

IPhO 2006

Theory problems and solutions · Singapore · 3 theory problems with complete solutions

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Problem 1: Gravity in a Neutron Interferometer

Enter all your answers into the **Answer Script**.

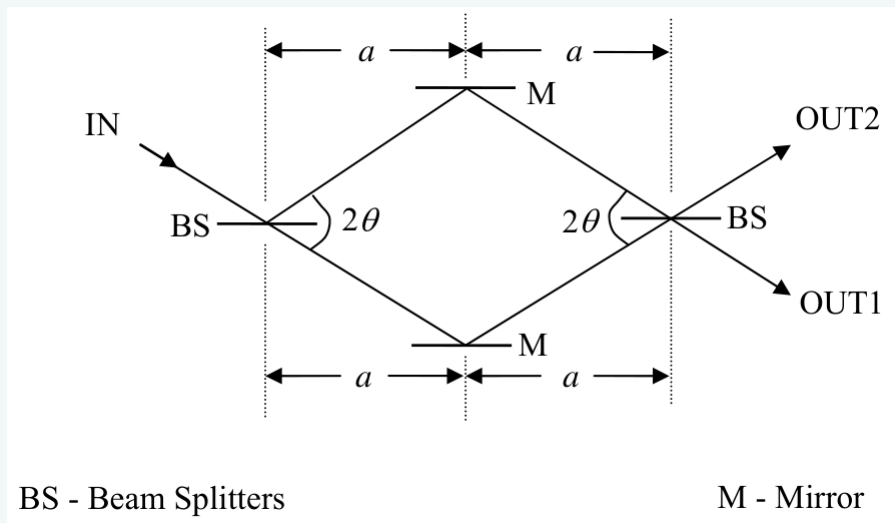


Figure 1a

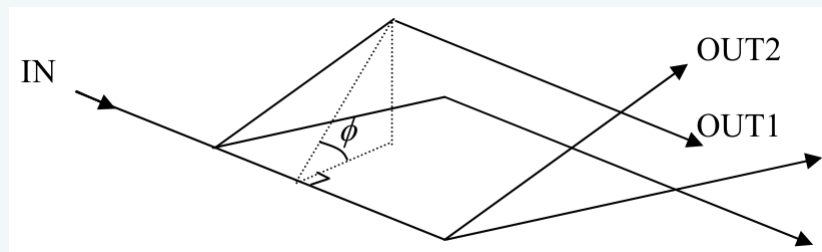


Figure 1b

Physical situation We consider the situation of the famous neutron-interferometer experiment by Colella, Overhauser and Werner, but idealize the set-up inasmuch as we shall assume perfect beam splitters and mirrors within the interferometer. The experiment studies the effect of the gravitational pull on the de Broglie waves of neutrons.

The symbolic representation of this interferometer in analogy to an optical interferometer is shown in Figure 1a. The neutrons enter the interferometer through the IN port and follow the two paths shown. The neutrons are detected at either one of the two output ports, OUT1 or OUT2. The two paths enclose a diamond-shaped area, which is typically a few cm^2 in size.

The neutron de Broglie waves (of typical wavelength of 10^{-10} m) interfere such that all neutrons emerge from the output port OUT1 if the interferometer plane is horizontal. But when the interferometer is tilted around the axis of the incoming neutron beam by angle ϕ (Figure 1b), one

observes a ϕ dependent redistribution of the neutrons between the two output ports OUT1 and OUT2.

Geometry For $\phi = 0^\circ$ the interferometer plane is horizontal; for $\phi = 90^\circ$ the plane is vertical with the output ports above the tilt axis.

1.1 (1.0) How large is the diamond-shaped area A enclosed by the two paths of the interferometer?

1.2 (1.0) What is the height H of output port OUT1 above the horizontal plane of the tilt axis?

Express A and H in terms of a , θ , and ϕ .

Optical path length The optical path length N_{opt} (a number) is the ratio of the geometrical path length (a distance) and the wavelength λ . If λ changes along the path, N_{opt} is obtained by integrating λ^{-1} along the path.

1.3 (3.0) What is the difference ΔN_{opt} in the optical path lengths of the two paths when the interferometer has been tilted by angle ϕ ? Express your answer in terms of a , θ , and ϕ as well as the neutron mass M , the de Broglie wavelength λ_0 of the incoming neutrons, the gravitational acceleration g , and Planck's constant h .

1.4 (1.0) Introduce the volume parameter

$$V = \frac{h^2}{gM^2}$$

and express ΔN_{opt} solely in terms of A , V , λ_0 , and ϕ . State the value of V for $M = 1.675 \times 10^{-27} \text{ kg}$, $g = 9.800 \text{ m s}^{-2}$, and $h = 6.626 \times 10^{-34} \text{ J s}$.

1.5 (2.0) How many cycles — from high intensity to low intensity and back to high intensity — are completed by output port OUT1 when ϕ is increased from $\phi = -90^\circ$ to $\phi = 90^\circ$?

Experimental data The interferometer of an actual experiment was characterized by $a = 3.600 \text{ cm}$ and $\theta = 22.10^\circ$, and 19.00 full cycles were observed.

1.6 (1.0) How large was λ_0 in this experiment?

1.7 (1.0) If one observed 30.00 full cycles in another experiment of the same kind that uses neutrons with $\lambda_0 = 0.2000 \text{ nm}$, how large would be the area A ?

Hint: If $|\alpha x| \ll 1$, it is permissible to replace $(1 + x)^\alpha$ by $1 + \alpha x$.

Official English text of the problem.

Answers

Define the signed difference as $\Delta N_{\text{opt}} = N_{\text{lower}} - N_{\text{upper}}$, with the paths named according to Figure 1a. The phase results retain first order in the gravitational energy change.

- **1.1:** $A = 2a^2 \tan \theta$.
- **1.2:** $H = 2a \sin \theta \sin \phi$.
- **1.3:** $\Delta N_{\text{opt}} \simeq \frac{2M^2 g a^2 \lambda_0}{h^2} \tan \theta \sin \phi$.

- **1.4:** $\Delta N_{\text{opt}} \simeq \frac{A\lambda_0}{V} \sin \phi$, $V = 1.597 \times 10^{-14} \text{ m}^3$.
- **1.5:** $C = \frac{2A\lambda_0}{V} = \frac{4M^2 g a^2 \lambda_0}{h^2} \tan \theta$ cycles.
- **1.6:** $\lambda_0 = 1.441 \times 10^{-10} \text{ m} = 0.1441 \text{ nm}$.
- **1.7:** $A = 1.198 \times 10^{-3} \text{ m}^2 = 11.98 \text{ cm}^2$.

