of the system in **rad/sec**?

To a good approximation, the receiving radiation from a star is a uniform black body radiation from a flat disc with a radius equal to the radius of the star. Therefore, the power received from the star is proportional to AT^4 where A is area of the disc and T is the surface temperature of the star.

1.2 — 1.6 points

Use the diagram in Figure 1 to find the ratios T_1/T_2 and R_1/R_2 .

Spectrometry of Binary Systems

In this section, we are going to calculate the astronomical properties of a binary star by using experimental spectrometric data of the binary system.

Atoms absorb or emit radiation at their certain characteristic wavelengths. Consequently, the observed spectrum of a star contains *absorption lines* due to the atoms in the star's atmosphere. Sodium has a characteristic yellow line spectrum (D_1 line) with a wavelength $5895.9 \text{ \AA}$ ($10 \text{ \AA} = 1 \text{ nm}$). We examine the absorption spectrum of atomic Sodium at this wavelength for the binary system of the previous section. The spectrum of the light that we receive from the binary star is Doppler-shifted, because the stars are moving with respect to us. Each star has a different speed. Accordingly the absorption wavelength for each star will be shifted by a different amount. Highly accurate wavelength measurements are required to observe the Doppler shift since the speed of the stars is much less than the speed of light. The speed of the center of mass of the binary system we consider in this problem is much smaller than the orbital velocities of the stars. Hence all the Doppler shifts can be attributed to the orbital velocity of the stars. Table 1 shows the measured spectrum of the stars in the binary system we have observed.

Table 1: Absorption spectrum of the binary star system for the Sodium D_1 line

t/days	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4
$\lambda_1 (\text{\AA})$	5897.5	5897.7	5897.2	5896.2	5895.1	5894.3	5894.1	5894.6
$\lambda_2 (\text{\AA})$	5893.1	5892.8	5893.7	5896.2	5897.3	5898.7	5899.0	5898.1

t/days	2.7	3.0	3.3	3.6	3.9	4.2	4.5	4.8
$\lambda_1 (\text{\AA})$	5895.6	5896.7	5897.3	5897.7	5897.2	5896.2	5895.0	5894.3
$\lambda_2 (\text{\AA})$	5896.4	5894.5	5893.1	5892.8	5893.7	5896.2	5897.4	5898.7

(Note: There is no need to make a graph of the data in this table)

2 Using Table 1,

2.1 — 1.8 points

Let v_1 and v_2 be the orbital velocity of each star. Find v_1 and v_2 .
The speed of light $c = 3.0 \times 10^8 \text{ m/s}$. Ignore all relativistic effects.

2.2 — 0.7 points

Find the mass ratio of the stars (m_1/m_2).

2.3 — 0.8 points

Let r_1 and r_2 be the distances of each star from their center of mass.

Find r_1 and r_2 .

2.4 — 0.2 points

Let r be the distance between the stars. Find r .

3 The gravitational force is the only force acting between the stars.

3.1 — 1.2 points

Find the mass of each star up to one significant digit.

The universal gravitational constant $G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.

General Characteristics of Stars

4 Most of the stars generate energy through the same mechanism. Because of this, there is an empirical relation between their mass, M , and their luminosity, L , which is the total radiant power of the star. This relation could be written in the form $L/L_{\text{Sun}} = (M/M_{\text{Sun}})^\alpha$.

Here, $M_{\text{Sun}} = 2.0 \times 10^{30} \text{ kg}$ is the solar mass and, $L_{\text{Sun}} = 3.9 \times 10^{26} \text{ W}$ is the solar luminosity. This relation is shown in a log-log diagram in Figure 2.

