

IPhO 2005

Theory problems and solutions · Salamanca, Spain · 3 theory problems with complete solutions

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Problem 1: An Ill Fated Satellite

Th 1 AN ILL FATED SATELLITE

The most frequent orbital manoeuvres performed by spacecraft consist of velocity variations along the direction of flight, namely accelerations to reach higher orbits or brakings done to initiate re-entering in the atmosphere. In this problem we will study the orbital variations when the engine thrust is applied in a radial direction.

To obtain numerical values use: Earth radius $R_T = 6.37 \cdot 10^6 \text{ m}$, Earth surface gravity $g = 9.81 \text{ m/s}^2$, and take the length of the sidereal day to be $T_0 = 24.0 \text{ h}$.

We consider a geosynchronous¹ communications satellite of mass m placed in an equatorial circular orbit of radius r_0 . These satellites have an "apogee engine" which provides the tangential thrusts needed to reach the final orbit.

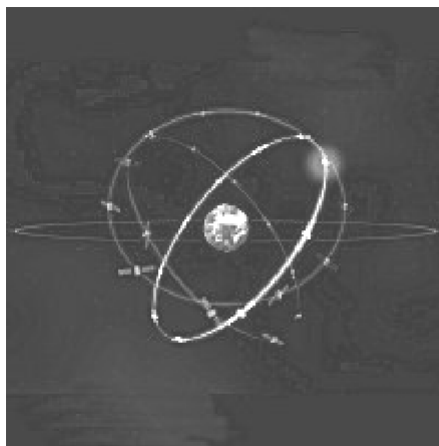


Image: ESA

Image: ESA

Marks are indicated at the beginning of each subquestion, in parenthesis.

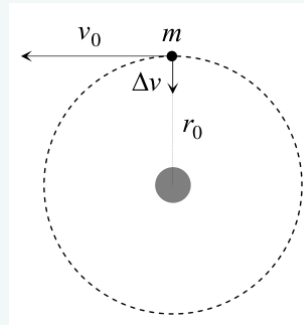
Question 1

1.1 (0.3) Compute the numerical value of r_0 .

1.2 (0.3+0.1) Give the analytical expression of the velocity v_0 of the satellite as a function of g , R_T , and r_0 , and calculate its numerical value.

1.3 (0.4+0.4) Obtain the expressions of its angular momentum L_0 and mechanical energy E_0 , as functions of v_0 , m , g and R_T .

Once this geosynchronous circular orbit has been reached (see Figure F-1), the satellite has been stabilised in the desired location, and is being readied to do its work, an error by the ground controllers causes the apogee engine to be fired again. The thrust happens to be directed towards the Earth and, despite the quick reaction of the ground crew to shut the engine off, an unwanted velocity variation Δv is imparted on the satellite. We characterize this boost by the parameter $\beta = \Delta v/v_0$. The duration of the engine burn is always negligible with respect to any other orbital times, so that it can be considered as instantaneous.



F-1

Question 2

Suppose $\beta < 1$.

2.1 (0.4+0.5) Determine the parameters of the new orbit², *semi-latus-rectum* l and *eccentricity* ε , in terms of r_0 and β .

2.2 (1.0) Calculate the angle α between the major axis of the new orbit and the position vector at the accidental misfire.

2.3 (1.0+0.2) Give the analytical expressions of the perigee $r_{\min}$ and apogee $r_{\max}$ distances to the Earth centre, as functions of r_0 and β , and calculate their numerical values for $\beta = 1/4$.

2.4 (0.5+0.2) Determine the period of the new orbit, T , as a function of T_0 and β , and calculate its numerical value for $\beta = 1/4$.

¹ Its revolution period is T_0 .

² See the "hint".

Question 3

3.1 (0.5) Calculate the minimum boost parameter, β_{esc} , needed for the satellite to escape Earth gravity.

3.2 (1.0) Determine in this case the closest approach of the satellite to the Earth centre in the new trajectory, $r'_{\min}$, as a function of r_0 .

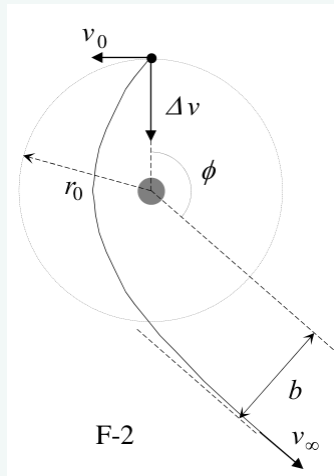
Question 4

Suppose $\beta > \beta_{\text{esc}}$.

4.1 (1.0) Determine the residual velocity at the infinity, v_{∞} , as a function of v_0 and β .

4.2 (1.0) Obtain the "impact parameter" b of the asymptotic escape direction in terms of r_0 and β . (See Figure F-2).

4.3 (1.0+0.2) Determine the angle ϕ of the asymptotic escape direction in terms of β . Calculate its numerical value for $\beta = \frac{3}{2}\beta_{\text{esc}}$.

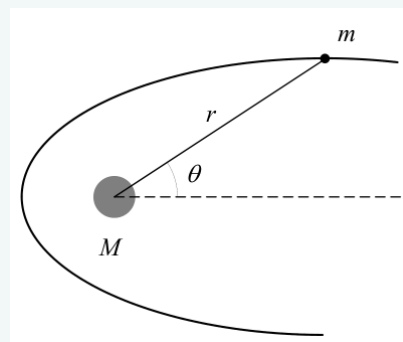


F-2

HINT

Under the action of central forces obeying the inverse-square law, bodies follow trajectories described by ellipses, parabolas or hyperbolas. In the approximation $m \ll M$ the gravitating mass M is at one of the foci. Taking the origin at this focus, the general polar equation of these curves can be written as (see Figure F-3)

$$r(\theta) = \frac{l}{1 - \varepsilon \cos \theta}$$



F-3

where l is a positive constant named the *semi-latus-rectum* and ε is the *eccentricity* of the curve. In terms of constants of motion:

$$l = \frac{L^2}{GMm^2} \quad \text{and} \quad \varepsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

where G is the Newton constant, L is the modulus of the angular momentum of the orbiting mass, with respect to the origin, and E is its mechanical energy, with zero potential energy at infinity.

We may have the following cases:

- i) If $0 \leq \varepsilon < 1$, the curve is an ellipse (circumference for $\varepsilon = 0$).
- ii) If $\varepsilon = 1$, the curve is a parabola.

iii) If $\varepsilon > 1$, the curve is a hyperbola.

Official English text of the problem.

