

IPhO 2003

Theory problems and solutions · Taipei, Taiwan · 3 theory problems with complete solutions

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Problem 1: A Swing with a Falling Weight

A rigid cylindrical rod of radius R is held horizontal above the ground. With a string of negligible mass and length L ($L > 2\pi R$), a pendulum bob of mass m is suspended from point A at the top of the rod as shown in Figure 1a. The bob is raised until it is level with A and then released from rest when the string is taut. Neglect any stretching of the string. Assume the pendulum bob may be treated as a mass point and swings only in a plane perpendicular to the axis of the rod. Accordingly, the pendulum bob is also referred to as the *particle*. The acceleration of gravity is $\vec{g}$.

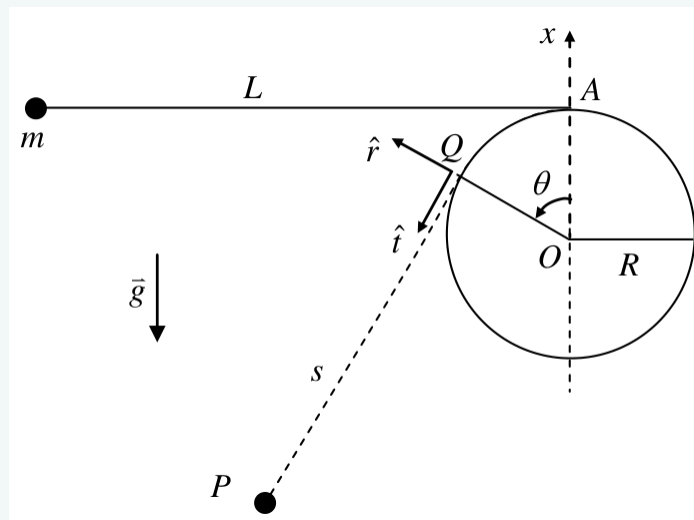


Figure 1a

Let O be the origin of the coordinate system. When the particle is at point P , the string is tangential to the cylindrical surface at Q . The length of the line segment QP is called s . The unit tangent vector and the unit radial vector at Q are given by $\hat{t}$ and $\hat{r}$, respectively. The angular displacement θ of the radius OQ , as measured counterclockwise from the vertical x -axis along OA , is taken to be positive.

When $\theta = 0$, the length s is equal to L and the gravitational potential energy U of the particle is zero. As the particle moves, the instantaneous time rates of change of θ and s are given by $\dot{\theta}$ and $\dot{s}$, respectively.

Unless otherwise stated, all the speeds and velocities are relative to the fixed point O .

Part A

In Part A, the string is taut as the particle moves. In terms of the quantities introduced above (i.e., $s, \theta, \dot{s}, \dot{\theta}, R, L, g, \hat{t}$ and $\hat{r}$): find:

(a) The relation between $\dot{\theta}$ and $\dot{s}$. (0.5 point)

- (b) The velocity $\vec{v}_Q$ of the moving point Q relative to O . (0.5 point)
- (c) The particle's velocity $\vec{v}'$ relative to the moving point Q when it is at P . (0.7 point)
- (d) The particle's velocity $\vec{v}$ relative to O when it is at P . (0.7 point)
- (e) The $\hat{t}$ -component of the particle's acceleration relative to O when it is at P . (0.7 point)
- (f) The particle's gravitational potential energy U when it is at P . (0.5 point)
- (g) The speed v_m of the particle at the lowest point of its trajectory. (0.7 point)

Part B

In Part B, the ratio L to R has the following value:

$$\frac{L}{R} = \frac{9\pi}{8} + \frac{2}{3} \cot \frac{\pi}{16} = 3.534 + 3.352 = 6.886$$

- (h) What is the speed v_s of the particle when the string segment from Q to P is both straight and shortest in length? (in terms of g and R) (2.4 points)
- (i) What is the speed v_H of the particle at its highest point H when it has swung to the other side of the rod? (in terms of g and R) (1.9 points)

Part C

In Part C, instead of being suspended from A , the pendulum bob of mass m is connected by a string over the top of the rod to a heavier weight of mass M , as shown in Figure 1b. The weight can also be treated as a particle.

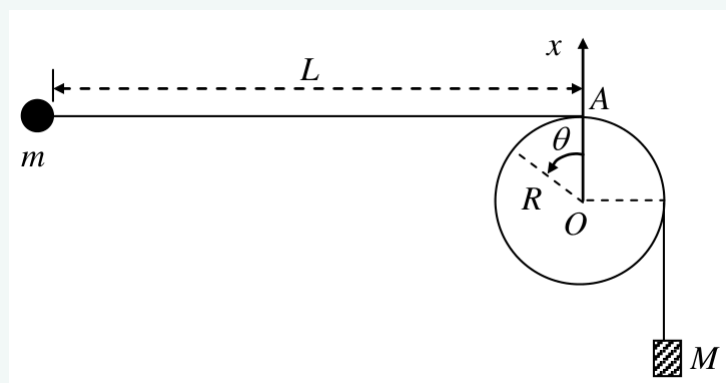


Figure 1b

Initially, the bob is held stationary at the same level as A so that, with the weight hanging below O , the string is taut with a horizontal section of length L . The bob is then released from rest and the weight starts falling. Assume that the bob remains in a vertical plane and can swing past the falling weight without any interruption.