Solution

1.1 — Enclosed area

Label the entrance beam splitter O , the upper mirror U , the lower mirror D , and the exit beam splitter R , according to Figure 1a.

Each side of the diamond has length

$$\ell = \frac{a}{\cos \theta}.$$

The horizontal diagonal OR is $2a$, while the perpendicular diagonal UD is $2a \tan \theta$. Therefore,

$$A = \frac{1}{2}(2a)(2a \tan \theta) = 2a^2 \tan \theta.$$

Tilting the apparatus does not change this area.

1.2 — Height of OUT1

The incoming beam is collinear with OD , so the tilt axis lies along OD . The opposite side UR is parallel to this axis.

The perpendicular distance, within the interferometer plane, from R to the tilt axis is

$$d = \ell \sin(2\theta) = \frac{a}{\cos \theta} 2 \sin \theta \cos \theta = 2a \sin \theta.$$

Rotating the plane by ϕ gives this distance a vertical component $d \sin \phi$. Hence

$$H = 2a \sin \theta \sin \phi.$$

Both U and R have height H , while O and D remain at height zero. The OUT1 beam is parallel to the tilt axis, so it stays at height H . For negative ϕ , H is negative.

1.3 — Difference in optical path lengths

Use nonrelativistic mechanics, a uniform gravitational field, and elastic interactions with the stationary ideal optical elements. Set the gravitational potential energy to zero on the tilt axis.

The incoming momentum and kinetic energy are

$$p_0 = \frac{h}{\lambda_0}, \quad K_0 = \frac{h^2}{2M\lambda_0^2}.$$

Energy conservation at height z gives

$$\frac{p(z)^2}{2M} + Mgz = \frac{p_0^2}{2M}.$$

Using the de Broglie relation, the reciprocal wavelength is therefore

$$\frac{1}{\lambda(z)} = \frac{p(z)}{h} = \frac{1}{\lambda_0} \sqrt{1 - \frac{2M^2 g \lambda_0^2 z}{h^2}}.$$

We retain first order in the small ratio $Mg|z|/K_0$. The hint then gives

$$\frac{1}{\lambda(z)} \simeq \frac{1}{\lambda_0} - \frac{M^2 g \lambda_0}{h^2} z.$$

At this accuracy we use the straight segments shown in the figure. For fixed endpoints, the extra geometrical length caused by a small gravitational deflection is second order in that deflection.

The two sloping segments OU and DR both have length ℓ , and their heights vary identically from 0 to H . Thus their optical path lengths are equal:

$$J = \int_0^\ell \frac{ds}{\lambda(Hs/\ell)}.$$

The remaining segments are at constant heights: OD is at zero and UR is at H . Consequently,

$$N_{\text{lower}} = \frac{\ell}{\lambda_0} + J, \quad N_{\text{upper}} = J + \frac{\ell}{\lambda(H)}.$$

Subtracting,

$$\begin{aligned} \Delta N_{\text{opt}} &= \ell \left(\frac{1}{\lambda_0} - \frac{1}{\lambda(H)} \right) \\ &= \frac{\ell}{\lambda_0} \left[1 - \sqrt{1 - \frac{2M^2 g \lambda_0^2 H}{h^2}} \right] \\ &\simeq \frac{M^2 g \lambda_0}{h^2} \ell H. \end{aligned}$$

Since

$$\ell H = \frac{a}{\cos \theta} 2a \sin \theta \sin \phi = 2a^2 \tan \theta \sin \phi,$$

the requested result is

$$\boxed{\Delta N_{\text{opt}} \simeq \frac{2M^2 g a^2 \lambda_0}{h^2} \tan \theta \sin \phi.}$$

For positive ϕ , the upper path has the smaller optical length because its elevated segment has a smaller neutron momentum and hence a longer wavelength. Reversing the subtraction convention reverses the sign without changing the interference predictions.

1.4 — Volume parameter

Using $A = 2a^2 \tan \theta$ and $V = h^2/(gM^2)$,

$$\boxed{\Delta N_{\text{opt}} \simeq \frac{A \lambda_0}{V} \sin \phi.}$$

The dimensions of V are

$$[V] = \frac{(\text{kg m}^2 \text{s}^{-1})^2}{(\text{m s}^{-2}) \text{kg}^2} = \text{m}^3.$$

Numerically,

$$V = \frac{(6.626 \times 10^{-34})^2}{9.800(1.675 \times 10^{-27})^2} = 1.5967876 \dots \times 10^{-14} \text{ m}^3.$$

Thus

$$\boxed{V = 1.597 \times 10^{-14} \text{ m}^3.}$$

1.5 — Number of intensity cycles

An optical path difference of one corresponds to a relative phase change of 2π :

$$\delta = 2\pi\Delta N_{\text{opt}}.$$

The given constructive interference at OUT1 when $\phi = 0$ fixes the constant phase offset. For the ideal equal-amplitude paths,

$$\frac{I_{\text{OUT1}}}{I_{\text{in}}} = \frac{1 + \cos \delta}{2} = \cos^2 \left(\pi \frac{A\lambda_0}{V} \sin \phi \right).$$

Maxima occur at integer ΔN_{opt} , and minima at half-integer values. Each increase of one in ΔN_{opt} therefore produces one intensity cycle.

As ϕ increases from -90° to 90° , $\sin \phi$ increases monotonically from -1 to 1 . Hence the total number of cycles is

$$\begin{aligned} C &= \Delta N_{\text{opt}}(90^\circ) - \Delta N_{\text{opt}}(-90^\circ) \\ &= \frac{A\lambda_0}{V} - \left(-\frac{A\lambda_0}{V} \right). \end{aligned}$$

Therefore, to the same first-order accuracy,

$$\boxed{C = \frac{2A\lambda_0}{V} = \frac{4M^2ga^2\lambda_0}{h^2} \tan \theta.}$$

1.6 — Incoming wavelength

The experimental area is

$$A = 2(0.03600 \text{ m})^2 \tan(22.10^\circ) = 1.052502 \dots \times 10^{-3} \text{ m}^2.$$