Answers

• 1.1

$$r_0 = \left(\frac{gR_T^2 T_0^2}{4\pi^2} \right)^{1/3} = \boxed{4.22 \times 10^7 \text{ m}}.$$

• 1.2

$$v_0 = \sqrt{\frac{gR_T^2}{r_0}} = \boxed{3.07 \times 10^3 \text{ m s}^{-1}}.$$

• 1.3

$$\boxed{L_0 = \frac{mgR_T^2}{v_0}}, \quad \boxed{E_0 = -\frac{1}{2}mv_0^2}.$$

• 2.1

$$\boxed{l = r_0, \quad \varepsilon = \beta}.$$

• 2.2

$$\boxed{\alpha = \frac{\pi}{2} = 90^\circ}.$$

This applies for $0 < \beta < 1$; at $\beta = 0$, the orbit is circular and has no distinguished major axis.

• 2.3

$$\boxed{r_{\min} = \frac{r_0}{1 + \beta}}, \quad \boxed{r_{\max} = \frac{r_0}{1 - \beta}}.$$

For $\beta = 1/4$,

$$\boxed{r_{\min} = 3.38 \times 10^7 \text{ m}}, \quad \boxed{r_{\max} = 5.63 \times 10^7 \text{ m}}.$$

• 2.4

$$\boxed{T = \frac{T_0}{(1 - \beta^2)^{3/2}}}.$$

For $\beta = 1/4$, $\boxed{T = 26.4 \text{ h}}.$

• 3.1

$$\boxed{\beta_{\text{esc}} = 1}.$$

- 3.2

$$r'_{\min} = \frac{r_0}{2}.$$

- 4.1

$$v_{\infty} = v_0 \sqrt{\beta^2 - 1}.$$

- 4.2

$$b = \frac{r_0}{\sqrt{\beta^2 - 1}}.$$

- 4.3

$$\phi = \frac{\pi}{2} + \arccos\left(\frac{1}{\beta}\right) = \pi - \arcsin\left(\frac{1}{\beta}\right).$$

For $\beta = \frac{3}{2}\beta_{\text{esc}} = \frac{3}{2}$,

$$\phi = 138.2^\circ.$$

Solution

Let

$$\mu = GM = gR_T^2,$$

where M is Earth's mass. We use Newtonian gravity for a fixed, spherical Earth, with $m \ll M$, and neglect atmospheric drag and other bodies. The engine burn is treated as instantaneous, and the satellite's mass as constant. Gravitational potential energy is zero at infinity.

1.1 — Radius of the geosynchronous orbit

For a circular orbit, gravitational acceleration provides the centripetal acceleration:

$$\frac{\mu}{r_0^2} = \frac{v_0^2}{r_0}.$$

The orbital period is T_0 , so

$$v_0 = \frac{2\pi r_0}{T_0}.$$

Combining these equations gives

$$r_0^3 = \frac{\mu T_0^2}{4\pi^2}.$$

Thus

$$r_0 = \left(\frac{gR_T^2 T_0^2}{4\pi^2} \right)^{1/3}.$$

Using $T_0 = 24.0 \times 3600 \text{ s}$,

$$r_0 = \left[\frac{9.81(6.37 \times 10^6)^2(24.0 \times 3600)^2}{4\pi^2} \right]^{1/3} = 4.222197 \dots \times 10^7 \text{ m}.$$

Therefore,

$$r_0 = 4.22 \times 10^7 \text{ m}.$$

1.2 — Initial orbital velocity

The circular-orbit condition gives

$$v_0^2 = \frac{\mu}{r_0} = \frac{gR_T^2}{r_0}.$$

Hence

$$v_0 = \sqrt{\frac{gR_T^2}{r_0}}.$$

Numerically,

$$v_0 = 3070.47 \dots \text{ m s}^{-1} = 3.07 \times 10^3 \text{ m s}^{-1}.$$

1.3 — Initial angular momentum and mechanical energy

The velocity is perpendicular to the radius, so the angular momentum magnitude is

$$L_0 = mr_0 v_0.$$

Since $\mu/r_0 = v_0^2$,

$$L_0 = \frac{m\mu}{v_0} = \frac{mgR_T^2}{v_0}.$$

The mechanical energy is

$$E_0 = \frac{1}{2}mv_0^2 - \frac{\mu m}{r_0}.$$

Using $\mu/r_0 = v_0^2$,

$$E_0 = -\frac{1}{2}mv_0^2.$$

2.1 — Semi-latus-rectum and eccentricity after the impulse

Let $\hat{\mathbf{r}}$ point radially outward at the misfire and $\hat{\mathbf{t}}$ point along the original velocity. Immediately after the impulse,

$$\mathbf{v} = v_0 \hat{\mathbf{t}} - \beta v_0 \hat{\mathbf{r}}.$$

The two components are perpendicular, giving

$$v^2 = v_0^2(1 + \beta^2).$$

The impulse is radial and therefore has zero angular impulse about Earth's centre. Consequently,

$$L = L_0 = mr_0 v_0.$$

The radius remains r_0 during the instantaneous burn, so the new energy is

$$E = \frac{1}{2}mv_0^2(1 + \beta^2) - \frac{\mu m}{r_0} = \frac{1}{2}mv_0^2(\beta^2 - 1).$$

Thus $E < 0$ for $\beta < 1$, as required for a bound orbit.

From the supplied orbital formulas,

$$l = \frac{L^2}{\mu m^2} = \frac{r_0^2 v_0^2}{\mu} = \boxed{r_0},$$

and

$$\varepsilon^2 = 1 + \frac{2EL^2}{\mu^2 m^3} = 1 + (\beta^2 - 1) = \beta^2.$$

Because β is a nonnegative speed ratio,

$$\boxed{\varepsilon = \beta}.$$

These expressions for l and ε also hold for the parabolic and hyperbolic cases considered later.

2.2 — Orientation of the major axis

At the instant of the misfire, $r = r_0 = l$. Substitution into the polar equation gives

$$r_0 = \frac{r_0}{1 - \beta \cos \theta_0}.$$

For $\beta > 0$, this requires

$$\cos \theta_0 = 0.$$

The polar reference axis is the major-axis direction, so the position vector at the misfire is perpendicular to that axis:

$$\boxed{\alpha = \frac{\pi}{2} = 90^\circ}.$$

To identify the branch, choose θ to increase in the direction of orbital motion. Then

$$\frac{dr}{d\theta} = -\frac{r_0 \beta \sin \theta}{(1 - \beta \cos \theta)^2}.$$

The initial radial velocity is inward, so $dr/d\theta < 0$. Therefore,

$$\theta_0 = \frac{\pi}{2}.$$

Perigee occurs subsequently at $\theta = \pi$. In the orientation of Figure F-1, the major axis is horizontal and the perigee is to the left of Earth.

For $\beta = 0$, the orbit remains circular, and no unique major-axis direction exists.