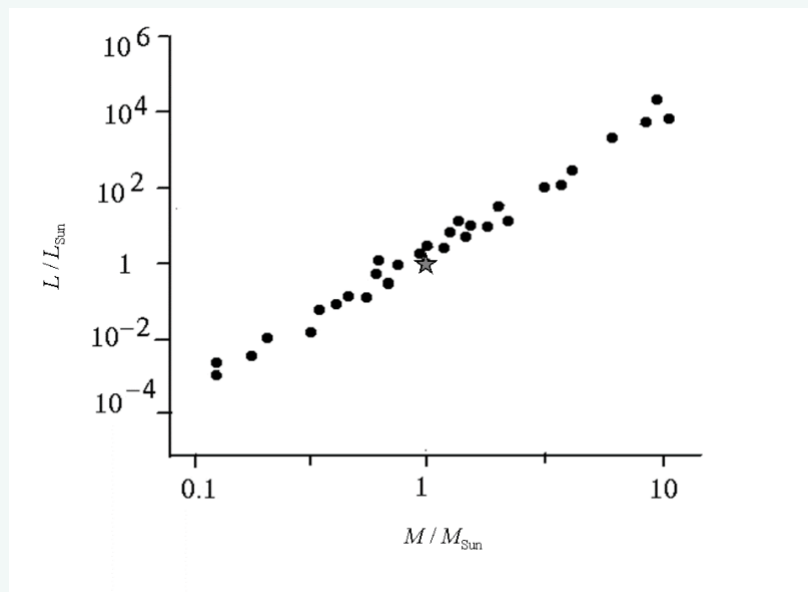


Figure 2. The luminosity of a star versus its mass varies as a power law. The diagram is log-log. The star-symbol represents Sun with a mass of $2.0 \times 10^{30} \text{ kg}$ and luminosity of $3.9 \times 10^{26} \text{ W}$.

4.1 — 0.6 points

Find α up to one significant digit.

4.2 — 0.6 points

Let L_1 and L_2 be the luminosity of the stars in the binary system studied in the previous sections. Find L_1 and L_2 .

4.3 — 0.9 points

What is the distance, d , of the star system from us in light years?

To find the distance you can use the diagram of Figure 1. One light year is the distance light travels in one year.

4.4 — 0.4 points

What is the maximum angular distance, θ , between the stars from our observation point?

4.5 — 0.4 points

What is the smallest aperture size for an optical telescope, D , that can resolve these two stars?

Official English text of the problem.

Answers

Later results use unrounded intermediate values.

- **1.1:** $P = 3.0 \text{ days} \simeq 2.6 \times 10^5 \text{ s}$; $\omega \simeq 2.4 \times 10^{-5} \text{ rad/s}$.
- **1.2:** $T_1/T_2 \simeq 1.39$; $R_1/R_2 \simeq 1.56$.
- **2.1:** $v_1 \simeq 9.2 \times 10^4 \text{ m/s}$; $v_2 \simeq 1.6 \times 10^5 \text{ m/s}$.
- **2.2:** $m_1/m_2 \simeq 1.7$.
- **2.3:** $r_1 \simeq 3.8 \times 10^9 \text{ m}$; $r_2 \simeq 6.5 \times 10^9 \text{ m}$.
- **2.4:** $r \simeq 1.0 \times 10^{10} \text{ m}$.
- **3.1:** To one significant digit, $m_1 \simeq 6 \times 10^{30} \text{ kg}$, $m_2 \simeq 4 \times 10^{30} \text{ kg}$.
- **4.1:** $\alpha \simeq 4$.
- **4.2:** $L_1 \simeq 3.2 \times 10^{28} \text{ W}$; $L_2 \simeq 3.7 \times 10^{27} \text{ W}$.
- **4.3:** $d \simeq 82 \text{ light years} \simeq 7.7 \times 10^{17} \text{ m}$.
- **4.4:** $\theta_{\max} \simeq 1.3 \times 10^{-8} \text{ rad}$, or $2.7 \times 10^{-3} \text{ arcseconds}$.
- **4.5:** $D_{\min} \simeq 54 \text{ m}$, using the Rayleigh criterion at the given sodium wavelength, 589.59 nm.

Solution

1.1

The orbital period is the interval between successive minima of the **same depth**. Figure 1 gives approximately 3.0 days. Adjacent minima of different depths correspond to the two different conjunctions and are separated by half a period.

Thus

$$P = 3.0 \times 86400 \text{ s} = 259200 \text{ s} \simeq \boxed{2.6 \times 10^5 \text{ s}}.$$

The angular frequency is

$$\omega = \frac{2\pi}{P} = 2.424 \dots \times 10^{-5} \text{ rad/s} \simeq \boxed{2.4 \times 10^{-5} \text{ rad/s}}.$$

1.2

Let the unobscured contributions to the intensity at Earth be

$$I_1 = CR_1^2 T_1^4, \quad I_2 = CR_2^2 T_2^4,$$

where C is common to both stars because they are effectively at the same distance. Outside eclipse,

$$I_0 = I_1 + I_2.$$

The edge-on geometry produces central eclipses. Since $R_1 > R_2$, the larger star can completely hide the smaller star. During this eclipse the lost intensity is I_2 .