Solving the cycle formula for the wavelength,

$$\lambda_0 = \frac{CV}{2A}.$$

Using $C = 19.00$ and retaining unrounded intermediate values,

$$\lambda_0 = \frac{19.00(1.5967876 \dots \times 10^{-14} \text{ m}^3)}{2(1.052502 \dots \times 10^{-3} \text{ m}^2)} = 1.441278 \dots \times 10^{-10} \text{ m}.$$

Thus

$$\boxed{\lambda_0 = 1.441 \times 10^{-10} \text{ m} = 0.1441 \text{ nm}.}$$

The approximation used above is well justified. These neutrons have speed

$$v_0 = \frac{h}{M\lambda_0} \simeq 2.745 \times 10^3 \text{ m s}^{-1},$$

and the largest height is $H_{\max} = 2a \sin \theta \simeq 0.02709 \text{ m}$. Therefore,

$$\frac{MgH_{\max}}{K_0} = \frac{2gH_{\max}}{v_0^2} \simeq 7.05 \times 10^{-8} \ll 1.$$

1.7 — Area for the second experiment

Rearranging the same cycle formula,

$$A = \frac{CV}{2\lambda_0}.$$

With $C = 30.00$ and $\lambda_0 = 0.2000 \text{ nm} = 2.000 \times 10^{-10} \text{ m}$,

$$A = \frac{30.00(1.5967876 \dots \times 10^{-14} \text{ m}^3)}{2(2.000 \times 10^{-10} \text{ m})} = 1.1975907 \dots \times 10^{-3} \text{ m}^2.$$

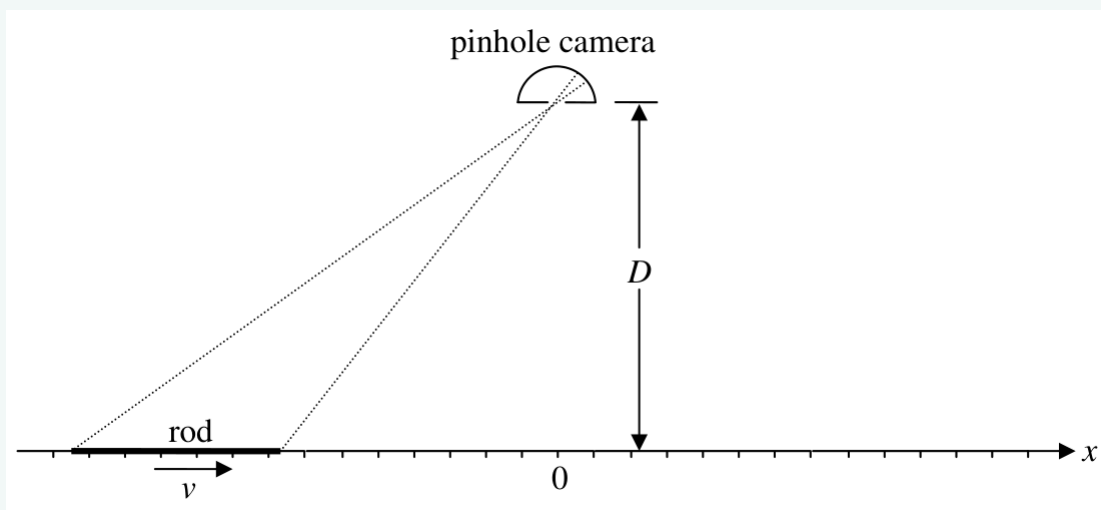
Hence

$$A = 1.198 \times 10^{-3} \text{ m}^2 = 11.98 \text{ cm}^2.$$

The solution was checked against the official solution.

Problem 2: Watching a Rod in Motion

Enter all your answers into the **Answer Script**.



Physical situation A pinhole camera, with the pinhole at $x = 0$ and at distance D from the x axis, takes pictures of a rod, by opening the pinhole for a very short time. There are equidistant marks along the x axis by which the *apparent length* of the rod, as it is seen on the picture, can be determined from the pictures taken by the pinhole camera. On a picture of the rod *at rest*, its length is L . However, the rod *is not* at rest, but is moving with constant velocity v along the x axis.

Basic relations A picture taken by the pinhole camera shows a tiny segment of the rod at position $\tilde{x}$.

2.1 (0.6) What is the *actual position* x of this segment at the time when the picture is taken? State your answer in terms of $\tilde{x}$, D , L , v , and the speed of light $c = 3.00 \times 10^8 \text{ m s}^{-1}$. Employ the quantities

$$\beta = \frac{v}{c} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

if they help to simplify your result.

2.2 (0.9) Find also the corresponding inverse relation, that is: express $\tilde{x}$ in terms of x , D , L , v , and c .

Note: The *actual position* is the position in the frame in which the camera is at rest

Apparent length of the rod The pinhole camera takes a picture at the instant when the actual position of the center of the rod is at some point x_0 .

2.3 (1.5) In terms of the given variables, determine the apparent length of the rod on this picture.

2.4 (1.5) Check one of the boxes in the **Answer Script** to indicate how the apparent length changes with time.

Symmetric picture One pinhole-camera picture shows both ends of the rod at the same distance from the pinhole.

2.5 (0.8) Determine the apparent length of the rod on this picture.

2.6 (1.0) What is the actual position of the middle of the rod at the time when this picture is taken?

2.7 (1.2) Where does the picture show the image of the middle of the rod?

Very early and very late pictures The pinhole camera took one picture very early, when the rod was very far away and approaching, and takes another picture very late, when the rod is very far away and receding. On one of the pictures the apparent length is 1.00 m, on the other picture it is 3.00 m.

2.8 (0.5) Check the box in the **Answer Script** to indicate which length is seen on which picture.

2.9 (1.0) Determine the velocity v .

2.10 (0.6) Determine the length L of the rod at rest.

2.11 (0.4) Infer the apparent length on the symmetric picture.

2.4	Check one: The apparent length ☐ increases first, reaches a maximum value, then decreases. ☐ decreases first, reaches a minimum value, then increases. ☐ decreases all the time. ☐ increases all the time.	1.5
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	Very early and very late pictures	For Examiners Use Only
2.8	Check one: ☐ The apparent length is 1 m on the early picture and 3 m on the late picture. ☐ The apparent length is 3 m on the early picture and 1 m on the late picture.	0.5

Official English text of the problem.