2.3 — Perigee and apogee

For

$$r(\theta) = \frac{r_0}{1 - \beta \cos \theta},$$

the smallest radius occurs when the denominator is largest, at $\theta = \pi$. The largest radius occurs at $\theta = 0$. Hence

$$\boxed{r_{\min} = \frac{r_0}{1 + \beta}}, \quad \boxed{r_{\max} = \frac{r_0}{1 - \beta}}.$$

For $\beta = 1/4$,

$$r_{\min} = \frac{4}{5}r_0 = \boxed{3.38 \times 10^7 \text{ m}},$$

$$r_{\max} = \frac{4}{3}r_0 = \boxed{5.63 \times 10^7 \text{ m}}.$$

Both are distances from Earth's centre.

2.4 — Period of the new orbit

The ellipse's semi-major axis is

$$a = \frac{r_{\min} + r_{\max}}{2} = \frac{r_0}{2} \left(\frac{1}{1 + \beta} + \frac{1}{1 - \beta} \right) = \frac{r_0}{1 - \beta^2}.$$

Kepler's third law for orbits around the same central mass gives

$$\frac{T^2}{a^3} = \frac{T_0^2}{r_0^3}.$$

Therefore,

$$T = T_0 \left(\frac{a}{r_0} \right)^{3/2} = \boxed{\frac{T_0}{(1 - \beta^2)^{3/2}}}.$$

For $\beta = 1/4$,

$$T = 24.0 \left(\frac{16}{15} \right)^{3/2} \text{ h} = 26.4396 \dots \text{ h},$$

so

$$\boxed{T = 26.4 \text{ h}}.$$

3.1 — Minimum impulse for escape

A trajectory can reach infinity only if its mechanical energy is nonnegative. The limiting escape trajectory has $E = 0$:

$$\frac{1}{2}mv_0^2(\beta_{\text{esc}}^2 - 1) = 0.$$

Thus

$$\boxed{\beta_{\text{esc}} = 1}.$$

At this threshold, $\varepsilon = 1$, so the orbit is parabolic and the speed tends to zero at infinity.

3.2 — Closest approach at the escape threshold

For $\beta = 1$,

$$l = r_0, \quad \varepsilon = 1.$$

The perigee distance of the parabola is therefore

$$r'_{\min} = \frac{l}{1 + \varepsilon} = \boxed{\frac{r_0}{2}}.$$

The satellite initially moves inward, so it reaches this closest approach before escaping. Numerically,

$$r'_{\min} = 2.11 \times 10^7 \text{ m} > R_T,$$

so the threshold trajectory clears Earth.

4.1 — Residual speed at infinity

For $\beta > 1$, the energy is positive. At infinity, the gravitational potential energy vanishes, and energy conservation gives

$$\frac{1}{2}mv_{\infty}^2 = \frac{1}{2}mv_0^2(\beta^2 - 1).$$

Hence

$$v_{\infty} = v_0 \sqrt{\beta^2 - 1}.$$

4.2 — Impact parameter

The impact parameter b is the perpendicular distance from Earth's centre to the outgoing asymptote.

Far from Earth, the motion approaches uniform motion along that asymptote. Its angular momentum magnitude is consequently

$$L = mbv_{\infty}.$$

Conservation of angular momentum gives

$$mbv_{\infty} = mr_0v_0.$$

Using the result of 4.1,

$$b = \frac{r_0v_0}{v_{\infty}} = \frac{r_0}{\sqrt{\beta^2 - 1}}.$$

4.3 — Direction of escape

Use the same polar convention as in 2.2:

$$r(\theta) = \frac{r_0}{1 - \beta \cos \theta}, \quad \theta_0 = \frac{\pi}{2},$$

with θ increasing along the motion and perigee at $\theta = \pi$.

At infinity, the denominator tends to zero:

$$\cos \theta_{\infty} = \frac{1}{\beta}.$$

Define

$$\delta = \arccos\left(\frac{1}{\beta}\right), \quad 0 < \delta < \frac{\pi}{2}.$$

After passing perigee, the outgoing branch reaches

$$\theta_{\infty} = 2\pi - \delta.$$

The tangential speed is $L/(mr)$, which tends to zero as $r \rightarrow \infty$. Thus the limiting velocity direction coincides with the limiting outward radial direction.

The angle ϕ marked on the right of Figure F-2 is the smaller angle between the original outward radius and this outgoing direction. Therefore,

$$\phi = 2\pi - (\theta_\infty - \theta_0) = 2\pi - \left(2\pi - \delta - \frac{\pi}{2}\right).$$

Hence

$$\phi = \frac{\pi}{2} + \arccos\left(\frac{1}{\beta}\right) = \pi - \arcsin\left(\frac{1}{\beta}\right).$$

The relevant branch satisfies

$$\frac{\pi}{2} < \phi < \pi.$$

For $\beta = \frac{3}{2}\beta_{\text{esc}} = \frac{3}{2}$,

$$\phi = 90^\circ + \arccos\left(\frac{2}{3}\right) = 90^\circ + 48.1897\dots^\circ,$$

giving

$$\phi = 138.2^\circ.$$

The escape formulas describe a collision-free Kepler trajectory. For a finite Earth this additionally requires $r_0/(1 + \beta) > R_T$, which holds for the specified numerical example.

The solution was checked against the official solution.

Problem 2: Absolute Measurements of Electrical Quantities

Th 2 ABSOLUTE MEASUREMENTS OF ELECTRICAL QUANTITIES

The technological and scientific transformations underwent during the XIX century produced a compelling need of universally accepted standards for the electrical quantities. It was thought the new absolute units should only rely on the standards of length, mass and time established after the French Revolution. An intensive experimental work to settle the values of these units was developed from 1861 until 1912. We propose here three case studies.

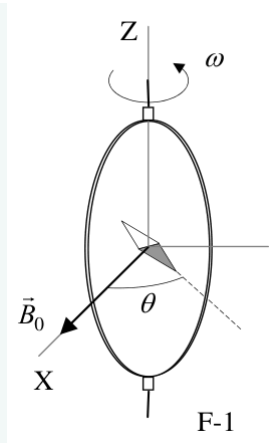
Marks are indicated at the beginning of each subquestion, in parenthesis.

Determination of the ohm (Kelvin)

A closed circular coil of N turns, radius a and total resistance R is rotated with uniform angular velocity ω about a vertical diameter in a horizontal magnetic field $\vec{B}_0 = B_0 \vec{i}$.

1. (0.5+1.0) Compute the electromotive force $\mathcal{E}$ induced in the coil, and also the mean power¹ $\langle P \rangle$ required for maintaining the coil in motion. Neglect the coil self inductance.

A small magnetic needle is placed at the center of the coil, as shown in Figure F-1. It is free to turn slowly around the Z axis in a horizontal plane, but it cannot follow the rapid rotation of the coil.