At the other eclipse, the smaller star blocks an equal-sized patch of the hotter star. The lost intensity is then

$$B = CR_2^2 T_1^4.$$

Because $T_1 > T_2$, $B > I_2$. Therefore, the shallower minimum is the eclipse in which star 2 is hidden.

The shallow minimum gives

$$I_1 = 0.90I_0, \quad I_2 = 0.10I_0.$$

The deep minimum gives

$$B = (1 - 0.63)I_0 = 0.37I_0.$$

Dividing B by I_2 ,

$$\left(\frac{T_1}{T_2}\right)^4 = \frac{CR_2^2 T_1^4}{CR_2^2 T_2^4} = \frac{0.37}{0.10} = 3.7,$$

so

$$\boxed{\frac{T_1}{T_2} = 3.7^{1/4} \simeq 1.39}.$$

Similarly,

$$\left(\frac{R_1}{R_2}\right)^2 = \frac{I_1}{B} = \frac{0.90}{0.37},$$

hence

$$\boxed{\frac{R_1}{R_2} = \sqrt{\frac{0.90}{0.37}} \simeq 1.56}.$$

2.1

For nonrelativistic motion, the Doppler shift is

$$\frac{\lambda - \lambda_0}{\lambda_0} = \frac{u}{c},$$

where u is the line-of-sight velocity, positive for recession, and

$$\lambda_0 = 5895.9 \text{ \AA}.$$

In an edge-on circular orbit, the line-of-sight velocity varies sinusoidally between $-v_i$ and $+v_i$. Thus the wavelength amplitude determines the orbital speed:

$$v_i = \frac{c}{\lambda_0} \frac{\lambda_{i,\max} - \lambda_{i,\min}}{2}.$$

Using the tabulated extrema,

$$\Delta\lambda_1 = \frac{5897.7 - 5894.1}{2} \text{ \AA} = 1.8 \text{ \AA},$$

$$\Delta\lambda_2 = \frac{5899.0 - 5892.8}{2} \text{ \AA} = 3.1 \text{ \AA}.$$

Therefore,

$$v_1 = 3.0 \times 10^8 \frac{1.8}{5895.9} = 9.159 \dots \times 10^4 \text{ m/s},$$

$$v_2 = 3.0 \times 10^8 \frac{3.1}{5895.9} = 1.577 \dots \times 10^5 \text{ m/s}.$$

Thus

$$v_1 \simeq 9.2 \times 10^4 \text{ m/s}, \quad v_2 \simeq 1.6 \times 10^5 \text{ m/s}.$$

2.2

In the center-of-mass frame, the stars have equal and opposite momenta:

$$m_1 v_1 = m_2 v_2.$$

Consequently,

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = \frac{3.1}{1.8} = 1.722 \dots \simeq 1.7.$$

2.3

Both stars have the same angular frequency. For uniform circular motion,

$$v_i = \omega r_i, \quad r_i = \frac{v_i P}{2\pi}.$$

Using the unrounded values from the preceding calculations,

$$r_1 = 3.778 \dots \times 10^9 \text{ m}, \quad r_2 = 6.507 \dots \times 10^9 \text{ m}.$$

Therefore,

$$r_1 \simeq 3.8 \times 10^9 \text{ m}, \quad r_2 \simeq 6.5 \times 10^9 \text{ m}.$$

2.4

The center of mass lies between the stars, so their separation is

$$r = r_1 + r_2 = 1.02854 \dots \times 10^{10} \text{ m}.$$

Thus, to two significant figures,

$$r \simeq 1.0 \times 10^{10} \text{ m}.$$

3.1

The gravitational force supplies the centripetal force for each star:

$$\frac{Gm_1 m_2}{r^2} = m_1 \omega^2 r_1 = m_2 \omega^2 r_2.$$

It follows that

$$m_1 = \frac{\omega^2 r_2 r^2}{G}, \quad m_2 = \frac{\omega^2 r_1 r^2}{G}.$$

Substitution, retaining intermediate digits, gives

$$m_1 = 6.037 \dots \times 10^{30} \text{ kg}, \quad m_2 = 3.5056 \dots \times 10^{30} \text{ kg}.$$

The requested answers to one significant digit are therefore

$$\boxed{m_1 \simeq 6 \times 10^{30} \text{ kg}, \quad m_2 \simeq 4 \times 10^{30} \text{ kg}}.$$

The unrounded calculated masses will be retained below, since raising prematurely rounded masses to a power would introduce substantial numerical error.

