

IPhO 2004

Theory problems and solutions · Pohang, Korea · 3 theory problems with complete solutions

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Problem 1: "Ping-Pong" Resistor

A capacitor consists of two circular parallel plates both with radius R separated by distance d , where $d \ll R$, as shown in Fig. 1.1(a). The top plate is connected to a *constant* voltage source at a potential V while the bottom plate is grounded. Then a thin and small disk of mass m with radius $r \ll R, d$ and thickness $t \ll r$ is placed on the center of the bottom plate, as shown in Fig. 1.1(b).

Let us assume that the space between the plates is in vacuum with the dielectric constant ϵ_0 ; the plates and the disk are made of perfect conductors; and all the electrostatic edge effects may be neglected. The inductance of the whole circuit and the relativistic effects can be safely disregarded. The image charge effect can also be neglected.

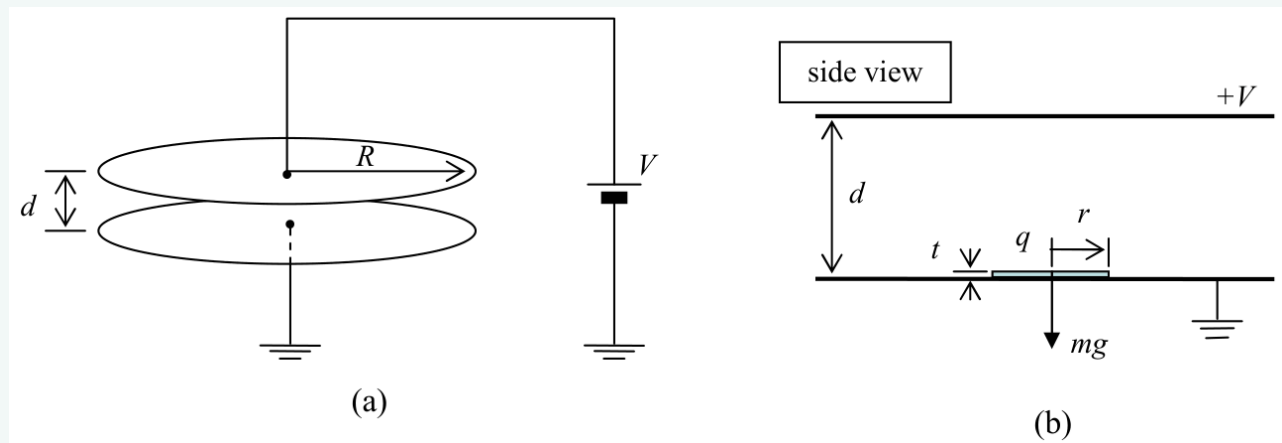


Figure 1.1 Schematic drawings of (a) a parallel plate capacitor connected to a constant voltage source and (b) a side view of the parallel plates with a small disk inserted inside the capacitor. (See text for details.)

- (a) [1.2 points] Calculate the electrostatic force F_p between the plates separated by d before inserting the disk in-between as shown in Fig. 1.1(a).
- (b) [0.8 points] When the disk is placed on the bottom plate, a charge q on the disk of Fig. 1.1(b) is related to the voltage V by $q = \chi V$. Find χ in terms of r , d , and ϵ_0 .
- (c) [0.5 points] The parallel plates lie perpendicular to a uniform gravitational field g . To lift up the disk at rest initially, we need to increase the applied voltage beyond a threshold voltage V_{th} . Obtain V_{th} in terms of m , g , d , and χ .
- (d) [2.3 points] When $V > V_{th}$, the disk makes an up-and-down motion between the plates. (Assume that the disk moves only vertically *without any wobbling*.) The collisions between the disk and the plates are inelastic with the restitution coefficient $\eta \equiv (v_{after}/v_{before})$, where v_{before}

and v_{after} are the speeds of the disk just before and after the collision respectively. The plates are stationarily fixed in position. The speed of the disk *just after* the collision at the bottom plate approaches a “steady-state speed” v_s , which depends on V as follows:

$$v_s = \sqrt{\alpha V^2 + \beta}. \quad (1.1)$$

Obtain the coefficients α and β in terms of m , g , χ , d , and η . Assume that the whole surface of the disk touches the plate evenly and simultaneously so that the complete charge exchange happens instantaneously at every collision.

(e) [2.2 points] After reaching its steady state, the time-averaged current I through the capacitor plates can be approximated by $I = \gamma V^2$ when $qV \gg mgd$. Express the coefficient γ in terms of m , χ , d , and η .

(f) [3 points] When the applied voltage V is decreased (extremely slowly), there exists a critical voltage V_c below which the charge will cease to flow. Find V_c and the corresponding current I_c in terms of m , g , χ , d , and η . By comparing V_c with the lift-up threshold V_{th} discussed in (c), make a rough sketch of the $I - V$ characteristics when V is increased and decreased in the range from $V = 0$ to $3V_{\text{th}}$.

Official English text of the problem.

Answers

Here $q = \chi V$ denotes the **magnitude** of the disk charge, consistently with $qV \gg mgd$ in (e). Its signed charge is $-q$ after contact with the bottom plate and $+q$ after contact with the top plate. The shuttling results below assume $0 < \eta < 1$.

- (a) The force is attractive, with magnitude

$$F_p = \frac{\varepsilon_0 \pi R^2 V^2}{2d^2}.$$

- (b)

$$\chi = \frac{\varepsilon_0 \pi r^2}{d}.$$

- (c)

$$V_{\text{th}} = \sqrt{\frac{2mgd}{\chi}}.$$

- (d)

$$\alpha = \frac{2\chi\eta^2}{m(1-\eta^2)}, \quad \beta = \frac{2gd\eta^2}{1+\eta^2}.$$

- (e)

$$\gamma = \frac{1}{d} \sqrt{\frac{\chi^3}{2m} \frac{1+\eta}{1-\eta}}.$$

• (f)

$$V_c = \sqrt{\frac{mgd}{\chi} \frac{1 - \eta^2}{1 + \eta^2}} = V_{th} \sqrt{\frac{1 - \eta^2}{2(1 + \eta^2)}},$$

$$I_c = \frac{2\eta g \sqrt{m\chi}}{1 + \eta^2} \sqrt{\frac{1 - \eta}{1 + \eta}}.$$

On increasing V , the current remains zero until V_{th} , then jumps to the conducting branch. On decreasing V , it follows that branch down to (V_c, I_c) , then drops to zero. At high voltage, the branch approaches $I = \gamma V^2$.