The kinetic friction between the string and the rod surface is negligible. But the static friction is assumed to be large enough so that the weight will remain stationary once it has come to a stop (i.e. zero velocity).

- (j) Assume that the weight indeed comes to a stop after falling a distance D and that $(L - D) \gg R$. If the particle can then swing around the rod to $\theta = 2\pi$ while both segments of the string free from the rod remain straight, the ratio $\alpha = D/L$ must not be smaller than a critical

value α_c . Neglecting terms of the order R/L or higher, obtain an estimate on α_c in terms of M/m . (3.4 points)

Official English text of the problem.

Answers

- (a) $\dot{s} = -R\dot{\theta}$.
- (b) $\vec{v}_Q = R\dot{\theta}\hat{t}$.
- (c) $\vec{v}' = \dot{s}\hat{t} - s\dot{\theta}\hat{r}$.
- (d) $\vec{v} = -s\dot{\theta}\hat{r}$.
- (e) $a_{\hat{t}} = -s\dot{\theta}^2$.
- (f) $U = mg[R(\cos\theta - 1) - s\sin\theta]$.
- (g) $v_m = \sqrt{2g\left[L + \left(1 - \frac{\pi}{2}\right)R\right]}$.
- (h) $v_s = \frac{2}{\sqrt{3}}\cos\frac{\pi}{16}\sqrt{gR} \approx 1.133\sqrt{gR}$, at $\theta_s = \frac{9\pi}{8}$.
- (i) $v_H = \frac{2}{\sqrt{3}}\cos\frac{\pi}{16}\sin\frac{\pi}{8}\sqrt{gR} \approx 0.4334\sqrt{gR}$.
- (j) $\alpha \geq \alpha_c$, $\alpha_c \simeq \frac{3}{2M/m + 3}$.

The speed factors $\sqrt{gR}$ have units of m s^{-1} when g and R are expressed in SI units.

Solution

Take $\hat{x}$ vertically upward and introduce $\hat{y}$ horizontally to the left. The unit vectors shown in Figure 1a are then

$$\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}, \quad \hat{t} = -\sin\theta\hat{x} + \cos\theta\hat{y}.$$

Consequently,

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{t}, \quad \frac{d\hat{t}}{dt} = -\dot{\theta}\hat{r},$$

and the particle's position is

$$\vec{OP} = R\hat{r} + s\hat{t}.$$

(a) The length wrapped around the rod is $R\theta$. Conservation of string length gives

$$s + R\theta = L.$$

Differentiating,

$$\boxed{\dot{s} = -R\dot{\theta}}.$$

(b) Since $\vec{OQ} = R\hat{r}$,

$$\boxed{\vec{v}_Q = \frac{d}{dt}(R\hat{r}) = R\dot{\theta}\hat{t}}.$$

Here Q denotes the changing geometrical contact point; it need not remain the same point on the stationary rod.

(c) The displacement of the particle relative to Q is $s\hat{t}$. Differentiating this vector in fixed axes gives

$$\vec{v}' = \frac{d}{dt}(s\hat{t}) = \dot{s}\hat{t} + s\frac{d\hat{t}}{dt}.$$

Thus

$$\boxed{\vec{v}' = \dot{s}\hat{t} - s\dot{\theta}\hat{r}}.$$

(d) Adding the velocity of Q ,

$$\vec{v} = \vec{v}_Q + \vec{v}' = (R\dot{\theta} + \dot{s})\hat{t} - s\dot{\theta}\hat{r}.$$

Using part (a), the $\hat{t}$ -component cancels:

$$\boxed{\vec{v} = -s\dot{\theta}\hat{r}}.$$

In particular, during the initial swing with $\dot{\theta} > 0$,

$$v = s\dot{\theta}.$$

(e) Differentiate the velocity:

$$\vec{a} = -(\dot{s}\dot{\theta} + s\ddot{\theta})\hat{r} - s\dot{\theta}^2\hat{t}.$$

Therefore,

$$\boxed{a_{\hat{t}} = -s\dot{\theta}^2 = -\frac{v^2}{s}}.$$

The negative sign indicates acceleration toward Q along the straight string.

(f) From the position vector, the particle's vertical coordinate is

$$x = R\cos\theta - s\sin\theta.$$

Initially $x = R$, where the potential energy is defined to be zero. Hence

$$\boxed{U = mg(x - R) = mg[R(\cos\theta - 1) - s\sin\theta]}.$$

(g) The tension is along $-\hat{t}$, whereas the velocity is along $\hat{r}$, so tension does no work. Conservation of mechanical energy gives

$$\frac{1}{2}mv^2 + U = 0,$$

or

$$v^2 = 2g[R(1 - \cos\theta) + s\sin\theta].$$

To locate the lowest point, differentiate the height using $ds/d\theta = -R$:

$$\frac{dx}{d\theta} = -R\sin\theta - \frac{ds}{d\theta}\sin\theta - s\cos\theta = -s\cos\theta.$$

The height decreases before $\theta = \pi/2$ and increases immediately afterward. Thus the lowest point occurs at

$$\theta_m = \frac{\pi}{2}, \quad s_m = L - \frac{\pi R}{2}, \quad x_m = -s_m.$$

The vertical drop from release is $R + s_m$, giving

$$v_m = \sqrt{2g(R + s_m)} = \sqrt{2g \left[L + \left(1 - \frac{\pi}{2} \right) R \right]}.$$

(h) Let T denote the tension. Gravity has component $mg \sin \theta$ along $\hat{t}$, so Newton's second law and part (e) give

$$mg \sin \theta - T = -m \frac{v^2}{s}.$$

Therefore,

$$T = m \left(\frac{v^2}{s} + g \sin \theta \right).$$

Substituting the energy expression,

$$\frac{T}{mg} = \frac{2R(1 - \cos \theta)}{s} + 3 \sin \theta.$$

As the bob winds around the rod, $s = L - R\theta$ decreases. A string cannot exert negative tension, so this constrained motion ends when T first returns to zero. At that instant the string is still straight and has its shortest length on the initial swing.