Answers

Use the abbreviations

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad a = \frac{L}{2\gamma}, \quad R = \sqrt{D^2 + a^2}.$$

- 2.1

$$x = \tilde{x} + \beta\sqrt{D^2 + \tilde{x}^2}.$$

- 2.2

$$\tilde{x} = \gamma^2 \left(x - \beta\sqrt{x^2 + \frac{D^2}{\gamma^2}} \right).$$

- 2.3

$$\tilde{L}(x_0) = \gamma L - \beta\gamma^2 \left[\sqrt{(x_0 + a)^2 + \frac{D^2}{\gamma^2}} - \sqrt{(x_0 - a)^2 + \frac{D^2}{\gamma^2}} \right].$$

- 2.4 Select the **third box**: the apparent length **decreases all the time**.

- 2.5

$$\tilde{L}_{\text{sym}} = \frac{L}{\gamma}.$$

- 2.6

$$x_{0,\text{sym}} = \beta R = \beta\sqrt{D^2 + \frac{L^2}{4\gamma^2}}.$$

- 2.7

$$\tilde{x}_{\text{mid}} = \beta\gamma^2 \left(R - \sqrt{D^2 + \beta^2 a^2} \right) = \frac{\beta a^2}{R + \sqrt{D^2 + \beta^2 a^2}}.$$

It lies to the right of the image's geometric center, toward the leading end.

- 2.8 Select the **second box**: 3.00 m **early**, 1.00 m **late**.

- 2.9

$$v = 0.500c = 1.50 \times 10^8 \text{ m s}^{-1}$$

in the positive x direction.

- 2.10

$$L = \sqrt{3} \text{ m} = 1.73 \text{ m}.$$

$$\tilde{L}_{\text{sym}} = 1.50 \text{ m.}$$

Solution

Work throughout in the camera's rest frame, with the rod moving in the positive x direction. The exposure is sufficiently short that all light forming one photograph arrives at the same time t . Light propagates along straight lines at speed c .

The stationary scale identifies the coordinate from which a ray was emitted. Thus $\tilde{x}$ is the segment's position at emission, whereas x is its position when the light reaches the pinhole.

2.1

Suppose the light was emitted at time t_e . Its distance to the pinhole is

$$r = \sqrt{D^2 + \tilde{x}^2},$$

so its travel time is

$$t - t_e = \frac{r}{c}.$$

During this interval the segment moves through $v(t - t_e)$. Therefore

$$x = \tilde{x} + v(t - t_e) = \tilde{x} + \frac{v}{c} \sqrt{D^2 + \tilde{x}^2},$$

giving

$$x = \tilde{x} + \beta \sqrt{D^2 + \tilde{x}^2}.$$

The relation concerns one segment and is independent of L .

2.2

Rearranging and squaring the previous result gives

$$(x - \tilde{x})^2 = \beta^2 (D^2 + \tilde{x}^2),$$

or

$$(1 - \beta^2) \tilde{x}^2 - 2x\tilde{x} + x^2 - \beta^2 D^2 = 0.$$

The quadratic roots are

$$\tilde{x} = \gamma^2 \left(x \pm \beta \sqrt{x^2 + \frac{D^2}{\gamma^2}} \right).$$

For positive velocity, the original, unsquared equation requires $x - \tilde{x} > 0$: the segment advances between emission and reception. This selects the minus sign. Hence define

$$\tilde{x} = f(x) = \gamma^2 \left(x - \beta \sqrt{x^2 + \frac{D^2}{\gamma^2}} \right).$$

2.3

The given L is the rod's rest length. Special relativity therefore gives its simultaneous length in the camera frame as

$$\ell = \frac{L}{\gamma}.$$

At the instant of the photograph, its endpoint positions are

$$x_- = x_0 - a, \quad x_+ = x_0 + a, \quad a = \frac{\ell}{2} = \frac{L}{2\gamma}.$$

Their apparent positions are $f(x_-)$ and $f(x_+)$. Since

$$f'(x) = \gamma^2 \left(1 - \frac{\beta x}{\sqrt{x^2 + D^2/\gamma^2}} \right) > 0,$$

the endpoint order is preserved, and the apparent length is

$$\tilde{L}(x_0) = f(x_0 + a) - f(x_0 - a).$$

Substitution yields

$$\tilde{L}(x_0) = \gamma L - \beta \gamma^2 \left[\sqrt{(x_0 + a)^2 + \frac{D^2}{\gamma^2}} - \sqrt{(x_0 - a)^2 + \frac{D^2}{\gamma^2}} \right].$$

This includes both Lorentz contraction and the different light travel times from the endpoints.

2.4

Differentiate $f'(x)$:

$$f''(x) = -\frac{\beta D^2}{(x^2 + D^2/\gamma^2)^{3/2}} < 0.$$

Thus $f'(x)$ strictly decreases with x . Consequently,

$$\frac{d\tilde{L}}{dx_0} = f'(x_0 + a) - f'(x_0 - a) < 0.$$

Since $dx_0/dt = v > 0$,

$$\frac{d\tilde{L}}{dt} < 0.$$

Therefore the apparent length **decreases all the time**. Select the **third box**.

2.5

In the symmetric photograph the two apparent endpoint positions are $-b$ and $+b$, because their distances from the pinhole are equal:

$$r_- = r_+ = \sqrt{D^2 + b^2}.$$

Equal travel distances and a common arrival time imply equal emission times. The endpoint separation at emission is therefore the simultaneous, Lorentz-contracted length:

$$2b = \frac{L}{\gamma}.$$

Thus $b = a$, and

$$\tilde{L}_{\text{sym}} = 2a = \frac{L}{\gamma}.$$

2.6

At the common endpoint emission time, the endpoints are at $-a$ and $+a$, so the rod's actual middle is at $x = 0$.

Their light takes time

$$\Delta t = \frac{R}{c}, \quad R = \sqrt{D^2 + a^2},$$

to reach the pinhole. During this interval the middle advances through $v\Delta t$. Its actual position when the photograph is taken is therefore

$$x_{0,\text{sym}} = \beta R = \beta \sqrt{D^2 + \frac{L^2}{4\gamma^2}}.$$

2.7

Apply the inverse relation to the actual position of the rod's middle:

$$\tilde{x}_{\text{mid}} = f(\beta R) = \beta \gamma^2 \left[R - \sqrt{\beta^2 R^2 + \frac{D^2}{\gamma^2}} \right].$$

Using $R^2 = D^2 + a^2$ and $1/\gamma^2 = 1 - \beta^2$,

$$\beta^2 R^2 + \frac{D^2}{\gamma^2} = D^2 + \beta^2 a^2.$$

Hence

$$\tilde{x}_{\text{mid}} = \beta \gamma^2 \left[R - \sqrt{D^2 + \beta^2 a^2} \right].$$

Rationalizing gives the equivalent expression

$$\tilde{x}_{\text{mid}} = \frac{\beta a^2}{\sqrt{D^2 + a^2} + \sqrt{D^2 + \beta^2 a^2}}.$$

For the stated motion, $0 < \tilde{x}_{\text{mid}} < a$. Thus the material midpoint appears between the image's geometric center and its right endpoint.