The exactly sticking endpoint $\eta = 0$ is singular: applying the contact condition after every bottom impact gives

$$V_c = V_{th}, \quad I_c = (3 - \sqrt{3}) g \sqrt{m\chi}$$

for the limiting current on the conducting side, with no hysteresis. At $\eta = 1$, there is no finite steady state.

Solution

Throughout, neglect the disk thickness compared with d , and use the stated approximations concerning edges and image charges. The applied field has magnitude

$$E = \frac{V}{d}$$

and points downward for $V > 0$.

(a)

The surface charge density on either capacitor plate has magnitude

$$\sigma = \varepsilon_0 E.$$

The force on a plate must exclude its own field. The field produced by the opposite plate is $E/2$, so the electrostatic pressure is

$$p = \sigma \frac{E}{2} = \frac{\varepsilon_0 E^2}{2}.$$

Multiplying by the plate area gives

$$F_p = p\pi R^2 = \frac{\varepsilon_0 \pi R^2 V^2}{2d^2}.$$

The oppositely charged plates attract.

(b)

While the disk rests on the bottom plate, it forms part of that conductor. Its exposed upper face carries surface charge density $-\varepsilon_0 E$. Thus the magnitude of its charge is

$$q = \varepsilon_0 E \pi r^2 = \frac{\varepsilon_0 \pi r^2}{d} V.$$

Consequently,

$$\chi = \frac{\varepsilon_0 \pi r^2}{d}.$$

If q were instead interpreted as the signed bottom-contact charge, its coefficient would be negative; the subsequent use of qV as positive electrical work establishes the magnitude convention used here.

(c)

At rest on the lower plate, the disk is part of the plate surface. Its upward electric force is therefore the pressure found in (a), multiplied by the disk area:

$$F_{\text{contact}} = \frac{\varepsilon_0 E^2}{2} \pi r^2 = \frac{qE}{2} = \frac{\chi V^2}{2d}.$$

If N is the upward normal reaction, equilibrium requires

$$N + \frac{\chi V^2}{2d} = mg.$$

Lift-off becomes possible when the required normal reaction reaches zero. Hence

$$\frac{\chi V_{\text{th}}^2}{2d} = mg,$$

giving

$$V_{\text{th}} = \sqrt{\frac{2mgd}{\chi}}.$$

The factor $1/2$ is essential here. Once the disk is in flight, neglecting image forces as instructed, its net force in the uniform applied field is qE . The contact force and the flight force must therefore be treated separately.

(d)

Define the magnitude of the electric acceleration during flight by

$$a_e = \frac{qE}{m} = \frac{\chi V^2}{md}.$$

After leaving the bottom plate, the disk has charge $-q$, so its upward acceleration is $a_e - g$. After touching the top plate, its charge becomes $+q$, so its downward acceleration is $a_e + g$.

Let v_n be the upward speed just after the n -th bottom collision. Let u_n be the speed immediately before the following top collision, and w_n the speed immediately before the next bottom collision.

The work–energy theorem on the upward flight gives

$$u_n^2 = v_n^2 + 2d(a_e - g).$$

The top collision reduces the speed to ηu_n . The downward flight therefore gives

$$w_n^2 = \eta^2 u_n^2 + 2d(a_e + g).$$

Finally, the bottom collision gives $v_{n+1} = \eta w_n$. Combining these relations,

$$v_{n+1}^2 = \eta^4 v_n^2 + 2d\eta^2 [(1 + \eta^2)a_e + (1 - \eta^2)g].$$

For $0 < \eta < 1$, deviations of v_n^2 from its fixed point decrease by a factor η^4 per cycle. At steady state,

$$(1 - \eta^4)v_s^2 = 2d\eta^2 [(1 + \eta^2)a_e + (1 - \eta^2)g].$$

Thus

$$v_s^2 = \frac{2d\eta^2 a_e}{1 - \eta^2} + \frac{2d\eta^2 g}{1 + \eta^2} = \frac{2\chi\eta^2}{m(1 - \eta^2)} V^2 + \frac{2gd\eta^2}{1 + \eta^2}.$$

Comparing with Eq. (1.1),

$$\boxed{\alpha = \frac{2\chi\eta^2}{m(1 - \eta^2)}, \quad \beta = \frac{2gd\eta^2}{1 + \eta^2}}.$$

(e)

In one complete cycle, the disk changes from $-q$ to $+q$ at the top plate and from $+q$ to $-q$ at the bottom plate. Thus a charge $2q$ passes through the source per complete cycle. If the period is T ,

$$I = \frac{2q}{T}.$$

When $qV \gg mgd$, we have $a_e \gg g$, so gravity may be neglected. The two flights then have the same duration τ , with $T = 2\tau$.

Let w be the speed immediately before either impact. The speed immediately after impact is ηw . During each flight,

$$w^2 - \eta^2 w^2 = 2a_e d, \quad w = \sqrt{\frac{2a_e d}{1 - \eta^2}}.$$

Because the acceleration is constant, the mean speed during a flight is $(1 + \eta)w/2$. Therefore

$$\tau = \frac{2d}{(1 + \eta)w},$$

and

$$I = \frac{q}{\tau} = \frac{\chi V(1 + \eta)}{2d} \sqrt{\frac{2\chi V^2}{m(1 - \eta^2)}}.$$

For $V \geq 0$,

$$I = \frac{1}{d} \sqrt{\frac{\chi^3}{2m} \frac{1 + \eta}{1 - \eta}} V^2.$$

Hence

$$\boxed{\gamma = \frac{1}{d} \sqrt{\frac{\chi^3}{2m} \frac{1 + \eta}{1 - \eta}}}.$$

(f)

The high-voltage approximation in (e) cannot be used to determine cessation. Gravity must be retained.