The tension is positive for $0 < \theta \leq \pi$. Write its first subsequent zero as

$$\theta_s = \pi + \delta.$$

The condition $T = 0$ becomes

$$2R(1 + \cos \delta) - 3s_s \sin \delta = 0,$$

hence

$$\frac{s_s}{R} = \frac{2}{3} \frac{1 + \cos \delta}{\sin \delta} = \frac{2}{3} \cot \frac{\delta}{2}.$$

Since $L = s_s + R\theta_s$,

$$\frac{L}{R} = \pi + \delta + \frac{2}{3} \cot \frac{\delta}{2}.$$

Comparison with the specified exact value of L/R gives the relevant first root

$$\boxed{\delta = \frac{\pi}{8}}, \quad \boxed{\theta_s = \frac{9\pi}{8}},$$

and

$$s_s = \frac{2R}{3} \cot \frac{\pi}{16} \approx 3.352R.$$

At zero tension,

$$v_s^2 = -gs_s \sin \theta_s = gs_s \sin \frac{\pi}{8}.$$

Using $\sin(2a) \cot a = 2 \cos^2 a$,

$$v_s^2 = \frac{4gR}{3} \cos^2 \frac{\pi}{16}.$$

Thus

$$v_s = \frac{2}{\sqrt{3}} \cos \frac{\pi}{16} \sqrt{gR} \approx 1.133 \sqrt{gR}.$$

(i) At the instant found in part (h),

$$\hat{r}_s = -\cos \frac{\pi}{8} \hat{x} - \sin \frac{\pi}{8} \hat{y}.$$

Since $\vec{v}_s = -v_s \hat{r}_s$, the upward and horizontal velocity components are

$$v_{x,s} = v_s \cos \frac{\pi}{8}, \quad v_{y,s} = v_s \sin \frac{\pi}{8}.$$

The string goes slack, and the particle follows a projectile trajectory. Its horizontal velocity stays constant, while its upward velocity reaches zero after

$$t_H = \frac{v_s}{g} \cos \frac{\pi}{8}.$$

At the highest point, its speed is therefore just its horizontal speed:

$$v_H = v_s \sin \frac{\pi}{8} = \frac{2}{\sqrt{3}} \cos \frac{\pi}{16} \sin \frac{\pi}{8} \sqrt{gR} \approx 0.4334 \sqrt{gR}.$$

The continued slackness during this ascent can also be checked geometrically. Define

$$k = \frac{s_s}{R}, \quad u = \frac{v_s}{\sqrt{gR}}, \quad c = \cos \frac{\pi}{8}, \quad b = \sin \frac{\pi}{8}, \quad \tau = t \sqrt{\frac{g}{R}}.$$

The projectile coordinates $X = x/R$, $Y = y/R$ are

$$X = -c + kb + uc\tau - \frac{1}{2}\tau^2, \quad Y = -b - kc + ub\tau, \quad 0 \leq \tau \leq uc.$$

For a point on this trajectory, let

$$\sigma = \sqrt{X^2 + Y^2 - 1}, \quad \phi = 2\pi + \operatorname{atan} 2(Y, X).$$

The string length required for a wrapped arc followed by a straight tangent is

$$\frac{\ell_{\text{req}}}{R} = \phi - \arctan \sigma + \sigma.$$

Substitution and differentiation show that this length decreases throughout the ascent: it starts at $L/R = 6.88585$ and reaches 6.75065 at H . Thus the string remains slack up to the highest point, as required for the projectile calculation.

(j) Neglecting the specified corrections due to the rod radius, the bob's distance from the effective pivot after the weight stops is

$$\ell = L - D.$$

The wrapped lengths and the vertical offsets associated with the rod are of order R ; neglecting them is justified by $(L - D) \gg R$.

The falling weight loses gravitational potential energy MgD . Kinetic friction is negligible, and the weight has zero kinetic energy when it stops. Thereafter, the stationary string contact does no work. Thus the bob's conserved mechanical energy, measured relative to its initial horizontal level, is MgD .