Its light travels a shorter distance than the endpoint light, so it is emitted later. By that later emission time, the material midpoint has moved to positive x .

2.8

For the very early and very late photographs, use the far-distance limit

$$\sqrt{D^2 + \tilde{x}^2} \simeq |\tilde{x}|.$$

In the early photograph the rod approaches from negative x . Equation 2.1 becomes

$$x \simeq (1 - \beta)\tilde{x}.$$

Applying this to both endpoints gives the early apparent length

$$A \equiv \tilde{L}_{\text{early}} = \frac{\ell}{1 - \beta} = \frac{L}{\gamma(1 - \beta)}.$$

In the late photograph the rod recedes toward positive x , so

$$x \simeq (1 + \beta)\tilde{x},$$

and

$$B \equiv \tilde{L}_{\text{late}} = \frac{\ell}{1 + \beta} = \frac{L}{\gamma(1 + \beta)}.$$

Because $\beta > 0$, $A > B$. Therefore

$$A = 3.00 \text{ m}, \quad B = 1.00 \text{ m}.$$

Select the **second box**.

2.9

Taking the ratio eliminates the rod's length:

$$\frac{A}{B} = \frac{1 + \beta}{1 - \beta}.$$

Solving,

$$\beta = \frac{A - B}{A + B} = \frac{3.00 - 1.00}{3.00 + 1.00} = 0.500.$$

Thus

$$v = \beta c = 0.500(3.00 \times 10^8) = 1.50 \times 10^8 \text{ m s}^{-1},$$

directed along positive x .

2.10

Multiplying the early and late apparent lengths gives

$$AB = \frac{L^2}{\gamma^2(1 - \beta)(1 + \beta)} = \frac{L^2}{\gamma^2(1 - \beta^2)} = L^2.$$

Therefore

$$L = \sqrt{AB} = \sqrt{(3.00 \text{ m})(1.00 \text{ m})} = 1.73 \text{ m}.$$

2.11

From part 2.5,

$$\tilde{L}_{\text{sym}} = \frac{L}{\gamma}.$$

With $\beta = 1/2$,

$$\gamma = \frac{1}{\sqrt{1 - 1/4}} = \frac{2}{\sqrt{3}}.$$

Using the unrounded rest length $L = \sqrt{3} \text{ m}$,

$$\tilde{L}_{\text{sym}} = \frac{\sqrt{3}}{2/\sqrt{3}} \text{ m} = 1.50 \text{ m}.$$

The solution was checked against the official solution.

Problem 3: Five order-of-magnitude estimates: Digital Camera, Hard-boiled Egg, Lightning, Capillary Vessels, Skyscraper

This question consists of five independent parts. Each of them asks for an estimate of an order of magnitude only, not for a precise answer. Enter all your answers into the **Answer Script**.

Digital Camera Consider a digital camera with a square CCD chip with linear dimension $L = 35 \text{ mm}$ having $N_p = 5 \text{ Mpix}$ ($1 \text{ Mpix} = 10^6 \text{ pixels}$). The lens of this camera has a focal length of $f = 38 \text{ mm}$. The well known sequence of numbers (2, 2.8, 4, 5.6, 8, 11, 16, 22) that appear on the lens refer to the so called F-number, which is denoted by $F\#$ and defined as the ratio of the focal length and the diameter D of the aperture of the lens, $F\# = f/D$.

3.1 (1.0) Find the best possible spatial resolution $\Delta x_{\min}$, at the chip, of the camera as limited by the lens. Express your result in terms of the wavelength λ and the F-number $F\#$ and give the numerical value for $\lambda = 500 \text{ nm}$.

3.2 (0.5) Find the necessary number N of Mpix that the CCD chip should possess in order to match this optimal resolution.

3.3 (0.5) Sometimes, photographers try to use a camera at the smallest practical aperture. Suppose that we now have a camera of $N_0 = 16 \text{ Mpix}$, with the chip size and focal length as given above. Which value is to be chosen for $F\#$ such that the image quality is not limited by the optics?

3.4 (0.5) Knowing that the human eye has an approximate angular resolution of $\phi = 2 \text{ arcmin}$ and that a typical photo printer will print a minimum of 300 dpi (dots per inch), at what minimal distance z should you hold the printed page from your eyes so that you do not see the individual dots?

Data

1 inch	= 25.4 mm
1 arcmin	= $2.91 \times 10^{-4} \text{ rad}$

Hard-boiled egg An egg, taken directly from the fridge at temperature $T_0 = 4^\circ\text{C}$, is dropped into a pot with water that is kept boiling at temperature T_1 .

3.5 (0.5) How large is the amount of energy U that is needed to get the egg coagulated?

3.6 (0.5) How large is the heat flow J that is flowing into the egg?

3.7 (0.5) How large is the heat power P transferred to the egg?

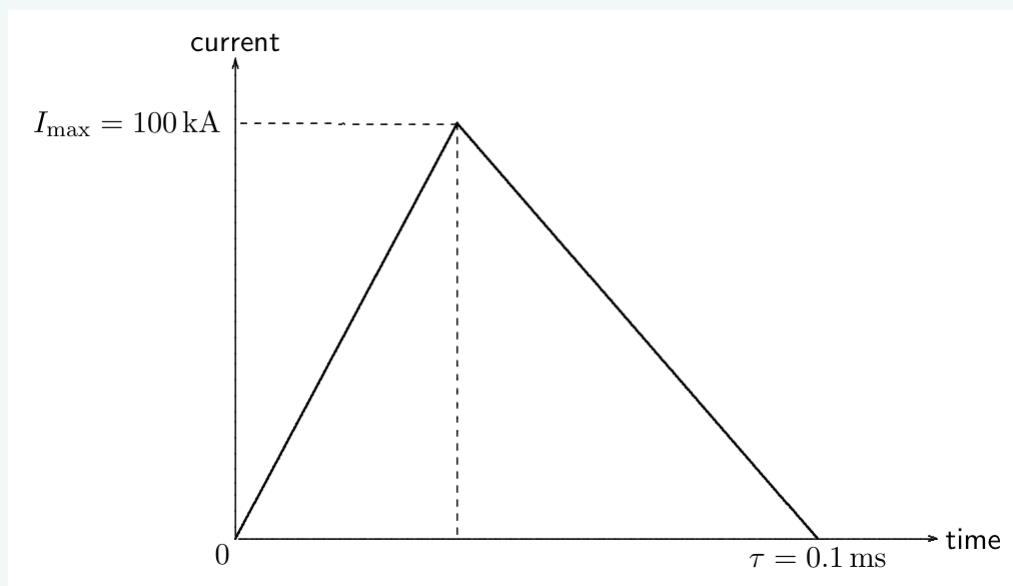
3.8 (0.5) For how long do you need to cook the egg so that it is hard-boiled?