Let u and w denote the steady-state speeds immediately before the top and bottom impacts, respectively. From (d), $v_s = \eta w$, and

$$w^2 = 2d \left(\frac{a_e}{1 - \eta^2} + \frac{g}{1 + \eta^2} \right).$$

Using the upward-flight energy equation,

$$u^2 = v_s^2 + 2d(a_e - g),$$

we obtain

$$u^2 = 2d \left(\frac{a_e}{1 - \eta^2} - \frac{g}{1 + \eta^2} \right).$$

As the voltage is reduced, the limiting periodic trajectory is the one that just reaches the upper plate with zero speed. Setting $u = 0$ gives

$$a_{e,c} = g \frac{1 - \eta^2}{1 + \eta^2}.$$

Since $a_e = \chi V^2 / (md)$,

$$V_c = \sqrt{\frac{mgd}{\chi} \frac{1 - \eta^2}{1 + \eta^2}}.$$

Below this voltage, the disk cannot maintain flights reaching the upper plate. Once it misses that plate, its charge does not reverse, and successive bottom collisions dissipate its remaining mechanical energy.

At the critical voltage,

$$u_c = 0, \quad w_c = 2\sqrt{\frac{gd}{1 + \eta^2}}, \quad v_{s,c} = \eta w_c.$$

The upward flight starts at speed ηw_c and ends at zero; the downward flight starts at zero and ends at w_c . Using the mean speeds,

$$T_c = \frac{2d}{\eta w_c} + \frac{2d}{w_c} = \frac{1 + \eta}{\eta} \sqrt{\frac{d(1 + \eta^2)}{g}}.$$

Therefore the current immediately before cessation is

$$I_c = \frac{2\chi V_c}{T_c} = \frac{2\eta\chi V_c}{1 + \eta} \sqrt{\frac{g}{d(1 + \eta^2)}}.$$

Substitution of V_c yields

$$I_c = \frac{2\eta g \sqrt{m\chi}}{1 + \eta^2} \sqrt{\frac{1 - \eta}{1 + \eta}}.$$

For every fixed $0 < \eta < 1$, this current is positive, so cessation produces a finite downward jump.

To specify the sketch precisely, the full conducting branch can be written without neglecting gravity. Its two flight times are

$$\tau_{\uparrow} = \frac{2d}{\eta w + u}, \quad \tau_{\downarrow} = \frac{2d}{w + \eta u}.$$

Thus

$$I_{\text{on}}(V) = \frac{2\chi V}{\frac{2d}{\eta w + u} + \frac{2d}{w + \eta u}},$$

where $u(V)$ and $w(V)$ are given above. These expressions remain valid when the upward flight is decelerated.

Comparison with (c) gives

$$\frac{V_c}{V_{\text{th}}} = \sqrt{\frac{1 - \eta^2}{2(1 + \eta^2)}} < 1.$$

The requested sketch therefore has the following features:

- **Increasing voltage:** follow $I = 0$ from the origin to V_{th} . Just above V_{th} , the disk lifts off and the steady current jumps to $I_{\text{on}}(V_{\text{th}})$. Continue upward along the conducting branch to $3V_{\text{th}}$.
- **Decreasing voltage:** retrace the conducting branch through V_{th} and continue to (V_c, I_c) . Just below V_c , drop to $I = 0$, then follow the voltage axis back to the origin.
- **High-voltage shape:** the conducting branch approaches the parabola $I = \gamma V^2$. At $3V_{\text{th}}$, $qV = 18mgd$, so a suitable approximate endpoint for the rough sketch is $I \approx 9\gamma V_{\text{th}}^2$.

Equivalently, away from the jump points:

Voltage range	Increasing V	Decreasing V
$0 \leq V < V_c$	0	0
$V_c < V < V_{\text{th}}$	0	$I_{\text{on}}(V)$
$V_{\text{th}} < V \leq 3V_{\text{th}}$	$I_{\text{on}}(V)$	$I_{\text{on}}(V)$

The interval $V_c < V < V_{\text{th}}$ is the hysteresis interval: a resting disk stays at rest, while an already moving disk continues to shuttle.

For completeness, **exactly** $\eta = 0$ requires separate treatment. Every bottom impact leaves the disk at rest, so the contact lift-off condition from (c) must be met anew. Consequently, the current stops at V_{th} , rather than at the limit of the positive- η critical-voltage formula. On the conducting side, at $V \rightarrow V_{\text{th}}^+$, $a_e \rightarrow 2g$, and

$$T \rightarrow \sqrt{\frac{2d}{g}} + \sqrt{\frac{2d}{3g}}.$$

Hence

$$I_c = \frac{2\chi V_{\text{th}}}{\sqrt{2d/g} + \sqrt{2d/(3g)}} = \boxed{(3 - \sqrt{3}) g \sqrt{m\chi}}.$$

Both switching voltages then coincide, so there is no hysteresis. At the other endpoint, $\eta = 1$, collisions dissipate no energy while the electric field adds $2qV$ per cycle; a finite steady-state speed cannot exist.

The solution was checked against the official solution.

Problem 2: Rising Balloon

A rubber balloon filled with helium gas goes up high into the sky where the pressure and temperature decrease with height. In the following questions, assume that the shape of the balloon remains spherical regardless of the payload, and neglect the payload volume. Also assume that the temperature of the helium gas inside of the balloon is always the same as that of the ambient air, and treat all gases as ideal gases. The universal gas constant is $R = 8.31 \text{ J/mol} \cdot \text{K}$ and the molar masses of helium and air are $M_H = 4.00 \times 10^{-3} \text{ kg/mol}$ and $M_A = 28.9 \times 10^{-3} \text{ kg/mol}$, respectively. The gravitational acceleration is $g = 9.8 \text{ m/s}^2$.