Its height at angle θ is approximately $-\ell \sin \theta$, so

$$\frac{1}{2}mv^2 - mg\ell \sin \theta = MgD.$$

Writing $\mu = M/m$, this gives

$$v^2 = 2g(\mu D + \ell \sin \theta).$$

For the subsequent circular motion, the tension is

$$T = m \left(\frac{v^2}{\ell} + g \sin \theta \right) = mg \left(\frac{2\mu D}{\ell} + 3 \sin \theta \right).$$

The restrictive position is the top of the loop, at $\theta = 3\pi/2$, where $\sin \theta = -1$. Keeping the string straight requires

$$T_{\text{top}} \geq 0,$$

or

$$\frac{2\mu D}{\ell} \geq 3.$$

Equivalently, at the top the bob must retain at least the speed $\sqrt{g\ell}$; its energy there must therefore be at least

$$mg\ell + \frac{1}{2}mg\ell = \frac{3}{2}mg\ell.$$

Using $\ell = L - D$ and $\alpha = D/L$,

$$\begin{aligned} 2\mu D &\geq 3(L - D), \\ (2\mu + 3)\alpha &\geq 3. \end{aligned}$$

Consequently,

$$\boxed{\alpha_c \simeq \frac{3}{2\mu + 3} = \frac{3}{2M/m + 3}, \quad \alpha \geq \alpha_c.}$$

Equality is the limiting case of zero tension at the top. The assumed static friction holds the heavier weight stationary during the subsequent swing.

The solution was checked against the official solution.

Problem 2: A Piezoelectric Crystal Resonator under an Alternating Voltage

Consider a uniform rod of unstressed length ℓ and cross-sectional area A (Figure 2a). Its length changes by $\Delta\ell$ when equal and opposite forces of magnitude F are applied to its ends faces normally. The stress T on the end faces is defined to be F/A . The fractional change in its length,

i.e., $\Delta\ell/\ell$, is called the strain S of the rod. In terms of stress and strain, Hooke's law may be expressed as

$$T = YS \quad \text{or} \quad \frac{F}{A} = Y \frac{\Delta\ell}{\ell}. \quad (1)$$

where Y is called the Young's modulus of the rod material. Note that a compressive stress T corresponds to $F < 0$ and a decrease in length (i.e., $\Delta\ell < 0$). Such a stress is thus negative in value and is related to the pressure p by $T = -p$.

For a uniform rod of density ρ , the speed of propagation of longitudinal waves (i.e., sound speed) along the rod is given by

$$u = \sqrt{Y/\rho}. \quad (2)$$

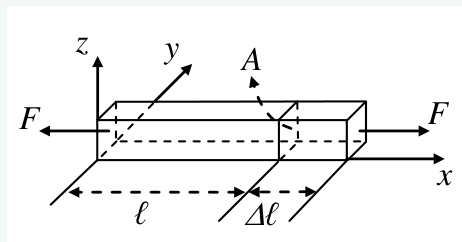


Figure 2a

The effect of damping and dissipation can be ignored in answering the following questions.

Part A: mechanical properties

A uniform rod of semi-infinite length, extending from $x = 0$ to ∞ (see Figure 2b), has a density ρ . It is initially stationary and unstressed. A piston then steadily exerts a small pressure p on its left face at $x = 0$ for a very short time Δt , causing a pressure wave to propagate with speed u to the right.

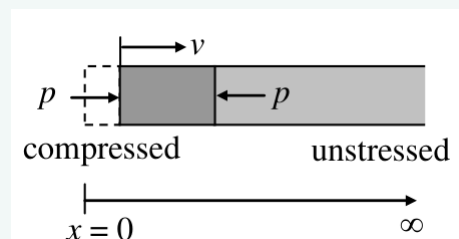


Figure 2b

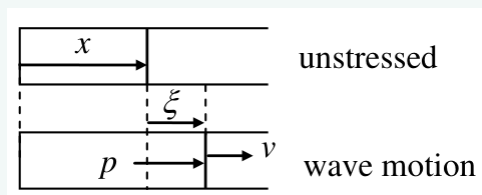


Figure 2c

(a) If the piston causes the rod's left face to move at a constant velocity v (Figure 2b), what are the strain S and pressure p at the left face during the time Δt ? *Answers must be given in terms of ρ , u , and v only.* (1.6 points)

(b) Consider a longitudinal wave traveling along the x direction in the rod. For a cross section at x when the rod is unstressed (Figure 2c), let $\xi(x, t)$ be its displacement at time t and assume

$$\xi(x, t) = \xi_0 \sin k(x - ut) \quad (3)$$

where ξ_0 and k are constants. Determine the corresponding velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ as a function of x and t . (2.4 points)

Part B: electromechanical properties (including piezoelectric effect)

Consider a quartz crystal slab of length b , thickness h , and width w (Figure 2d). Its length and thickness are along the x -axis and z -axis. Electrodes are formed by thin metallic coatings at its top and bottom surfaces. Electrical leads that also serve as mounting support (Figure 2e) are soldered to the electrode's centers, which may be assumed to be stationary for longitudinal oscillations along the x direction.

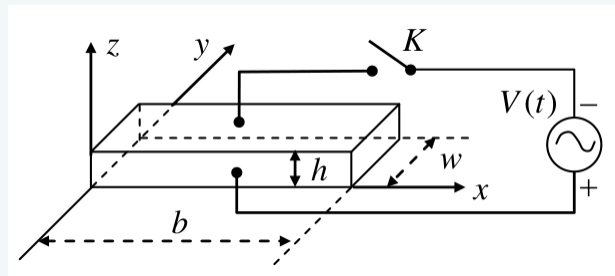


Figure 2d

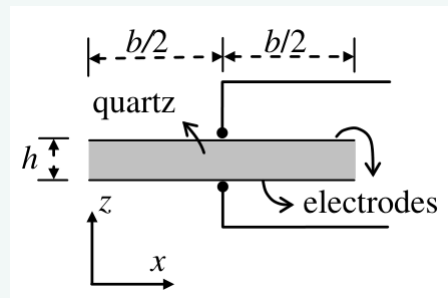


Figure 2e

The quartz crystal under consideration has a density ρ of $2.65 \times 10^3 \text{ kg/m}^3$ and Young's modulus Y of $7.87 \times 10^{10} \text{ N/m}^2$. The length b of the slab is 1.00 cm and the width w and height h of the slab are such that $h \ll w$ and $w \ll b$. With switch K left open, we assume only longitudinal modes of standing wave oscillation in the x direction are excited in the quartz slab.