Hint You may use the simplified form of Fourier's Law $J = \kappa \Delta T / \Delta r$, where ΔT is the temperature difference associated with Δr , the typical length scale of the problem. The heat flow J is in units of W m^{-2} .

Data

Mass density of the egg:	$\mu = 10^3 \text{ kg m}^{-3}$
Specific heat capacity of the egg:	$C = 4.2 \text{ J K}^{-1} \text{ g}^{-1}$
Radius of the egg:	$R = 2.5 \text{ cm}$
Coagulation temperature of albumen (egg protein):	$T_c = 65^\circ\text{C}$
Heat transport coefficient:	$\kappa = 0.64 \text{ W K}^{-1} \text{ m}^{-1}$ (assumed to be the same for liquid and solid albumen)

Lightning An oversimplified model of lightning is presented. Lightning is caused by the build-up of electrostatic charge in clouds. As a consequence, the bottom of the cloud usually gets positively charged and the top gets negatively charged, and the ground below the cloud gets negatively charged. When the corresponding electric field exceeds the breakdown strength value of air, a disruptive discharge occurs: this is lightning.



Idealized current pulse flowing between the cloud and the ground during lightning.

Answer the following questions with the aid of this simplified curve for the current as a function of time and these data:

Distance between the bottom of the cloud and the ground:	$h = 1 \text{ km};$
Breakdown electric field of humid air:	$E_0 = 300 \text{ kV m}^{-1};$
Total number of lightning striking Earth per year:	$32 \times 10^6;$
Total human population:	$6.5 \times 10^9 \text{ people.}$

3.9 (0.5) What is the total charge Q released by lightning?

3.10 (0.5) What is the average current I flowing between the bottom of the cloud and the ground during lightning?

3.11 (1.0) Imagine that the energy of all storms of one year is collected and equally shared among all people. For how long could you continuously light up a 100 W light bulb for your share?

Capillary Vessels Let us regard blood as an incompressible viscous fluid with mass density μ similar to that of water and dynamic viscosity $\eta = 4.5 \text{ g m}^{-1} \text{ s}^{-1}$. We model blood vessels as circular straight pipes with radius r and length L and describe the blood flow by Poiseuille's law,

$$\Delta p = RD,$$

the Fluid Dynamics analog of Ohm's law in Electricity. Here Δp is the pressure difference between the entrance and the exit of the blood vessel, $D = Sv$ is the volume flow through the cross-sectional area S of the blood vessel and v is the blood velocity. The hydraulic resistance R is given by

$$R = \frac{8\eta L}{\pi r^4}.$$

For the systemic blood circulation (the one flowing from the left ventricle to the right auricle of the heart), the blood flow is $D \approx 100 \text{ cm}^3 \text{ s}^{-1}$ for a man at rest. Answer the following questions under the assumption that all capillary vessels are connected in parallel and that each of them has radius $r = 4 \text{ }\mu\text{m}$ and length $L = 1 \text{ mm}$ and operates under a pressure difference $\Delta p = 1 \text{ kPa}$.

3.12 (1.0) How many capillary vessels are in the human body?

3.13 (0.5) How large is the velocity v with which blood is flowing through a capillary vessel?

Skyscraper At the bottom of a 1000 m high skyscraper, the outside temperature is $T_{\text{bot}} = 30^\circ\text{C}$. The objective is to estimate the outside temperature T_{top} at the top. Consider a thin slab of air (ideal nitrogen gas with adiabatic coefficient $\gamma = 7/5$) rising slowly to height z where the pressure is lower, and assume that this slab expands adiabatically so that its temperature drops to the temperature of the surrounding air.

3.14 (0.5) How is the fractional change in temperature dT/T related to dp/p , the fractional change in pressure?

3.15 (0.5) Express the pressure difference dp in terms of dz , the change in height.

3.16 (1.0) What is the resulting temperature at the top of the building?

Data

Boltzmann constant:	$k = 1.38 \times 10^{-23} \text{ J K}^{-1}$
Mass of a nitrogen molecule:	$m = 4.65 \times 10^{-26} \text{ kg}$
Gravitational acceleration:	$g = 9.80 \text{ m s}^{-2}$

Official English text of the problem.

Answers

The main estimates use the diffraction scale $\lambda F\#$, boiling water at 100°C , and $E_0 h$ as the characteristic lightning voltage.

- **3.1:** $\Delta x_{\min} \sim \lambda F\#$. At the largest available aperture, $F\# = 2$, this is $1.0 \mu\text{m}$. The circular-aperture Rayleigh value is $1.22 \mu\text{m}$.
- **3.2:** $N \sim 1.2 \times 10^3 \text{ Mpix}$, or $8.2 \times 10^2 \text{ Mpix}$ with the Rayleigh coefficient.
- **3.3:** $F\# = 16$ using $\Delta x \sim \lambda F\#$. A strict Rayleigh comparison gives $F\# = 11$.
- **3.4:** $z_{\min} \approx 0.15 \text{ m} = 15 \text{ cm}$.
- **3.5:** $U \approx 1.7 \times 10^4 \text{ J}$.
- **3.6:** $J \sim 2.5 \times 10^3 \text{ W m}^{-2}$.
- **3.7:** $P \sim 19 \text{ W}$.
- **3.8:** $t \sim 8.7 \times 10^2 \text{ s} \approx 15 \text{ min}$.
- **3.9:** $Q = 5 \text{ C}$.
- **3.10:** $I = 5.0 \times 10^4 \text{ A} = 50 \text{ kA}$.
- **3.11:** $t \sim 7.4 \times 10^4 \text{ s} \approx 20 \text{ h}$. A completely discharging capacitor model gives approximately 10 h.
- **3.12:** $N_{\text{cap}} \approx 4.5 \times 10^9$ capillary vessels.
- **3.13:** $v \approx 4.4 \times 10^{-4} \text{ m s}^{-1} = 0.44 \text{ mm s}^{-1}$.
- **3.14:** $\frac{dT}{T} = \frac{\gamma - 1}{\gamma} \frac{dp}{p} = \frac{2}{7} \frac{dp}{p}$.
- **3.15:** $dp = -\rho g dz = -\frac{mpg}{kT} dz$, where ρ is the local air density.
- **3.16:** $T_{\text{top}} \approx 294 \text{ K} \approx 21^\circ\text{C}$.