[Part A]

(a) [1.5 points] Let the pressure of the ambient air be P and the temperature be T . The pressure inside of the balloon is higher than that of outside due to the surface tension of the balloon. The balloon contains n moles of helium gas and the pressure inside is $P + \Delta P$. Find the buoyant force F_B acting on the balloon as a function of P and ΔP .

(b) [2 points] On one particular summer day in Korea, the air temperature T at the height z from the sea level was found to be $T(z) = T_0(1 - z/z_0)$ in the range of $0 < z < 15 \text{ km}$ with $z_0 = 49 \text{ km}$ and $T_0 = 303 \text{ K}$. The pressure and density at the sea level were $P_0 = 1.0 \text{ atm} = 1.01 \times 10^5 \text{ Pa}$ and $\rho_0 = 1.16 \text{ kg/m}^3$, respectively. For this height range, the pressure takes the form

$$P(z) = P_0(1 - z/z_0)^\eta. \quad (2.1)$$

Express η in terms of z_0 , ρ_0 , P_0 , and g , and find its numerical value to the two significant digits. Treat the gravitational acceleration as a constant, independent of height.

[Part B]

When a rubber balloon of spherical shape with un-stretched radius r_0 is inflated to a sphere of radius r ($\geq r_0$), the balloon surface contains extra elastic energy due to the stretching. In a simplistic theory, the elastic energy at constant temperature T can be expressed by

$$U = 4\pi r_0^2 \kappa R T \left(2\lambda^2 + \frac{1}{\lambda^4} - 3 \right) \quad (2.2)$$

where $\lambda \equiv r/r_0$ (≥ 1) is the size-inflation ratio and κ is a constant in units of mol/m^2 .

(c) [2 points] Express ΔP in terms of parameters given in Eq. (2.2), and sketch ΔP as a function of $\lambda = r/r_0$.

(d) [1.5 points] The constant κ can be determined from the amount of the gas needed to inflate the balloon. At $T_0 = 303 \text{ K}$ and $P_0 = 1.0 \text{ atm} = 1.01 \times 10^5 \text{ Pa}$, an un-stretched balloon ($\lambda = 1$) contains $n_0 = 12.5$ moles of helium. It takes $n = 3.6n_0 = 45$ moles in total to inflate the balloon to $\lambda = 1.5$ at the same T_0 and P_0 . Express the balloon parameter a , defined as $a = \kappa/\kappa_0$, in terms of n , n_0 , and λ , where $\kappa_0 \equiv r_0 P_0 / (4RT_0)$. Evaluate a to the two significant digits.

[Part C]

A balloon is prepared as in (d) at the sea level (inflated to $\lambda = 1.5$ with $n = 3.6n_0 = 45$ moles of helium gas at $T_0 = 303 \text{ K}$ and $P_0 = 1 \text{ atm} = 1.01 \times 10^5 \text{ Pa}$). The total mass including gas, balloon itself, and other payloads is $M_T = 1.12 \text{ kg}$. Now let the balloon rise from the sea level.

(e) [3 points] Suppose that the balloon eventually stops at the height z_f where the buoyant force balances the total weight. Find z_f and the inflation ratio λ_f at that height. Give the answers in two significant digits. Assume there are no drift effect and no gas leakage during the upward flight.

Official English text of the problem.

Answers

- (a) The upward buoyant force is

$$F_B = nM_A g \frac{P}{P + \Delta P}.$$

- (b)

$$\eta = \frac{\rho_0 g z_0}{P_0} = 5.5.$$

- (c)

$$\Delta P = \frac{4\kappa RT}{r_0} (\lambda^{-1} - \lambda^{-7}).$$

The curve starts at $(1, 0)$, reaches its single maximum at $\lambda = 7^{1/6} \approx 1.38$, and then decreases toward zero from above.

- (d)

$$a = \frac{n/n_0 - \lambda^3}{\lambda^2 - \lambda^{-4}} = 0.11.$$

- (e)

$$z_f = 1.1 \times 10^4 \text{ m} = 11 \text{ km}, \quad \lambda_f = 2.1.$$

The unrounded results are $z_f = 11.095 \text{ km}$ and $\lambda_f = 2.14594$; using the rounded values $\eta = 5.5$ and $a = 0.11$ instead gives the same answers to two significant digits.

Solution

(a)

By Archimedes' principle, the buoyant force equals the weight of the displaced air. Treating ambient density as uniform over the balloon's dimensions,

$$F_B = \rho_A V g.$$

The ideal-gas equations for the surrounding air and the helium give

$$\rho_A = \frac{PM_A}{RT}, \quad V = \frac{nRT}{P + \Delta P}.$$

The common temperature therefore cancels:

$$F_B = nM_A g \frac{P}{P + \Delta P}.$$

This is the buoyant force alone. The weights of the helium, envelope, and payload enter separately in the total force balance.

(b)

The atmosphere is in hydrostatic equilibrium:

$$\frac{dP}{dz} = -\rho(z)g.$$

For ideal air of constant composition,

$$\frac{\rho(z)}{\rho_0} = \frac{P(z)}{P_0} \frac{T_0}{T(z)}.$$

Using the prescribed temperature and pressure profiles,

$$\rho(z) = \rho_0 \left(1 - \frac{z}{z_0}\right)^{\eta-1}.$$

Differentiating Eq. (2.1) and substituting into the hydrostatic equation gives

$$-\frac{\eta P_0}{z_0} \left(1 - \frac{z}{z_0}\right)^{\eta-1} = -\rho_0 g \left(1 - \frac{z}{z_0}\right)^{\eta-1}.$$

Consequently,

$$\eta = \frac{\rho_0 g z_0}{P_0}.$$

Numerically,

$$\eta = \frac{(1.16)(9.8)(49 \times 10^3)}{1.01 \times 10^5} = 5.51517 \dots,$$

so, to two significant digits,

$$\eta = 5.5.$$

(c)

For a quasistatic expansion at fixed temperature, the work done by the excess internal pressure supplies the increase in the modeled elastic energy:

$$\Delta P dV = dU.$$

The balloon volume is

$$V = \frac{4\pi}{3} r_0^3 \lambda^3, \quad \frac{dV}{d\lambda} = 4\pi r_0^3 \lambda^2.$$