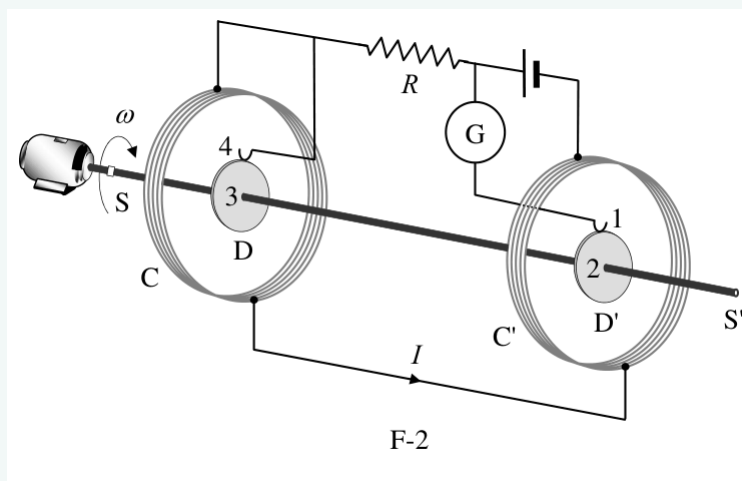
F-1

2. (2.0) Once the stationary regime is reached, the needle will set at a direction making a small angle θ with $\vec{B}_0$. Compute the resistance R of the coil in terms of this angle and the other parameters of the system.

Lord Kelvin used this method in the 1860s to set the absolute standard for the ohm. To avoid the rotating coil, Lorenz devised an alternative method used by Lord Rayleigh and Ms. Sidgwick, that we analyze in the next paragraphs.

Determination of the ohm (Rayleigh, Sidgwick).

The experimental setup is shown in Figure F-2. It consists of two identical metal disks D and D' of radius b mounted on the conducting shaft SS' . A motor rotates the set at an angular velocity ω , which can be adjusted for measuring R . Two identical coils C and C' (of radius a and with N turns each) surround the disks. They are connected in such a form that the current I flows through them in opposite directions. The whole apparatus serves to measure the resistance R .



F-2

¹ The mean value $\langle X \rangle$ of a quantity $X(t)$ in a periodic system of period T is

$$\langle X \rangle = \frac{1}{T} \int_0^T X(t) dt.$$

You may need one or more of these integrals:

$$\int_0^{2\pi} \sin x \, dx = \int_0^{2\pi} \cos x \, dx = \int_0^{2\pi} \sin x \cos x \, dx = 0,$$

$$\int_0^{2\pi} \sin^2 x \, dx = \int_0^{2\pi} \cos^2 x \, dx = \pi, \quad \text{and later} \quad \int x^n \, dx = \frac{1}{n+1} x^{n+1}.$$

3. (2.0) Assume that the current I flowing through the coils C and C' creates a uniform magnetic field B around D and D' , equal to the one at the centre of the coil. Compute¹ the electromotive force $\mathcal{E}$ induced between the rims 1 and 4, assuming that the distance between the coils is much larger than the radius of the coils and that $a \gg b$.

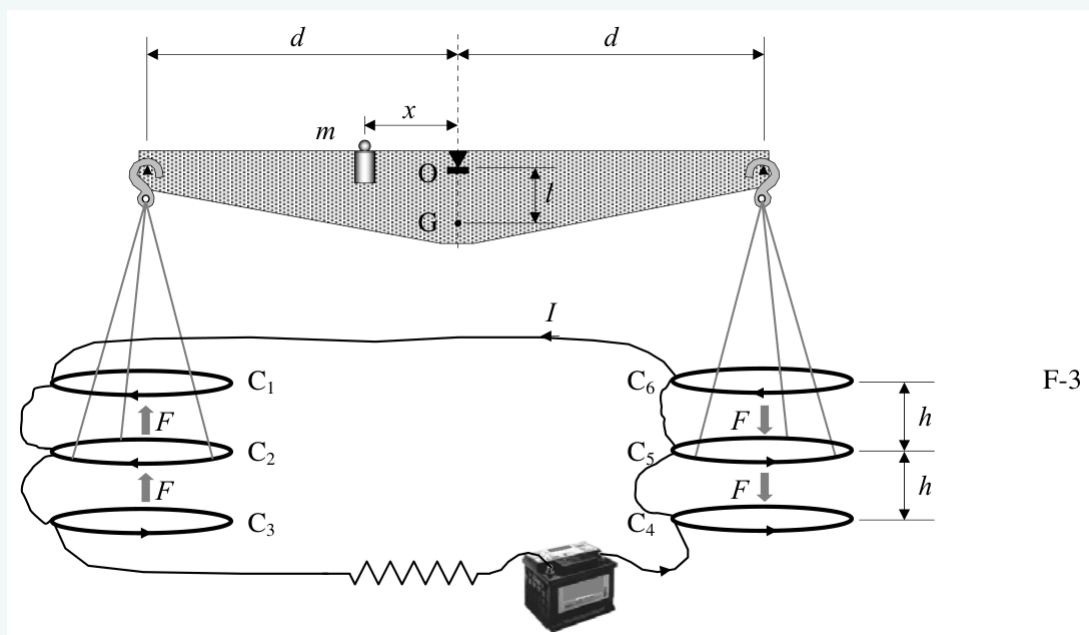
The disks are connected to the circuit by brush contacts at their rims 1 and 4. The galvanometer G detects the flow of current through the circuit 1-2-3-4.

4. (0.5) The resistance R is measured when G reads zero. Give R in terms of the physical parameters of the system.

Determination of the ampere

Passing a current through two conductors and measuring the force between them provides an absolute determination of the current itself. The "Current Balance" designed by Lord Kelvin in 1882 exploits this method. It consists of six identical single turn coils $C_1 \dots C_6$ of radius a , connected in series. As shown in Figure F-3, the fixed coils C_1, C_3, C_4 , and C_6 are on two horizontal planes separated by a small distance $2h$. The coils C_2 and C_5 are carried on balance arms of length d , and they are, in equilibrium, equidistant from both planes.

The current I flows through the various coils in such a direction that the magnetic force on C_2 is upwards while that on C_5 is downwards. A mass m at a distance x from the fulcrum O is required to restore the balance to the equilibrium position described above when the current flows through the circuit.



F-3

5. (1.0) Compute the force F on C_2 due to the magnetic interaction with C_1 . For simplicity assume that the force per unit length is the one corresponding to two long, straight wires carrying parallel currents.

6. (1.0) The current I is measured when the balance is in equilibrium. Give the value of I in terms of the physical parameters of the system. The dimensions of the apparatus are such that we can neglect the mutual effects of the coils on the left and on the right.