Solution

3.1. For an ideal lens, diffraction at the aperture limits the angular separation of distinguishable objects to the scale

$$\theta_{\min} \sim \frac{\lambda}{D}.$$

For distant objects, an angular separation θ corresponds to a separation $f\theta$ in the focal plane. Thus

$$\Delta x_{\min} \sim f \frac{\lambda}{D} = \lambda F\#.$$

At $\lambda = 500 \text{ nm}$,

$$\Delta x_{\min} \sim 0.50(F\#) \mu\text{m}.$$

The best resolution occurs at the largest aperture, corresponding to the smallest listed F-number, $F\# = 2$. Its diameter is $D = f/2 = 19 \text{ mm}$, giving

$$\Delta x_{\min} \sim 1.0 \mu\text{m}.$$

For a circular aperture, the Rayleigh criterion gives the more specific expression

$$\Delta x_{\text{Rayleigh}} = 1.22\lambda F\#,$$

and hence $1.22 \mu\text{m}$ at $F\# = 2$. The coefficient is not essential to the requested order-of-magnitude estimate.

3.2. For a square chip with square pixels of pitch a , the total number of pixels is approximately $(L/a)^2$. Matching the pixel pitch to the optical resolution scale gives

$$N_{\text{pixels}} \sim \left(\frac{L}{\Delta x_{\min}} \right)^2.$$

Therefore,

$$N_{\text{pixels}} \sim \left(\frac{35 \times 10^{-3}}{1.0 \times 10^{-6}} \right)^2 = 1.225 \times 10^9.$$

In megapixels,

$$\boxed{N \sim 1.2 \times 10^3 \text{ Mpix.}}$$

Using the Rayleigh value instead gives

$$N_{\text{Rayleigh}} = \frac{1}{10^6} \left(\frac{35 \times 10^{-3}}{1.22 \times 10^{-6}} \right)^2 \approx 8.2 \times 10^2 \text{ Mpix.}$$

Both estimates require about 10^3 Mpix.

3.3. The pixel pitch of the 16 Mpix chip is

$$a = \frac{L}{\sqrt{16 \times 10^6}} = \frac{35 \text{ mm}}{4000} = 8.75 \mu\text{m}.$$

For the optics not to set a coarser resolution than the pixels, require

$$\lambda F\# \lesssim a.$$

Thus

$$F\# \lesssim \frac{8.75 \mu\text{m}}{0.500 \mu\text{m}} = 17.5.$$

The smallest aperture corresponds to the largest allowed F-number. Among the available settings, this gives

$$\boxed{F\# = 16.}$$

At this setting the estimated diffraction scale is $8.0 \mu\text{m}$; at $F\# = 22$, it would be $11 \mu\text{m}$.

If the Rayleigh coefficient is retained and the inequality is enforced strictly,

$$1.22\lambda F\# \leq a \implies F\# \leq 14.3.$$

The largest listed setting then becomes $F\# = 11$. This discrete choice depends on the resolution convention.

3.4. At 300 dpi, adjacent printed dots have separation

$$d = \frac{25.4 \text{ mm}}{300} = 8.47 \times 10^{-5} \text{ m}.$$

At viewing distance z , their angular separation is approximately d/z . They cease to be individually resolved when

$$\frac{d}{z} \lesssim \phi.$$

Using

$$\phi = 2(2.91 \times 10^{-4}) = 5.82 \times 10^{-4} \text{ rad},$$

we obtain

$$z_{\min} \approx \frac{d}{\phi} = \frac{8.47 \times 10^{-5}}{5.82 \times 10^{-4}} \approx 0.15 \text{ m}.$$

3.5. Treat the egg as a sphere with uniform density and specific heat capacity. Neglect the shell's thermal effects and any additional enthalpy of coagulation, for which no data are supplied.

The egg's mass is

$$M = \frac{4}{3}\pi R^3 \mu = \frac{4}{3}\pi (2.5 \times 10^{-2})^3 (10^3) = 6.54 \times 10^{-2} \text{ kg}.$$

Convert the specific heat capacity:

$$C = 4.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}.$$

The energy needed to raise the egg from T_0 to the coagulation temperature is estimated by

$$U \approx MC(T_c - T_0).$$

Hence

$$U \approx (6.54 \times 10^{-2})(4.2 \times 10^3)(65 - 4) = 1.68 \times 10^4 \text{ J},$$

so

$$U \approx 1.7 \times 10^4 \text{ J}.$$

3.6. Assume ordinary atmospheric boiling, so $T_1 \approx 100^\circ\text{C}$. For a characteristic heat flux, take the conduction length to be the egg radius and the temperature contrast to be the initial water-to-egg contrast:

$$\Delta r \sim R, \quad \Delta T \sim T_1 - T_0 = 96 \text{ K}.$$

Fourier's law then gives the inward heat-flux magnitude

$$J \sim \kappa \frac{T_1 - T_0}{R} = 0.64 \frac{96}{2.5 \times 10^{-2}}.$$

Therefore,

$$J \sim 2.5 \times 10^3 \text{ W m}^{-2}.$$

The actual temperature gradient changes during cooking; this is its characteristic scale.

3.7. Heat enters through the egg's surface, whose area is

$$A = 4\pi R^2 = 7.85 \times 10^{-3} \text{ m}^2.$$

Consequently,

$$P \sim JA = (2.46 \times 10^3) (7.85 \times 10^{-3}) \approx 19.3 \text{ W}.$$

Thus

$$P \sim 19 \text{ W}.$$

3.8. Divide the required heating energy by the characteristic heat-transfer power:

$$t \sim \frac{U}{P}.$$

Combining the previous expressions also gives

$$t \sim \frac{\frac{4}{3}\pi R^3 \mu C (T_c - T_0)}{4\pi R^2 \kappa (T_1 - T_0)/R} = \frac{\mu C R^2}{3\kappa} \frac{T_c - T_0}{T_1 - T_0}.$$

Numerically,

$$t \sim \frac{1.68 \times 10^4}{19.3} \approx 8.7 \times 10^2 \text{ s} \approx 15 \text{ min.}$$

This estimate has the expected thermal-diffusion scaling $t \propto R^2$.