From Eq. (2.2),

$$\left(\frac{\partial U}{\partial \lambda}\right)_T = 16\pi r_0^2 \kappa R T (\lambda - \lambda^{-5}).$$

Thus

$$\Delta P = \frac{(\partial U / \partial \lambda)_T}{dV/d\lambda} = \frac{4\kappa RT}{r_0} (\lambda^{-1} - \lambda^{-7}).$$

To specify the sketch, define the pressure scale

$$C = \frac{4\kappa RT}{r_0}.$$

Then

$$\frac{d\Delta P}{d\lambda} = C\lambda^{-8}(7 - \lambda^6).$$

The curve has the following features:

- Its physical domain is $\lambda \geq 1$, and it starts at $\Delta P = 0$ when $\lambda = 1$.
- It rises with positive slope until

$$\lambda_{\max} = 7^{1/6} = 1.38309 \dots$$

- Its maximum value is

$$\Delta P_{\max} = C \frac{6}{7^{7/6}} \approx 0.620 C.$$

- After the maximum, it decreases monotonically, remaining positive. At large inflation,

$$\Delta P \sim \frac{C}{\lambda} \longrightarrow 0^+,$$

so the horizontal axis is an asymptote.

(d)

Let

$$V_0 = \frac{4\pi}{3} r_0^3.$$

At $\lambda = 1$, part (c) gives $\Delta P = 0$. Hence the un-stretched balloon satisfies

$$P_0 V_0 = n_0 R T_0.$$

Since

$$a = \frac{\kappa}{\kappa_0} = \frac{4\kappa R T_0}{r_0 P_0},$$

the excess pressure at temperature T_0 is

$$\frac{\Delta P}{P_0} = a (\lambda^{-1} - \lambda^{-7}).$$

The inflated balloon satisfies

$$(P_0 + \Delta P) V_0 \lambda^3 = n R T_0.$$

Dividing by $P_0 V_0 = n_0 R T_0$,

$$\frac{n}{n_0} = \lambda^3 [1 + a (\lambda^{-1} - \lambda^{-7})] = \lambda^3 + a (\lambda^2 - \lambda^{-4}).$$

Therefore,

$$a = \frac{n/n_0 - \lambda^3}{\lambda^2 - \lambda^{-4}}.$$

Substitution gives

$$a = \frac{3.6 - 1.5^3}{1.5^2 - 1.5^{-4}} = 0.109624\dots,$$

and hence

$$a = 0.11.$$

(e)

Define the dimensionless temperature

$$t = \frac{T}{T_0} = 1 - \frac{z}{z_0}.$$

Then

$$P = P_0 t^\eta, \quad \Delta P = a P_0 t (\lambda^{-1} - \lambda^{-7}).$$

Because no helium escapes, n remains constant. Its ideal-gas equation is

$$[P_0 t^\eta + a P_0 t (\lambda^{-1} - \lambda^{-7})] V_0 \lambda^3 = n R T_0 t.$$

Using $P_0 V_0 = n_0 R T_0$ and dividing by t ,

$$\frac{n}{n_0} = t^{\eta-1} \lambda^3 + a (\lambda^2 - \lambda^{-4}). \quad (1)$$

The displaced air mass follows from its ideal-gas density:

$$\rho_A V = \frac{M_A P}{R T} V_0 \lambda^3 = n_0 M_A t^{\eta-1} \lambda^3.$$

At the final height, buoyancy balances the entire weight:

$$n_0 M_A g t_f^{\eta-1} \lambda_f^3 = M_T g,$$

so

$$t_f^{\eta-1} \lambda_f^3 = \frac{M_T}{n_0 M_A}. \quad (2)$$

The helium's weight is already included in $M_T g$.

Eliminating the atmospheric factor between (1) and (2) gives an equation for the final inflation alone:

$$a (\lambda_f^2 - \lambda_f^{-4}) = \frac{n}{n_0} - \frac{M_T}{n_0 M_A}.$$

Keeping unrounded intermediate values,

$$\frac{M_T}{n_0 M_A} = \frac{1.12}{(12.5)(28.9 \times 10^{-3})} = 3.100346\dots,$$

and therefore

$$\lambda_f^2 - \lambda_f^{-4} = \frac{3.6 - 3.100346\dots}{0.109624\dots} = 4.557886\dots$$

The left side increases strictly for $\lambda_f \geq 1$, because its derivative is $2\lambda_f + 4\lambda_f^{-5} > 0$. Thus there is a unique physical root:

$$\lambda_f = 2.145936 \dots$$

Equation (2) now gives

$$t_f = \left(\frac{M_T}{n_0 M_A \lambda_f^3} \right)^{1/(\eta-1)} = 0.773571 \dots$$

Consequently,

$$z_f = z_0(1 - t_f) = (49 \text{ km})(1 - 0.773571 \dots) = 11.0950 \dots \text{ km}.$$

To two significant digits,

$$\boxed{\lambda_f = 2.1, \quad z_f = 11 \text{ km}}.$$

The final height lies within the specified atmospheric range. The balloon also has positive initial lift: at sea level, its displaced air mass is

$$n_0 M_A (1.5)^3 = 1.2192 \dots \text{ kg} > M_T.$$

The solution was checked against the official solution.

Problem 3: Atomic Probe Microscope

Atomic probe microscopes (APMs) are powerful tools in the field of nano-science. The motion of a cantilever in APM can be detected by a photo-detector monitoring the reflected laser beam, as shown in Fig. 3.1. The cantilever can move only in the vertical direction and its displacement z as a function of time t can be described by the equation

$$m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + kz = F, \tag{3.1}$$

where m is the cantilever mass, $k = m\omega_0^2$ is the spring constant of the cantilever, b is a small damping coefficient satisfying $\omega_0 \gg (b/m) > 0$, and finally F is an external driving force of the piezoelectric tube.