Let M be the mass of the balance (except for m and the hanging parts), G its centre of mass and l the distance $\overline{OG}$.

7. (2.0) The balance equilibrium is stable against deviations producing small changes δz in the height of C_2 and $-\delta z$ in C_5 . Compute² the maximum value $\delta z_{\max}$ so that the balance still returns towards the equilibrium position when it is released.

² Consider that the coils centres remain approximately aligned.

Use the approximations

$$\frac{1}{1 \pm \beta} \approx 1 \mp \beta + \beta^2 \quad \text{or} \quad \frac{1}{1 \pm \beta^2} \approx 1 \mp \beta^2$$

for $\beta \ll 1$, and $\sin \theta \approx \tan \theta$ for small θ .

Official English text of the problem.

Answers

Here μ_0 is the permeability of free space and g is gravitational acceleration. No numerical parameter values are supplied, so the answers are symbolic.

1. Taking the coil normal parallel to $\vec{B}_0$ at $t = 0$,

$$\mathcal{E}(t) = N\pi a^2 B_0 \omega \sin(\omega t), \quad \langle P \rangle = \frac{N^2 \pi^2 a^4 B_0^2 \omega^2}{2R}.$$

2.

$$R = \frac{\mu_0 \pi N^2 a \omega}{4 \tan \theta} \approx \frac{\mu_0 \pi N^2 a \omega}{4 \theta},$$

with θ in radians.

3. The magnitude of the emf between the rims is

$$|\mathcal{E}_{14}| = B \omega b^2 = \frac{\mu_0 N I \omega b^2}{2a}.$$

4.

$$R = \frac{\mu_0 N \omega b^2}{2a}.$$

5.

$$F = \frac{\mu_0 a I^2}{h}, \quad \text{directed upwards.}$$

6.

$$I = \sqrt{\frac{mgxh}{4\mu_0 ad}}.$$

7. To the order of the prescribed small-displacement approximation,

$$\delta z_{\max} \approx \frac{Mlh^2}{mgd} = \frac{Mglh^3}{4\mu_0 a I^2 d^2}.$$

A positive displacement returns towards equilibrium when $0 < \delta z < \delta z_{\max}$; equality is the limiting boundary.

Solution

1. Induced emf and mean mechanical power

Let $A = \pi a^2$. Choose the positive normal to the coil as

$$\hat{n}(t) = \cos(\omega t) \hat{i} + \sin(\omega t) \hat{j},$$

so that it is parallel to the applied field at $t = 0$.

The magnetic flux linkage through the N turns is

$$\Lambda(t) = NAB_0 \cos(\omega t).$$

Faraday's law therefore gives

$$\mathcal{E}(t) = -\frac{d\Lambda}{dt} = NAB_0\omega \sin(\omega t),$$

or

$$\mathcal{E}(t) = N\pi a^2 B_0 \omega \sin(\omega t).$$

A different initial orientation changes only the phase.

Since self inductance is neglected, the instantaneous current is

$$i(t) = \frac{\mathcal{E}(t)}{R}.$$

At constant angular speed, the mechanical power supplied against the magnetic braking torque equals the Joule heating:

$$P(t) = i^2(t)R = \frac{N^2 A^2 B_0^2 \omega^2}{R} \sin^2(\omega t).$$

Using $\langle \sin^2(\omega t) \rangle = \frac{1}{2}$,

$$\langle P \rangle = \frac{N^2 \pi^2 a^4 B_0^2 \omega^2}{2R}.$$

2. Resistance from the magnetic needle's deflection

The field at the centre of a circular N -turn coil is

$$\vec{B}_c(t) = \frac{\mu_0 N i(t)}{2a} \hat{n}(t).$$

Substituting the current found in part 1 and writing $\phi = \omega t$,

$$\vec{B}_c(t) = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} (\sin \phi \cos \phi \hat{i} + \sin^2 \phi \hat{j}).$$

The needle changes direction negligibly during one coil revolution. Consequently, its mean torque is its magnetic moment crossed with the **vector average** of the total field. Since

$$\langle \sin \phi \cos \phi \rangle = 0, \quad \langle \sin^2 \phi \rangle = \frac{1}{2},$$

the coil's mean field is

$$\langle \vec{B}_c \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{4R} \hat{j}.$$

It is perpendicular to the applied field.

The stationary needle aligns with

$$\langle \vec{B}_{\text{total}} \rangle = B_0 \hat{i} + \frac{\mu_0 N^2 \pi a B_0 \omega}{4R} \hat{j}.$$

Thus

$$\tan \theta = \frac{\langle B_{c,y} \rangle}{B_0} = \frac{\mu_0 N^2 \pi a \omega}{4R},$$

giving

$$R = \frac{\mu_0 \pi N^2 a \omega}{4 \tan \theta}.$$

For the specified small angle,

$$R \approx \frac{\mu_0 \pi N^2 a \omega}{4\theta}.$$

3. Emf between the rotating disks' rims

The magnetic field at the centre of either surrounding coil has magnitude

$$B = \frac{\mu_0 N I}{2a}.$$

The large separation of the coils allows their mutual field contributions to be neglected. The condition $a \gg b$ permits this central field to be treated as uniform across each disk.

At radius r , a disk element moves with tangential speed

$$v = \omega r.$$

In an axial magnetic field, the magnetic force per unit charge, $\vec{v} \times \vec{B}$, is radial. Hence the magnitude of the motional emf from the hub to the rim of one disk is

$$\mathcal{E}_d = \int_0^b B \omega r \, dr = \frac{1}{2} B \omega b^2.$$

The disks rotate in the same sense but their magnetic fields are opposite. Their rim-to-hub emfs therefore have opposite signs. Since the hubs are joined by the conducting shaft, the rim-to-rim emfs add in magnitude:

$$|\mathcal{E}_{14}| = 2\mathcal{E}_d = B \omega b^2.$$

Thus

$$|\mathcal{E}_{14}| = \frac{\mu_0 N I \omega b^2}{2a}.$$

The polarity is determined by the direction of $\vec{\omega}$ relative to the axial field in each disk.

4. Resistance at galvanometer null

When the galvanometer reads zero, no current flows through the disk–shaft–galvanometer branch. There is therefore no resistive voltage drop along that branch.

The disk emf balances the voltage IR across the resistance:

$$IR = |\mathcal{E}_{14}| = \frac{\mu_0 N I \omega b^2}{2a}.$$

Cancelling the nonzero current,

$$R = \frac{\mu_0 N \omega b^2}{2a}.$$

5. Force between C_1 and C_2

For two long parallel wires separated by h , each carrying current I , the magnetic force per unit length is

$$f = \frac{\mu_0 I^2}{2\pi h}.$$

Using the prescribed approximation, this applies locally around the circumference $2\pi a$ of the coils. The force is therefore

$$F = f(2\pi a) = \frac{\mu_0 I^2}{2\pi h}(2\pi a),$$

so

$$F = \frac{\mu_0 a I^2}{h}.$$

The currents in C_1 and C_2 circulate in the same direction, so the interaction is attractive. Since C_1 is above C_2 , the force on C_2 is upwards.