For a standing wave of frequency $f = \omega/2\pi$, the displacement $\xi(x, t)$ at time t of a cross section of the slab with equilibrium position x may be written as

$$\xi(x, t) = 2\xi_0 g(x) \cos \omega t, \quad (0 \leq x \leq b) \quad (4a)$$

where ξ_0 is a positive constant and the spatial function $g(x)$ is of the form

$$g(x) = B_1 \sin k \left(x - \frac{b}{2} \right) + B_2 \cos k \left(x - \frac{b}{2} \right). \quad (4b)$$

$g(x)$ has the maximum value of one and $k = \omega/u$. Keep in mind that the centers of the electrodes are stationary and the left and right faces of the slab are free and must have zero stress (or pressure).

(c) Determine the values of B_1 and B_2 in Eq. (4b) for a longitudinal standing wave in the quartz slab. (1.2 point)

(d) What are the two lowest frequencies at which longitudinal standing waves may be excited in the quartz slab? (1.2 point)

The piezoelectric effect is a special property of a quartz crystal. Compression or dilatation of the crystal generates an electric voltage across the crystal, and conversely, an external voltage applied across the crystal causes the crystal to expand or contract depending on the polarity of the voltage. Therefore, mechanical and electrical oscillations can be coupled and made to resonate through a quartz crystal.

To account for the piezoelectric effect, let the surface charge densities on the upper and lower electrodes be $-\sigma$ and $+\sigma$, respectively, when the quartz slab is under an electric field E in the z direction. Denote the slab's strain and stress in the x direction by S and T , respectively. Then the piezoelectric effect of the quartz crystal can be described by the following set of equations:

$$S = (1/Y)T + d_p E \quad (5a)$$

$$\sigma = d_p T + \varepsilon_T E \quad (5b)$$

where $1/Y = 1.27 \times 10^{-11} \text{ m}^2/\text{N}$ is the elastic compliance (i.e., inverse of Young's modulus) at constant electric field and $\varepsilon_T = 4.06 \times 10^{-11} \text{ F/m}$ is the permittivity at constant stress, while $d_p = 2.25 \times 10^{-12} \text{ m/V}$ is the piezoelectric coefficient.

Let switch K in Fig. 2d be closed. The alternating voltage $V(t) = V_m \cos \omega t$ now acts across the electrodes and a uniform electric field $E(t) = V(t)/h$ in the z direction appears in the quartz slab. When a steady state is reached, a longitudinal standing wave oscillation of angular frequency ω appears in the slab in the x direction.

With E being uniform, the wavelength λ and the frequency f of a longitudinal standing wave in the slab are still related by $\lambda = u/f$ with u given by Eq. (2). But, as Eq. (5a) shows, $T = YS$ is no longer valid, although the definitions of strain and stress remain unchanged and the end faces of the slab remain free with zero stress.

(e) Taking Eqs. (5a) and (5b) into account, the surface charge density σ on the lower electrode as a function of x and t is of the form,

$$\sigma(x, t) = \left[D_1 \cos k \left(x - \frac{b}{2} \right) + D_2 \right] \frac{V(t)}{h}$$

where $k = \omega/u$. Find the expressions for D_1 and D_2 . (2.2 points)

(f) The total surface charge $Q(t)$ on the lower electrode is related to $V(t)$ by

$$Q(t) = \left[1 + \alpha^2 \left(\frac{2}{kb} \tan \frac{kb}{2} - 1 \right) \right] C_0 V(t). \quad (6)$$

Find the expression for C_0 and the expression and numerical value of α^2 . (1.4 points)

Official English text of the problem.

Answers

- (a)

$$S = -\frac{v}{u}, \quad p = \rho uv.$$

- (b)

$$\begin{aligned} v(x, t) &= -ku\xi_0 \cos k(x - ut), \\ S(x, t) &= k\xi_0 \cos k(x - ut), \\ p(x, t) &= -\rho u^2 k\xi_0 \cos k(x - ut). \end{aligned}$$

- (c)

$$B_2 = 0, \quad B_1 = \pm 1.$$

The overall sign is an arbitrary oscillation phase; one may choose $B_1 = 1$.

- (d)

$$f_{\text{lowest}} = \frac{1}{2b} \sqrt{\frac{Y}{\rho}} = 2.72 \times 10^5 \text{ Hz}, \quad f_{\text{next}} = \frac{3}{2b} \sqrt{\frac{Y}{\rho}} = 8.17 \times 10^5 \text{ Hz}.$$

- (e)

$$D_1 = \frac{Yd_p^2}{\cos(kb/2)}, \quad D_2 = \varepsilon_T - Yd_p^2.$$

- (f)

$$C_0 = \frac{\varepsilon_T bw}{h}, \quad \alpha^2 = \frac{Yd_p^2}{\varepsilon_T} = \frac{d_p^2}{\varepsilon_T(1/Y)} \simeq 9.82 \times 10^{-3}.$$

Here α^2 is dimensionless.