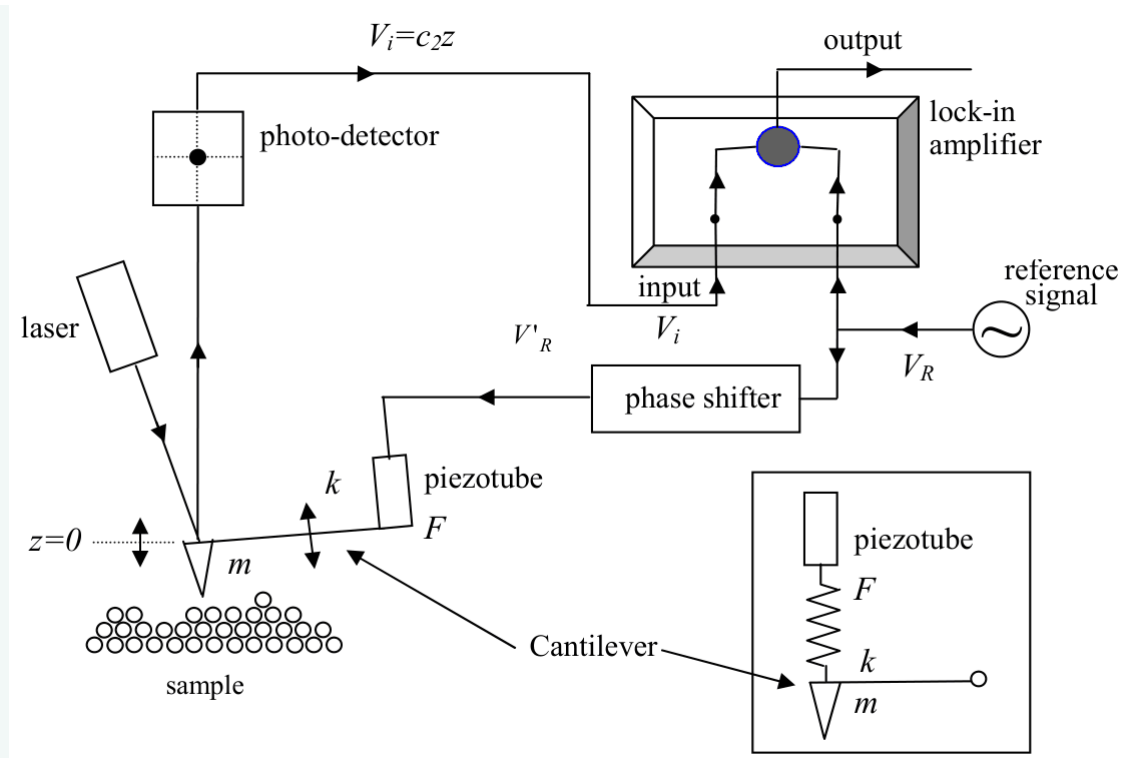


Figure 3.1 A schematic diagram for a scanning probe microscope (SPM). The inset in the lower right corner represents a simplified mechanical model to describe the coupling of the piezotube with the cantilever.

[Part A]

- (a) [1.5 points] When $F = F_0 \sin \omega t$, $z(t)$ satisfying Eq. (3.1) can be written as $z(t) = A \sin(\omega t - \phi)$, where $A > 0$ and $0 \leq \phi \leq \pi$. Find the expression of the amplitude A and $\tan \phi$ in terms of F_0 , m , ω , ω_0 , and b . Obtain A and the phase ϕ at the resonance frequency $\omega = \omega_0$.
- (b) [1 point] A lock-in amplifier shown in Fig.3.1 multiplies an input signal by the lock-in reference signal, $V_R = V_{R0} \sin \omega t$, and then passes *only* the dc (direct current) component of the multiplied signal. Assume that the input signal is given by $V_i = V_{i0} \sin(\omega_i t - \phi_i)$. Here V_{R0} , V_{i0} , ω_i , and ϕ_i are all positive given constants. Find the condition on ω (>0) for a non-vanishing output signal. What is the expression for the magnitude of the non-vanishing *dc output signal* at this frequency?
- (c) [1.5 points] Passing through the phase shifter, the lock-in reference voltage $V_R = V_{R0} \sin \omega t$ changes to $V'_R = V_{R0} \sin(\omega t + \pi/2)$. V'_R , applied to the piezoelectric tube, drives the cantilever with a force $F = c_1 V'_R$. Then, the photo-detector converts the displacement of the cantilever, z , into a voltage $V_i = c_2 z$. Here c_1 and c_2 are constants. Find the expression for the magnitude of the *dc output signal* at $\omega = \omega_0$.
- (d) [2 points] The small change Δm of the cantilever mass shifts the resonance frequency by $\Delta \omega_0$. As a result, the phase ϕ at the original resonance frequency ω_0 shifts by $\Delta \phi$. Find the mass change Δm corresponding to the phase shift $\Delta \phi = \pi/1800$, which is a typical resolution in phase measurements. The physical parameters of the cantilever are given by $m = 1.0 \times 10^{-12}$ kg, $k = 1.0$ N/m, and $(b/m) = 1.0 \times 10^3$ s $^{-1}$. Use the approximations $(1+x)^a \approx 1+ax$ and $\tan(\pi/2+x) \approx -1/x$ when $|x| \ll 1$.

[Part B]

From now on let us consider the situation that some forces, besides the driving force discussed in Part A, act on the cantilever due to the sample as shown in Fig.3.1.

(e) [1.5 points] Assuming that the additional force $f(h)$ depends only on the distance h between the cantilever and the sample surface, one can find a new equilibrium position h_0 . Near $h = h_0$, we can write $f(h) \approx f(h_0) + c_3(h - h_0)$, where c_3 is a constant in h . Find the new resonance frequency ω'_0 in terms of ω_0 , m , and c_3 .