6. Current from the balance equilibrium

The upper coil C_1 attracts C_2 upwards, while the lower coil C_3 , carrying the opposite current, repels it upwards. Each contributes the force found in part 5. Thus the total upward force on C_2 is $2F$.

Similarly, the total force on C_5 is $2F$ downwards. These two forces produce torques in the same rotational sense:

$$\tau_{\text{mag}} = (2F)d + (2F)d = 4Fd.$$

The added mass provides the opposing torque mgx . At equilibrium,

$$mgx = 4Fd = \frac{4\mu_0 a I^2 d}{h}.$$

Consequently,

$$I = \sqrt{\frac{mgxh}{4\mu_0 ad}}.$$

7. Maximum displacement that produces a restoring torque

Write $z = \delta z$, positive when C_2 rises and C_5 falls. Let α be the corresponding tilt of the balance, so that

$$z = d \sin \alpha.$$

We retain the assumed alignment of the coil centres and treat the current as fixed.

Define

$$K = \mu_0 a I^2.$$

After the displacement, the separations between C_2 and its upper and lower fixed coils are $h - z$ and $h + z$. Both magnetic forces remain upwards, so their sum is

$$F_L(z) = \frac{K}{h - z} + \frac{K}{h + z} = \frac{2Kh}{h^2 - z^2}.$$

On the right, the two separations are interchanged; the total downward force has the same magnitude.

Taking positive torque to mean a tendency to increase z , the total magnetic torque is

$$\tau_{\text{mag}} = 2d \cos \alpha F_L(z) = \frac{4Khd}{h^2 - z^2} \cos \alpha.$$

The added mass supplies the opposing torque

$$\tau_m = -mgx \cos \alpha.$$

Because the balance's centre of mass lies a distance l below its fulcrum, its weight supplies the restoring torque

$$\tau_M = -Mgl \sin \alpha.$$

The equal weights of the hanging parts cancel in the torque balance.

Hence the net torque is

$$\tau(z) = \left(\frac{4Khd}{h^2 - z^2} - mgx \right) \cos \alpha - Mgl \sin \alpha.$$

From the equilibrium condition in part 6,

$$mgx = \frac{4Kd}{h}.$$

Substitution gives

$$\tau(z) = mgx \frac{z^2}{h^2 - z^2} \cos \alpha - Mgl \sin \alpha.$$

For small tilt, $\cos \alpha > 0$, so the torque has the same sign as

$$\frac{\tau(z)}{\cos \alpha} = mgx \frac{z^2}{h^2 - z^2} - Mgl \tan \alpha.$$

Using the prescribed approximations,

$$\frac{z^2}{h^2 - z^2} \approx \frac{z^2}{h^2}, \quad \tan \alpha \approx \sin \alpha = \frac{z}{d},$$

we obtain

$$\frac{\tau(z)}{\cos \alpha} \approx \frac{mgx}{h^2} z^2 - \frac{Mgl}{d} z.$$

For $z > 0$, the balance initially moves back towards equilibrium if $\tau(z) < 0$. Therefore

$$\frac{mgx}{h^2} z < \frac{Mgl}{d},$$

or

$$z < \frac{Mlh^2}{mxd}.$$

The limiting displacement is thus

$$\delta z_{\max} \approx \frac{Mlh^2}{mxd}.$$

Using the equilibrium relation once more,

$$\delta z_{\max} \approx \frac{Mglh^3}{4\mu_0 a I^2 d^2}.$$

At this boundary the net torque vanishes to the retained order; a larger positive displacement produces a torque away from equilibrium. For a negative small displacement, both terms in the approximate torque point towards increasing z , initially returning the balance towards equilibrium.

The stated limit is the leading small-displacement result, requiring $\delta z_{\max} \ll h$ and $\delta z_{\max} \ll d$ for quantitative use of the approximations.

The solution was checked against the official solution.

Problem 3: Neutrons in a Gravitational Field

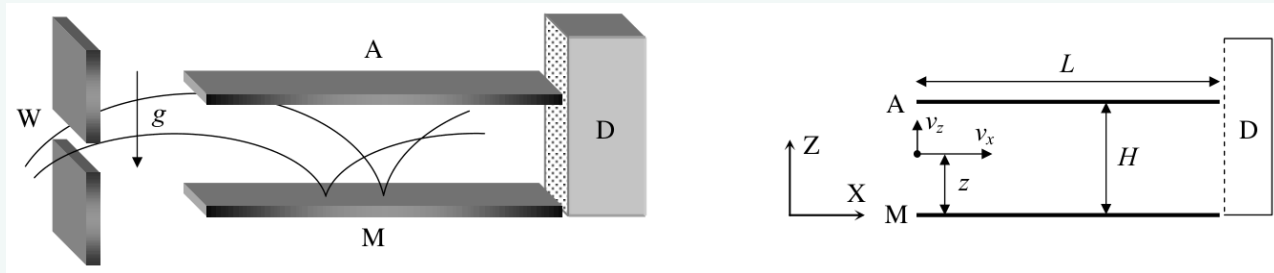
Th 3 NEUTRONS IN A GRAVITATIONAL FIELD

In the familiar classical world, an elastic bouncing ball on the Earth's surface is an ideal example for perpetual motion. The ball is trapped: it can not go below the surface or above its turning point. It will remain bounded in this state, turning down and bouncing up once and again, forever. Only air drag or inelastic bounces could stop the process and will be ignored in the following.

A group of physicists from the Institute Laue - Langevin in Grenoble reported¹ in 2002 experimental evidence on the behaviour of neutrons in the gravitational field of the Earth. In the experiment, neutrons moving to the right were allowed to fall towards a horizontal crystal surface acting as a neutron mirror, where they bounced back elastically up to the initial height once and again.

The setup of the experiment is sketched in Figure F-1. It consists of the opening W, the neutron mirror M (at height $z = 0$), the neutron absorber A (at height $z = H$ and with length L) and the neutron detector D. The beam of neutrons flies with constant horizontal velocity component v_x from W to D through the cavity between A and M. All the neutrons that reach the surface of A are

absorbed and disappear from the experiment. Those that reach the surface of M are reflected elastically. The detector D counts the transmission rate $N(H)$, that is, the total number of neutrons that reach D per unit time.



F-1

Marks are indicated at the beginning of each subquestion, in parenthesis.

The neutrons enter the cavity with a wide range of positive and negative vertical velocities, v_z . Once in the cavity, they fly between the mirror below and the absorber above.