Solution

We use the one-dimensional, small-strain approximation specified in the problem. Since x labels a cross section in its unstressed position, its velocity and longitudinal strain are

$$v(x, t) = \frac{\partial \xi}{\partial t}, \quad S(x, t) = \frac{\partial \xi}{\partial x}.$$

For the purely mechanical model, $T = YS$, $p = -T$, and $Y = \rho u^2$.

- (a)

During the time Δt , the disturbance travels a distance $u\Delta t$. To first order in the small deformation, the mass set into motion at speed v is therefore

$$m = \rho A u \Delta t.$$

The external impulse on the rod equals its acquired momentum:

$$p A \Delta t = mv = \rho A u \Delta t v.$$

Thus

$$p = \rho uv.$$

The stress is compressive, so $T = -p$. Hooke's law then gives

$$S = -\frac{p}{Y} = -\frac{\rho uv}{\rho u^2},$$

and hence

$$S = -\frac{v}{u}.$$

The sign also follows geometrically: the originally unstressed length $u\Delta t$ is shortened by the piston displacement $v\Delta t$.

(b)

Differentiating the displacement with respect to time gives the material velocity:

$$v(x, t) = \frac{\partial}{\partial t} [\xi_0 \sin k(x - ut)] = \boxed{-ku\xi_0 \cos k(x - ut)}.$$

Differentiating with respect to the unstressed coordinate gives the strain:

$$S(x, t) = \frac{\partial}{\partial x} [\xi_0 \sin k(x - ut)] = \boxed{k\xi_0 \cos k(x - ut)}.$$

Finally,

$$p(x, t) = -YS(x, t),$$

so

$$\boxed{p(x, t) = -\rho u^2 k \xi_0 \cos k(x - ut)}.$$

In particular, $p(x, t) = \rho u v(x, t)$, consistent with part (a) for a wave traveling to the right.

(c)

The midpoint is stationary, so

$$\xi(b/2, t) = 0 \implies g(b/2) = 0.$$

Substituting $x = b/2$ in Eq. (4b) gives

$$\boxed{B_2 = 0}.$$

Consequently,

$$g(x) = B_1 \sin k \left(x - \frac{b}{2} \right).$$

In the mechanical model, the free ends have zero stress and hence zero strain. Therefore,

$$g'(0) = g'(b) = 0.$$

Since

$$g'(x) = B_1 k \cos k \left(x - \frac{b}{2} \right),$$

a nontrivial oscillatory mode requires

$$\cos \frac{kb}{2} = 0.$$

For these allowed values of k , the sine function attains both $+1$ and -1 within the slab. The normalization $\max g(x) = 1$ thus implies

$$\boxed{|B_1| = 1}.$$

Either sign is possible. Changing B_1 to $-B_1$ is equivalent to shifting the oscillation phase by π ; we may choose $B_1 = 1$.

(d)

The free-end condition obtained above gives

$$\frac{kb}{2} = \frac{(2n+1)\pi}{2}, \quad n = 0, 1, 2, \dots$$

or

$$k_n = \frac{(2n+1)\pi}{b}.$$

Using $\omega = uk$ and $f = \omega/(2\pi)$,

$$f_n = \frac{(2n+1)u}{2b}.$$

Thus the permitted modes are the odd harmonics. The two lowest frequencies correspond to $n = 0$ and $n = 1$.

The sound speed is

$$u = \sqrt{\frac{7.87 \times 10^{10}}{2.65 \times 10^3}} = 5449.6 \text{ m s}^{-1}.$$

With $b = 1.00 \times 10^{-2} \text{ m}$,

$$f_{\text{lowest}} = \frac{5449.6}{2(1.00 \times 10^{-2})} = \boxed{2.72 \times 10^5 \text{ Hz}},$$

and

$$f_{\text{next}} = \frac{3(5449.6)}{2(1.00 \times 10^{-2})} = \boxed{8.17 \times 10^5 \text{ Hz}}.$$

(e)

With the piezoelectric effect included, Eq. (5a) gives

$$T = Y(S - d_p E) = Y \left(\frac{\partial \xi}{\partial x} - d_p E \right).$$

Newton's second law for an element of the slab is

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T}{\partial x}.$$

Because E is spatially uniform,

$$\rho \frac{\partial^2 \xi}{\partial t^2} = Y \frac{\partial^2 \xi}{\partial x^2}.$$

For the harmonic response at the driving frequency, write

$$\xi(x, t) = H(x)E(t).$$

Since $\ddot{E} = -\omega^2 E$, the spatial function satisfies

$$H'' + k^2 H = 0, \quad k = \frac{\omega}{u}.$$

The fixed midpoint requires $H(b/2) = 0$, so

$$H(x) = C \sin k \left(x - \frac{b}{2} \right).$$

The strain is therefore

$$S(x, t) = Ck \cos k \left(x - \frac{b}{2} \right) E(t).$$

The free-end condition is now $T = 0$, which means

$$S(0, t) = S(b, t) = d_p E(t).$$

It follows that

$$Ck \cos \frac{kb}{2} = d_p,$$

and thus

$$S(x, t) = d_p \frac{\cos k(x - b/2)}{\cos(kb/2)} E(t).$$

The stress becomes

$$T(x, t) = Y d_p \left[\frac{\cos k(x - b/2)}{\cos(kb/2)} - 1 \right] E(t).$$

Substitution into Eq. (5b) gives

$$\begin{aligned} \sigma(x, t) &= d_p T(x, t) + \varepsilon_T E(t) \\ &= \left[\frac{Y d_p^2}{\cos(kb/2)} \cos k \left(x - \frac{b}{2} \right) + \varepsilon_T - Y d_p^2 \right] \frac{V(t)}{h}. \end{aligned}$$