4.1

Taking logarithms of the empirical relation gives

$$\log\left(\frac{L}{L_{\text{Sun}}}\right) = \alpha \log\left(\frac{M}{M_{\text{Sun}}}\right).$$

Thus α is the slope of the trend in the log-log diagram.

The trend passes approximately through

$$\left(\frac{M}{M_{\text{Sun}}}, \frac{L}{L_{\text{Sun}}}\right) = (1, 1) \quad \text{and} \quad (10, 10^4).$$

Hence

$$\alpha \simeq \frac{\log_{10}(10^4) - \log_{10}(1)}{\log_{10}(10) - \log_{10}(1)} = \boxed{4}.$$

4.2

With $\alpha = 4$,

$$L_i = L_{\text{Sun}} \left(\frac{m_i}{M_{\text{Sun}}}\right)^4.$$

Using the unrounded dynamical masses,

$$\begin{aligned} L_1 &= 3.9 \times 10^{26} \left(\frac{6.037 \dots \times 10^{30}}{2.0 \times 10^{30}}\right)^4 = 3.238 \dots \times 10^{28} \text{ W}, \\ L_2 &= 3.9 \times 10^{26} \left(\frac{3.5056 \dots \times 10^{30}}{2.0 \times 10^{30}}\right)^4 = 3.681 \dots \times 10^{27} \text{ W}. \end{aligned}$$

Therefore,

$$\boxed{L_1 \simeq 3.2 \times 10^{28} \text{ W}, \quad L_2 \simeq 3.7 \times 10^{27} \text{ W}}.$$

As a consistency check, these give $L_1/L_2 \simeq 8.8$, close to the photometric ratio $I_1/I_2 = 9$, as expected for the approximate empirical law and measured data.

4.3

For isotropic radiation, luminosity L produces intensity $L/(4\pi d^2)$ at distance d . Outside eclipse, both stars contribute their full flux:

$$I_0 = \frac{L_1 + L_2}{4\pi d^2}.$$

Thus

$$d = \sqrt{\frac{L_1 + L_2}{4\pi I_0}} = \sqrt{\frac{3.6065 \dots \times 10^{28}}{4\pi(4.8 \times 10^{-9})}} = 7.7325 \dots \times 10^{17} \text{ m}.$$

Taking one year as approximately 365 days,

$$1 \text{ light year} = c(365 \times 86400 \text{ s}) \simeq 9.46 \times 10^{15} \text{ m}.$$

Therefore,

$$d \simeq 82 \text{ light years} \simeq 7.7 \times 10^{17} \text{ m}.$$

4.4

The projected separation is largest when the line joining the stars is perpendicular to the line of sight. Its value is then r .

Since $r \ll d$, the small-angle approximation gives

$$\theta_{\max} \simeq \frac{r}{d} = \frac{1.02854 \dots \times 10^{10}}{7.7325 \dots \times 10^{17}} = 1.330 \dots \times 10^{-8} \text{ rad}.$$

Hence

$$\theta_{\max} \simeq 1.3 \times 10^{-8} \text{ rad}$$

or approximately

$$\theta_{\max} \simeq 2.7 \times 10^{-3} \text{ arcseconds}.$$

4.5

For an ideal diffraction-limited telescope with a circular aperture, the Rayleigh criterion gives the minimum resolvable angular separation:

$$\theta_R = 1.22 \frac{\lambda}{D}.$$

The smallest aperture capable of resolving the stars at their maximum separation therefore satisfies

$$D_{\min} = \frac{1.22\lambda}{\theta_{\max}}.$$

Using the sodium wavelength supplied in the problem,

$$\lambda = 5895.9 \text{ \AA} = 5.8959 \times 10^{-7} \text{ m},$$

we obtain

$$D_{\min} = \frac{1.22(5.8959 \times 10^{-7})}{1.330 \dots \times 10^{-8}} \text{ m} \simeq \boxed{54 \text{ m}}.$$

More generally, the wavelength dependence is

$$D_{\min}(\lambda) \simeq 54 \text{ m} \left(\frac{\lambda}{589.59 \text{ nm}} \right).$$

The solution was checked against the official solution.