1. (1.5) Compute classically the range of vertical velocities $v_z(z)$ of the neutrons that, entering at a height z , can arrive at the detector D. Assume that L is much larger than any other length in the problem.
2. (1.5) Calculate classically the minimum length L_c of the cavity to ensure that all neutrons outside the previous velocity range, regardless of the values of z , are absorbed by A. Use $v_x = 10 \text{ m s}^{-1}$ and $H = 50 \text{ }\mu\text{m}$.

The neutron transmission rate $N(H)$ is measured at D. We expect that it increases monotonically with H .

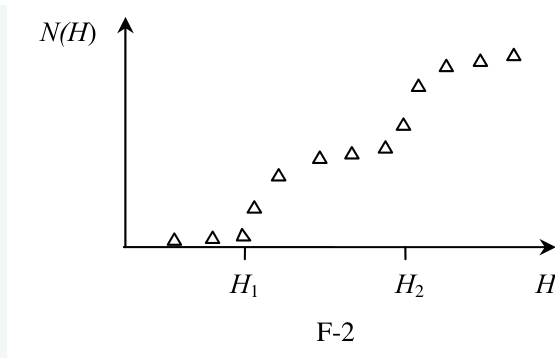
3. (2.5) Compute the classical rate $N_c(H)$ assuming that neutrons arrive at the cavity with vertical velocity v_z and at height z , being all the values of v_z and z equally probable. Give the answer in terms of ρ , the constant number of neutrons per unit time, per unit vertical velocity, per unit height, that enter the cavity with vertical velocity v_z and at height z .

¹ V. V. Nesvizhevsky et al. "Quantum states of neutrons in the Earth's gravitational field." Nature, 415 (2002) 297. Phys Rev D 67, 102002 (2003).

The experimental results obtained by the Grenoble group disagree with the above classical predictions, showing instead that the value of $N(H)$ experiences sharp increases when H crosses some critical heights $H_1, H_2 \dots$ (Figure F-2 shows a sketch). In other words, the experiment showed that the vertical motion of neutrons bouncing on the mirror is quantized. In the language that Bohr and Sommerfeld used to obtain the energy levels of the hydrogen atom, this can be written as: "The action S of these neutrons along the vertical direction is an integer multiple of the Planck action constant h ". Here S is given by

$$S = \int p_z(z) dz = nh, \quad n = 1, 2, 3 \dots \quad (\text{Bohr-Sommerfeld quantization rule})$$

where p_z is the vertical component of the classical momentum, and the integral covers a whole bouncing cycle. Only neutrons with these values of S are allowed in the cavity.



F-2

4. (2.5) Compute the turning heights H_n and energy levels E_n (associated to the vertical motion) using the Bohr-Sommerfeld quantization condition. Give the numerical result for H_1 in μm and for E_1 in eV.

The uniform initial distribution ρ of neutrons at the entrance changes, during the flight through a long cavity, into the step-like distribution detected at D (see Figure F-2). From now on, we consider for simplicity the case of a long cavity with $H < H_2$. Classically, all neutrons with energies in the range considered in question 1 were allowed through it, while quantum mechanically only neutrons in the energy level E_1 are permitted. According to the time-energy Heisenberg uncertainty principle, this reshuffling requires a minimum time of flight. The uncertainty of the vertical motion energy will be significant if the cavity length is small. This phenomenon will give rise to the widening of the energy levels.

5. (2.0) Estimate the minimum time of flight t_q and the minimum length L_q of the cavity needed to observe the first sharp increase in the number of neutrons at D. Use $v_x = 10 \text{ m s}^{-1}$.

Data:

Quantity	Value
Planck action constant	$h = 6.63 \cdot 10^{-34} \text{ J s}$
Speed of light in vacuum	$c = 3.00 \cdot 10^8 \text{ m s}^{-1}$
Elementary charge	$e = 1.60 \cdot 10^{-19} \text{ C}$
Neutron mass	$M = 1.67 \cdot 10^{-27} \text{ kg}$
Acceleration of gravity on Earth	$g = 9.81 \text{ m s}^{-2}$

If necessary, use the expression:

$$\int (1-x)^{1/2} dx = -\frac{2(1-x)^{3/2}}{3}$$

Official English text of the problem.

Answers

1. For $0 \leq z < H$, the transmitted velocities satisfy

$$-\sqrt{2g(H-z)} < v_z(z) < \sqrt{2g(H-z)}.$$

The endpoints reach the absorber and are therefore excluded.

2. The minimum classical cavity length is

$$L_c = 2v_x \sqrt{\frac{2H}{g}} = 6.39 \times 10^{-2} \text{ m} = 6.39 \text{ cm}.$$

3. The classical transmission rate is

$$N_c(H) = \frac{4}{3} \rho \sqrt{2g} H^{3/2}.$$

4. Using precisely the quantization rule stated in the problem,

$$H_n = \left(\frac{9n^2 \hbar^2}{32M^2 g} \right)^{1/3} = H_1 n^{2/3},$$

$$E_n = MgH_n = \left(\frac{9n^2 \hbar^2 Mg^2}{32} \right)^{1/3} = E_1 n^{2/3}.$$

Numerically,

$$H_1 = 16.5 \text{ } \mu\text{m}, \quad E_1 = 1.69 \times 10^{-12} \text{ eV}.$$

5. Using $\Delta E t_q \gtrsim \hbar/2$ and the resolution scale $\Delta E \sim E_1$, the uncertainty-limit estimates are

$$t_q \gtrsim \frac{\hbar}{4\pi E_1} = 1.95 \times 10^{-4} \text{ s}, \quad L_q \gtrsim 1.95 \times 10^{-3} \text{ m}.$$

A sharply resolved step requires longer flight. If the rough uncertainty estimate is instead written as $\Delta E t_q \sim \hbar$, the corresponding estimates are

$$t_q \sim 2.45 \times 10^{-3} \text{ s}, \quad L_q \sim 2.45 \times 10^{-2} \text{ m}.$$

The numerical prefactor is not fixed without a quantitative sharpness criterion.

Solution

1. Classical range of transmitted vertical velocities

The horizontal motion is uniform and does not affect the vertical energy. Between reflections, conservation of mechanical energy gives

$$E_z = \frac{1}{2} M v_z^2 + Mgz.$$

An elastic reflection at the mirror reverses v_z without changing E_z . At the upper turning point z_t , the vertical velocity vanishes, so

$$Mgz_t = \frac{1}{2}Mv_z^2 + Mgz,$$

and therefore

$$z_t = z + \frac{v_z^2}{2g}.$$

In a sufficiently long cavity, every neutron whose trajectory reaches $z = H$ is absorbed. Transmission consequently requires $z_t < H$, or

$$v_z^2 < 2g(H - z).$$

Thus

$$\boxed{-\sqrt{2g(H - z)} < v_z < \sqrt{2g(H - z)}, \quad 0 \leq z < H.}$$

Both signs occur: an initially downward-moving neutron first reflects from the mirror, then rises to the same turning height. Equality is excluded because the problem states that every neutron reaching the absorber is absorbed.