3.9. The transferred charge is the area under the current–time graph:

$$Q = \int_0^\tau I(t) dt.$$

The pulse is triangular, so its area is independent of when the peak occurs:

$$Q = \frac{1}{2} I_{\max} \tau.$$

With $I_{\max} = 100 \text{ kA} = 10^5 \text{ A}$ and $\tau = 0.1 \text{ ms} = 10^{-4} \text{ s}$,

$$Q = \frac{1}{2} (10^5) (10^{-4}) = 5 \text{ C.}$$

3.10. The average current over the duration of the pulse is

$$I = \frac{Q}{\tau} = \frac{I_{\max}}{2}.$$

Therefore,

$$I = 5.0 \times 10^4 \text{ A} = 50 \text{ kA.}$$

3.11. Approximating the electric field between cloud and ground as uniform, the voltage at breakdown is

$$V_0 \approx E_0 h = (300 \times 10^3)(10^3) = 3.0 \times 10^8 \text{ V.}$$

The electrical energy released is

$$W = \int V(t) I(t) dt.$$

Using V_0 as its characteristic voltage gives

$$W \sim V_0 Q = (3.0 \times 10^8)(5) = 1.5 \times 10^9 \text{ J}$$

per lightning strike.

The annual energy share per person is therefore

$$W_{\text{person}} \sim \frac{32 \times 10^6}{6.5 \times 10^9} (1.5 \times 10^9) \approx 7.4 \times 10^6 \text{ J.}$$

A 100 W bulb consumes 100 J each second, so

$$t \sim \frac{W_{\text{person}}}{100} \approx 7.4 \times 10^4 \text{ s} \approx 20 \text{ h.}$$

For a cloud–ground system modeled as a capacitor that discharges completely, the voltage falls with the remaining charge. Its released energy is then

$$W = \frac{1}{2} Q V_0,$$

giving approximately $3.7 \times 10^4 \text{ s}$, or 10 h, per person. Both models give a lighting time of order one day.

3.12. The relevant SI quantities are

$$\eta = 4.5 \times 10^{-3} \text{ Pa s}, \quad r = 4 \times 10^{-6} \text{ m}, \quad L = 10^{-3} \text{ m}, \\ \Delta p = 10^3 \text{ Pa}, \quad D_{\text{total}} = 100 \text{ cm}^3 \text{ s}^{-1} = 10^{-4} \text{ m}^3 \text{ s}^{-1}.$$

Poiseuille's law gives the volume flow through one capillary:

$$D_{\text{cap}} = \frac{\Delta p}{R} = \frac{\pi r^4 \Delta p}{8 \eta L}.$$

Substitution yields

$$D_{\text{cap}} = \frac{\pi(4 \times 10^{-6})^4(10^3)}{8(4.5 \times 10^{-3})(10^{-3})} \approx 2.23 \times 10^{-14} \text{ m}^3 \text{ s}^{-1}.$$

Since the identical capillaries are connected in parallel, each experiences the same pressure difference, and their volume flows add:

$$D_{\text{total}} = N_{\text{cap}} D_{\text{cap}}.$$

Thus

$$N_{\text{cap}} = \frac{10^{-4}}{2.23 \times 10^{-14}} \approx 4.5 \times 10^9.$$

3.13. The velocity appearing in $D = S v$ is the mean velocity over the cross-section. Therefore,

$$v = \frac{D_{\text{cap}}}{\pi r^2} = \frac{\Delta p r^2}{8 \eta L}.$$

Numerically,

$$v = \frac{(10^3)(4 \times 10^{-6})^2}{8(4.5 \times 10^{-3})(10^{-3})}.$$

Hence

$$v \approx 4.4 \times 10^{-4} \text{ m s}^{-1} = 0.44 \text{ mm s}^{-1}.$$

3.14. For a fixed quantity of ideal gas undergoing a quasistatic adiabatic expansion,

$$p V^\gamma = \text{constant}.$$

Taking logarithmic differentials,

$$\frac{dp}{p} + \gamma \frac{dV}{V} = 0.$$

The ideal-gas law gives $T \propto p V$, with T measured in kelvin, so

$$\frac{dT}{T} = \frac{dp}{p} + \frac{dV}{V} = \left(1 - \frac{1}{\gamma}\right) \frac{dp}{p}.$$

Thus

$$\boxed{\frac{dT}{T} = \frac{\gamma - 1}{\gamma} \frac{dp}{p} = \frac{2}{7} \frac{dp}{p}.$$

Pressure and temperature both decrease as the slab rises.

3.15. Hydrostatic force balance on a thin horizontal layer gives

$$dp = -\rho g dz,$$

where ρ is the local mass density.

For nitrogen molecules of mass m , the ideal-gas law gives

$$p = \frac{\rho}{m} kT, \quad \rho = \frac{mp}{kT}.$$

Therefore,

$$\boxed{dp = -\rho g dz = -\frac{mpg}{kT} dz.}$$

3.16. Substitute the hydrostatic pressure gradient into the adiabatic temperature relation:

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma} \left(-\frac{mg}{kT} dz \right).$$

The temperature cancels from the right-hand side after multiplication by T , giving a constant lapse rate:

$$\frac{dT}{dz} = -\frac{\gamma - 1}{\gamma} \frac{mg}{k}.$$

For the supplied constants,

$$\frac{dT}{dz} = -\frac{2}{7} \frac{(4.65 \times 10^{-26})(9.80)}{1.38 \times 10^{-23}} \approx -9.43 \times 10^{-3} \text{ K m}^{-1}.$$

Over $H = 1000$ m, the temperature drop is approximately 9.4 K. Hence

$$T_{\text{top}} = T_{\text{bot}} - \frac{\gamma - 1}{\gamma} \frac{mgH}{k} \approx 303.15 - 9.43 = 293.72 \text{ K}.$$

Therefore,

$$\boxed{T_{\text{top}} \approx 294 \text{ K} \approx 21^\circ \text{C}.$$

The solution was checked against the official solution.