(f) [2.5 points] While scanning the surface by moving the sample horizontally, the tip of the cantilever charged with $Q = 6e$ encounters an electron of charge $q = e$ trapped (localized in space) at some distance below the surface. During the scanning around the electron, the maximum shift of the resonance frequency $\Delta\omega_0 (= \omega'_0 - \omega_0)$ is observed to be much smaller than ω_0 . Express the distance d_0 from the cantilever to the trapped electron at the maximum shift in terms of m , q , Q , ω_0 , $\Delta\omega_0$, and the Coulomb constant k_e . Evaluate d_0 in nm ($1 \text{ nm} = 1 \times 10^{-9} \text{ m}$) for $\Delta\omega_0 = 20 \text{ s}^{-1}$. The physical parameters of the cantilever are $m = 1.0 \times 10^{-12} \text{ kg}$ and $k = 1.0 \text{ N/m}$. Disregard any polarization effect in both the cantilever tip and the surface. Note that $k_e = 1/(4\pi\epsilon_0) = 9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$ and $e = -1.6 \times 10^{-19} \text{ C}$.

Official English text of the problem.

Answers

Write $S_{\text{dc}} = \langle V_R V_i \rangle$ for the dc output under the stated multiplication rule.

- (a)

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}, \quad \tan \phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}.$$

At $\omega = \omega_0$,

$$A = \frac{F_0}{b\omega_0}, \quad \phi = \frac{\pi}{2}.$$

- (b) A nonzero dc output requires

$$\omega = \omega_i, \quad \cos \phi_i \neq 0.$$

Its magnitude is

$$|S_{\text{dc}}| = \frac{V_{R0} V_{i0}}{2} |\cos \phi_i|.$$

- (c)

$$|S_{\text{dc}}| = \frac{|c_1 c_2| V_{R0}^2}{2b\omega_0}.$$

- (d)

$$\Delta m \simeq \frac{b}{\omega_0} \Delta \phi \simeq 1.7 \times 10^{-18} \text{ kg}.$$

The specified positive phase shift corresponds to an increase in mass.

- (e)

$$\omega'_0 = \sqrt{\omega_0^2 - \frac{c_3}{m}},$$

provided $m\omega_0^2 - c_3 > 0$.

- (f) The maximum upward frequency shift occurs directly above the electron, and

$$d_0 \simeq \left(\frac{k_e q Q}{m \omega_0 \Delta \omega_0} \right)^{1/3} \simeq 41 \text{ nm}.$$

Solution

We consider the steady periodic motion after the damped transient has decayed. Throughout, we use the problem's weak-damping convention that the resonance frequency is $\omega_0 = \sqrt{k/m}$.

- (a)

Let $\theta = \omega t - \phi$. Substitution of $z = A \sin \theta$ into the equation of motion gives

$$A(k - m\omega^2) \sin \theta + Ab\omega \cos \theta = F_0 \sin(\theta + \phi).$$

Expanding the right-hand side and equating the coefficients of $\sin \theta$ and $\cos \theta$,

$$A(k - m\omega^2) = F_0 \cos \phi, \quad Ab\omega = F_0 \sin \phi.$$

Squaring and adding these equations yields

$$A^2 [(k - m\omega^2)^2 + b^2 \omega^2] = F_0^2.$$

Using $k = m\omega_0^2$,

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2 \omega^2}}.$$

Dividing the coefficient equations gives

$$\tan \phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}.$$

The quadrant follows from the coefficient equations: for positive driving frequency, $\sin \phi > 0$, while $\cos \phi$ has the sign of $\omega_0^2 - \omega^2$. Thus the lag is below $\pi/2$ below resonance and above $\pi/2$ above resonance.

At $\omega = \omega_0$, the elastic and inertial terms cancel. Consequently,

$$\phi = \frac{\pi}{2}, \quad A = \frac{F_0}{b\omega_0}.$$

(b)

Using the product-to-sum identity,

$$\begin{aligned} V_R V_i &= V_{R0} V_{i0} \sin \omega t \sin(\omega_i t - \phi_i) \\ &= \frac{V_{R0} V_{i0}}{2} [\cos((\omega - \omega_i)t + \phi_i) - \cos((\omega + \omega_i)t - \phi_i)]. \end{aligned}$$

The ideal dc filter retains only terms independent of time. Since both frequencies are positive, the sum-frequency term always averages to zero. The difference-frequency term is constant only when

$$\boxed{\omega = \omega_i}.$$

At this frequency,

$$S_{\text{dc}} = \frac{V_{R0} V_{i0}}{2} \cos \phi_i,$$

so the magnitude is

$$\boxed{|S_{\text{dc}}| = \frac{V_{R0} V_{i0}}{2} |\cos \phi_i|}.$$

Frequency matching is necessary but does not ensure a nonzero result: if $\phi_i = \pi/2 + n\pi$, the input is in quadrature with the reference and the dc output vanishes. Hence non-vanishing output also requires $\cos \phi_i \neq 0$.

(c)

At the specified frequency, the shifted reference drives the cantilever with

$$F = c_1 V_{R0} \sin(\omega_0 t + \pi/2) = c_1 V_{R0} \cos \omega_0 t.$$

The cantilever displacement lags the force by $\pi/2$. The phase shifter therefore makes the displacement in phase with the original reference:

$$z(t) = \frac{c_1 V_{R0}}{b\omega_0} \sin \omega_0 t.$$

This expression also follows directly by substitution into the equation of motion. The detector voltage is

$$V_i(t) = \frac{c_1 c_2 V_{R0}}{b\omega_0} \sin \omega_0 t.$$

Multiplication by the original reference gives

$$V_R V_i = \frac{c_1 c_2 V_{R0}^2}{b\omega_0} \sin^2 \omega_0 t.$$

Since $\langle \sin^2 \omega_0 t \rangle = 1/2$,

$$S_{\text{dc}} = \frac{c_1 c_2 V_{R0}^2}{2b\omega_0}, \quad \boxed{|S_{\text{dc}}| = \frac{|c_1 c_2| V_{R0}^2}{2b\omega_0}}.$$

The absolute value allows for either sign of the conversion constants.