Comparing with the prescribed form,

$$\boxed{D_1 = \frac{Y d_p^2}{\cos(kb/2)}, \quad D_2 = \varepsilon_T - Y d_p^2.}$$

Both coefficients have units of permittivity, F/m.

These expressions describe the harmonic particular solution away from exact resonance. When $\cos(kb/2) = 0$, the boundary condition cannot be satisfied with a finite harmonic amplitude under a nonzero applied voltage. The poles therefore express the absence of a bounded steady-state response at resonance in the undamped model.

(f)

To linear order, an electrode strip of length dx has area $w dx$. Hence the total charge on the lower electrode is

$$Q(t) = w \int_0^b \sigma(x, t) dx.$$

The spatial integral needed is

$$\int_0^b \cos k \left(x - \frac{b}{2} \right) dx = \frac{2}{k} \sin \frac{kb}{2}.$$

Using the result from part (e),

$$Q(t) = \frac{wV(t)}{h} \left[\frac{2Yd_p^2}{k} \tan \frac{kb}{2} + b(\varepsilon_T - Yd_p^2) \right].$$

Factoring out $\varepsilon_T bw/h$,

$$Q(t) = \frac{\varepsilon_T bw}{h} \left[1 + \frac{Yd_p^2}{\varepsilon_T} \left(\frac{2}{kb} \tan \frac{kb}{2} - 1 \right) \right] V(t).$$

Comparison with Eq. (6) yields

$$\boxed{C_0 = \frac{\varepsilon_T bw}{h}}, \quad \boxed{\alpha^2 = \frac{Yd_p^2}{\varepsilon_T}}.$$

Using the stated elastic compliance,

$$\begin{aligned} \alpha^2 &= \frac{d_p^2}{\varepsilon_T(1/Y)} \\ &= \frac{(2.25 \times 10^{-12})^2}{(4.06 \times 10^{-11})(1.27 \times 10^{-11})} \\ &= 0.00981828 \simeq \boxed{9.82 \times 10^{-3}}. \end{aligned}$$

With the given length, the capacitance expression can also be written as

$$C_0 = (4.06 \times 10^{-13} \text{ F}) \frac{w}{h}.$$

As a consistency check, in the quasistatic limit $kb \rightarrow 0$,

$$\frac{2}{kb} \tan \frac{kb}{2} \longrightarrow 1,$$

so $Q(t) \rightarrow C_0 V(t)$, as expected for a stress-free slab with permittivity ε_T .

The solution was checked against the official solution.

Problem 3: Neutrino Mass and Neutron Decay (Part A); Light Levitation (Part B)

Part A

Neutrino Mass and Neutron Decay

A free neutron of mass m_n decays at rest in the laboratory frame of reference into three non-interacting particles: a proton, an electron, and an anti-neutrino. The rest mass of the proton is m_p , while the rest mass of the anti-neutrino m_v is assumed to be nonzero and much smaller than the rest mass of the electron m_e . Denote the speed of light in vacuum by c . The measured values of mass are as follows:

$$m_n = 939.56563 \text{ MeV}/c^2, \quad m_p = 938.27231 \text{ MeV}/c^2, \quad m_e = 0.5109907 \text{ MeV}/c^2$$

In the following, all energies and velocities are referred to the laboratory frame. Let E be the total energy of the electron coming out of the decay.

(a) Find the maximum possible value $E_{\max}$ of E and the speed v_m of the anti-neutrino when $E = E_{\max}$. Both answers must be expressed in terms of the rest masses of the particles and the speed of light. Given that $m_v < 7.3 \text{ eV}/c^2$, compute $E_{\max}$ and the ratio v_m/c to 3 significant digits.

(4.0 points)

Part B

Light Levitation

A transparent glass hemisphere with radius R and mass m has an index of refraction n . In the medium outside the hemisphere, the index of refraction is equal to one. A parallel beam of monochromatic laser light is incident uniformly and normally onto the central portion of its planar surface, as shown in Figure 3. The acceleration of gravity $\vec{g}$ is vertically downwards. The radius δ of the circular cross-section of the laser beam is much smaller than R . Both the glass hemisphere and the laser beam are axially symmetric with respect to the z -axis.

The glass hemisphere does not absorb any laser light. Its surface has been coated with a thin layer of transparent material so that reflections are negligible when light enters and leaves the glass hemisphere. The optical path traversed by laser light passing through the non-reflecting surface layer is also negligible.

(b) Neglecting terms of the order $(\delta/R)^3$ or higher, find the laser power P needed to balance the weight of the glass hemisphere.

(4.0 points)

Hint: $\cos \theta \approx 1 - \theta^2/2$ when θ is much smaller than one.