2. Minimum classical cavity length

We must find the longest possible time before a neutron outside this range reaches the absorber.

Write its conserved vertical energy as $E_z = Mga$, where $a \geq H$ is the turning height it would have without the absorber. Its vertical speed at height z is

$$|v_z(z)| = \sqrt{2g(a - z)}.$$

The time to travel upward from the mirror to the absorber is

$$\begin{aligned} \tau(a) &= \int_0^H \frac{dz}{\sqrt{2g(a - z)}} \\ &= \sqrt{\frac{2}{g}} \left(\sqrt{a} - \sqrt{a - H} \right). \end{aligned}$$

This decreases as a increases, and its largest value is

$$\tau(H) = \sqrt{\frac{2H}{g}}.$$

An initially upward-moving neutron reaches the absorber within one such upward passage. An initially downward-moving neutron can first descend to the mirror and then rise to the absorber. Its total time is at most $2\tau(a)$.

The limiting longest trajectory starts just below the absorber, moving downward very slowly, with energy just sufficient to return to height H . Hence the supremum of the absorption time is

$$t_c = 2\sqrt{\frac{2H}{g}}.$$

Since the horizontal flight time is L/v_x , all such neutrons are removed when $L/v_x \geq t_c$. Any smaller length allows some trajectories sufficiently close to the limiting one to escape absorption. Therefore

$$L_c = 2v_x \sqrt{\frac{2H}{g}}.$$

Numerically,

$$t_c = 2\sqrt{\frac{2(50 \times 10^{-6} \text{ m})}{9.81 \text{ m s}^{-2}}} = 6.39 \times 10^{-3} \text{ s},$$

so

$$L_c = (10 \text{ m s}^{-1})t_c = 6.39 \times 10^{-2} \text{ m}.$$

3. Classical transmission rate

By definition, the incident rate in an element $dz dv_z$ is

$$dN = \rho dz dv_z.$$

For a long cavity, only the velocity interval found in part 1 contributes. Integrating over this interval and over all entrance heights,

$$N_c(H) = \rho \int_0^H \int_{-\sqrt{2g(H-z)}}^{+\sqrt{2g(H-z)}} dv_z dz.$$

Thus

$$\begin{aligned} N_c(H) &= 2\rho\sqrt{2g} \int_0^H (H-z)^{1/2} dz \\ &= 2\rho\sqrt{2g} \frac{2}{3} H^{3/2}. \end{aligned}$$

Therefore

$$N_c(H) = \frac{4}{3}\rho\sqrt{2g} H^{3/2}.$$

This predicts a smooth, increasing curve starting at $N_c(0) = 0$, with no steps.

4. Quantized turning heights and energies

For a neutron whose vertical turning height is H_n ,

$$E_n = MgH_n.$$

The vertical momentum on the upward part of its trajectory is

$$p_z(z) = \sqrt{2M(E_n - Mgz)} = M\sqrt{2g(H_n - z)}.$$

During descent, both p_z and dz are negative, so the downward contribution to the action equals the upward contribution. The instantaneous reflection at the mirror contributes no displacement.

Consequently, the action over one full bounce is

$$\begin{aligned} S &= 2 \int_0^{H_n} M\sqrt{2g(H_n - z)} dz \\ &= \frac{4}{3}M\sqrt{2g} H_n^{3/2}. \end{aligned}$$

Applying the supplied Bohr-Sommerfeld rule $S = nh$,

$$\frac{4}{3}M\sqrt{2g}H_n^{3/2} = nh.$$

Solving,

$$H_n = \left(\frac{3nh}{4M\sqrt{2g}} \right)^{2/3} = \left(\frac{9n^2h^2}{32M^2g} \right)^{1/3}.$$

Multiplication by Mg gives

$$E_n = \left(\frac{9n^2h^2Mg^2}{32} \right)^{1/3}.$$

Both quantities scale as $n^{2/3}$.

For $n = 1$,

$$H_1 = \left[\frac{3(6.63 \times 10^{-34})}{4(1.67 \times 10^{-27})\sqrt{2(9.81)}} \right]^{2/3} = 1.653 \times 10^{-5} \text{ m}.$$

Therefore

$$H_1 = 16.5 \text{ } \mu\text{m}.$$

The corresponding energy is

$$E_1 = MgH_1 = 2.708 \times 10^{-31} \text{ J}.$$

Using $1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$,

$$E_1 = \frac{2.708 \times 10^{-31}}{1.60 \times 10^{-19}} = 1.69 \times 10^{-12} \text{ eV}.$$

5. Minimum quantum flight time and cavity length

The first onset of transmission occurs near $MgH = E_1$. If the energy width of the first level is comparable to E_1 , this onset is substantially smeared. The relevant energy scale for an initial resolution estimate is therefore

$$\Delta E \sim E_1.$$

Using the time-energy uncertainty estimate in the form

$$\Delta E t_q \gtrsim \frac{\hbar}{2} = \frac{h}{4\pi},$$

a necessary flight-time scale is

$$t_q \gtrsim \frac{h}{4\pi E_1}.$$

Substituting the energy in joules,

$$t_q \gtrsim \frac{6.63 \times 10^{-34}}{4\pi(2.708 \times 10^{-31})} = 1.95 \times 10^{-4} \text{ s}.$$

The horizontal motion then gives

$$L_q = v_x t_q \gtrsim (10)(1.95 \times 10^{-4}) = 1.95 \times 10^{-3} \text{ m}.$$

These values characterize an uncertainty bound, rather than guaranteeing a particular experimental sharpness. To make that distinction quantitative, suppose the permitted energy width is $\Delta E = \epsilon E_1$. Since $E = Mgh$, this also corresponds to a relative height width $\Delta H/H_1 = \epsilon$. The estimates become

$$t_q \gtrsim \frac{1.95 \times 10^{-4}}{\epsilon} \text{ s}, \quad L_q \gtrsim \frac{1.95 \times 10^{-3}}{\epsilon} \text{ m}.$$

A sharp step, with $\epsilon \ll 1$, requires times and lengths appreciably larger than the basic bound.

For an order-of-magnitude treatment using $\Delta E t_q \sim h$, one instead quotes

$$t_q \sim \frac{h}{E_1} = 2.45 \times 10^{-3} \text{ s}, \quad \boxed{L_q \sim 2.45 \times 10^{-2} \text{ m} = 2.45 \text{ cm}.}$$

The solution was checked against the official solution.