(d)

The spring constant is unchanged when the mass changes. The new natural frequency is therefore

$$\omega_0 + \Delta\omega_0 = \sqrt{\frac{k}{m + \Delta m}} = \omega_0 \left(1 + \frac{\Delta m}{m}\right)^{-1/2}.$$

For $|\Delta m| \ll m$, the supplied binomial approximation gives

$$\Delta\omega_0 \simeq -\frac{\omega_0}{2} \frac{\Delta m}{m}.$$

The measurement frequency remains at the original value ω_0 . Taking b unchanged, the new phase satisfies

$$\tan\left(\frac{\pi}{2} + \Delta\phi\right) = \frac{b\omega_0}{k - (m + \Delta m)\omega_0^2}.$$

Because $k = m\omega_0^2$,

$$\tan\left(\frac{\pi}{2} + \Delta\phi\right) = -\frac{b}{\omega_0\Delta m}.$$

Applying $\tan(\pi/2 + x) \simeq -1/x$,

$$-\frac{1}{\Delta\phi} \simeq -\frac{b}{\omega_0\Delta m},$$

and hence

$$\Delta m \simeq \frac{b}{\omega_0} \Delta\phi.$$

The given parameters yield

$$\omega_0 = \sqrt{\frac{1.0}{1.0 \times 10^{-12}}} = 1.0 \times 10^6 \text{ s}^{-1},$$

$$b = m \frac{b}{m} = (1.0 \times 10^{-12})(1.0 \times 10^3) = 1.0 \times 10^{-9} \text{ kg s}^{-1}.$$

Thus

$$\Delta m \simeq \frac{1.0 \times 10^{-9}}{1.0 \times 10^6} \frac{\pi}{1800} \text{ kg} = 1.7453 \times 10^{-18} \text{ kg},$$

or, to two significant figures,

$$\Delta m \simeq 1.7 \times 10^{-18} \text{ kg}.$$

Here $\Delta m/m \simeq 1.7 \times 10^{-6}$ and $\Delta\phi \simeq 1.7 \times 10^{-3}$, consistent with both small-change approximations. The associated frequency change is approximately -0.87 s^{-1} .

(e)

Take positive displacement and positive sample force to point away from the sample. Let z_{eq} be the new static displacement, corresponding to the gap h_0 . Static force balance gives

$$kz_{\text{eq}} = f(h_0).$$

For a small displacement from this equilibrium, define

$$x = z - z_{\text{eq}} = h - h_0.$$

Including the sample force, the equation of motion is

$$m\ddot{z} + b\dot{z} + kz = F + f(h).$$

Insert the linearized force $f(h) \simeq f(h_0) + c_3x$, then subtract the static equilibrium equation:

$$m\ddot{x} + b\dot{x} + kx = F + c_3x.$$

Therefore,

$$m\ddot{x} + b\dot{x} + (k - c_3)x = F.$$

The force gradient changes the effective spring constant to $k - c_3$, whereas the constant force merely shifts the equilibrium. Thus

$$\omega'_0 = \sqrt{\frac{k - c_3}{m}} = \sqrt{\omega_0^2 - \frac{c_3}{m}}.$$

A stable oscillatory equilibrium requires $k - c_3 > 0$; the resonance interpretation also retains the weak-damping assumption.

(f)

Since $q = e$ and $Q = 6e$,

$$qQ = 6e^2 > 0.$$

The interaction is repulsive. To identify where the frequency shift is greatest, let ρ be the horizontal separation between the tip and electron, and let d be their vertical separation. Coulomb's law gives the upward force component

$$f_z(\rho, d) = \frac{k_e qQ d}{(\rho^2 + d^2)^{3/2}}.$$

A vertical cantilever displacement changes d by the same amount. The relevant force gradient is therefore

$$c_3 = \frac{\partial f_z}{\partial d} = k_e qQ \frac{\rho^2 - 2d^2}{(\rho^2 + d^2)^{5/2}}.$$

For a small frequency shift, part (e) gives

$$\omega'_0 = \omega_0 \sqrt{1 - \frac{c_3}{m\omega_0^2}} \simeq \omega_0 - \frac{c_3}{2m\omega_0},$$

so

$$\Delta\omega_0(\rho) \simeq \frac{k_e qQ}{2m\omega_0} \frac{2d^2 - \rho^2}{(\rho^2 + d^2)^{5/2}}.$$

To leading order in the weak interaction, the scan height can be treated as fixed. This expression has its positive maximum at $\rho = 0$: while the numerator is positive, it decreases with ρ and the denominator increases; beyond $\rho = \sqrt{2}d$, the shift is nonpositive.

Thus the maximum occurs directly above the electron, where the tip–electron distance is d_0 . At that point,

$$f_z = \frac{k_e qQ}{d_0^2}, \quad c_3 = -\frac{2k_e qQ}{d_0^3}.$$

Substitution into part (e) yields

$$(\omega_0 + \Delta\omega_0)^2 = \omega_0^2 + \frac{2k_e qQ}{md_0^3}.$$

Consequently,

$$2\omega_0\Delta\omega_0 + (\Delta\omega_0)^2 = \frac{2k_e qQ}{md_0^3}.$$

Since $\Delta\omega_0 \ll \omega_0$, we neglect the squared shift:

$$d_0 \simeq \left(\frac{k_e qQ}{m\omega_0\Delta\omega_0} \right)^{1/3}.$$

Numerically,

$$\omega_0 = 1.0 \times 10^6 \text{ s}^{-1},$$

and

$$k_e qQ = (9.0 \times 10^9) 6(1.6 \times 10^{-19})^2 = 1.3824 \times 10^{-27} \text{ N m}^2.$$

Therefore,

$$d_0^3 \simeq \frac{1.3824 \times 10^{-27}}{(1.0 \times 10^{-12})(1.0 \times 10^6)(20)} \text{ m}^3 = 6.912 \times 10^{-23} \text{ m}^3,$$

giving

$$d_0 \simeq 4.1039 \times 10^{-8} \text{ m} \implies \boxed{d_0 \simeq 41 \text{ nm}}.$$

The relative frequency shift is $20/10^6 = 2.0 \times 10^{-5}$, confirming the validity of the small-shift approximation.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.