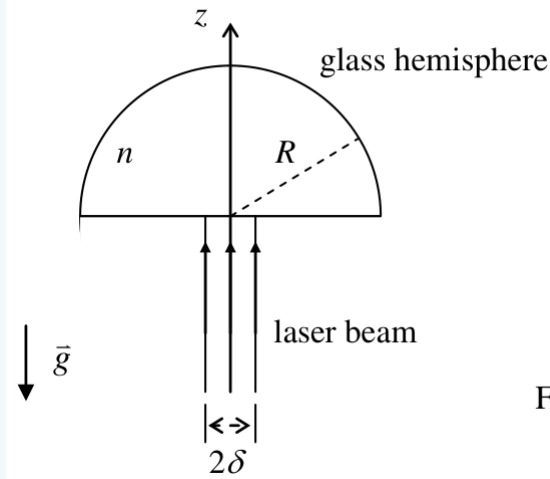


Figure 3

Figure 3

Official English text of the problem.

Answers

- (a) The maximum total electron energy is

$$E_{\max} = \frac{m_n^2 + m_e^2 - (m_p + m_v)^2}{2m_n} c^2.$$

At this endpoint, the anti-neutrino and proton have the same velocity, with speed

$$v_m = c \frac{\sqrt{[m_n^2 - (m_p + m_v + m_e)^2][m_n^2 - (m_p + m_v - m_e)^2]}}{m_n^2 + (m_p + m_v)^2 - m_e^2}.$$

To three significant digits,

$$E_{\max} = 1.29 \text{ MeV}, \quad \frac{v_m}{c} = 1.27 \times 10^{-3}.$$

- (b) To the requested order,

$$P = \frac{4mgcR^2}{(n-1)^2\delta^2}.$$

Solution

(a) Neutrino Mass and Neutron Decay

Let the electron momentum have magnitude p_e . Conservation of energy and momentum gives

$$E_p + E_v = m_n c^2 - E, \quad \mathbf{p}_p + \mathbf{p}_v = -\mathbf{p}_e.$$

Denote the invariant mass of the proton–anti-neutrino pair by $\mathcal{M}$. Using the electron energy–momentum relation,

$$p_e^2 c^2 = E^2 - m_e^2 c^4,$$

we obtain

$$\begin{aligned}\mathcal{M}^2 c^4 &= (E_p + E_v)^2 - c^2 |\mathbf{p}_p + \mathbf{p}_v|^2 \\ &= (m_n c^2 - E)^2 - p_e^2 c^2 \\ &= (m_n^2 + m_e^2) c^4 - 2m_n c^2 E.\end{aligned}$$

Consequently,

$$E = \frac{m_n^2 + m_e^2 - \mathcal{M}^2}{2m_n} c^2.$$

The electron energy is therefore largest when $\mathcal{M}$ is smallest.

In the pair's centre-of-momentum frame, the proton and anti-neutrino have equal and opposite momenta of magnitude q . Thus

$$\mathcal{M}c^2 = \sqrt{m_p^2 c^4 + q^2 c^2} + \sqrt{m_v^2 c^4 + q^2 c^2} \geq (m_p + m_v) c^2.$$

Equality holds when $q = 0$: both particles are at rest in this frame, so they have the **same velocity** in the laboratory frame. This configuration is kinematically allowed because $m_n > m_p + m_v + m_e$.

Writing $M = m_p + m_v$, the endpoint energy is therefore

$$E_{\max} = \frac{m_n^2 + m_e^2 - M^2}{2m_n} c^2.$$

At the endpoint, the pair has energy $m_n c^2 - E_{\max}$ and momentum magnitude p_e . Its common speed, which is the anti-neutrino speed, follows from $\mathbf{v} = c^2 \mathbf{p} / E_{\text{total}}$:

$$\frac{v_m}{c} = \frac{p_e c}{m_n c^2 - E_{\max}} = \frac{\sqrt{E_{\max}^2 - m_e^2 c^4}}{m_n c^2 - E_{\max}}.$$

Substitution gives

$$v_m = c \frac{\sqrt{[m_n^2 - (M + m_e)^2][m_n^2 - (M - m_e)^2]}}{m_n^2 + M^2 - m_e^2}, \quad M = m_p + m_v.$$

For the numerical evaluation, the limit $m_v \rightarrow 0^+$ gives

$$E_{\max} \simeq 1.29256882 \text{ MeV}, \quad \frac{v_m}{c} \simeq 1.26538356 \times 10^{-3}.$$

Increasing $m_v c^2$ from zero to its upper bound of 7.3 eV decreases these values by less than 7.30 eV and 8.5×10^{-9} , respectively. Hence the entire permitted mass range gives, to three significant digits,

$$E_{\max} = 1.29 \text{ MeV}, \quad v_m/c = 1.27 \times 10^{-3}.$$

Here $E_{\max}$ is the electron's **total** energy. The anti-neutrino is slow at this endpoint despite its tiny mass: its momentum is also tiny, as required for it to share the proton's velocity.

(b) Light Levitation

Consider a ray entering the hemisphere at distance r from the z -axis, where $0 \leq r \leq \delta$. It enters the planar surface normally and therefore continues vertically inside the glass.

At the spherical surface, let θ be the angle between the incident ray and the outward normal. Geometry gives

$$\sin \theta = \frac{r}{R}.$$

If θ' is the angle of the emerging ray relative to that normal, Snell's law gives

$$\sin \theta' = n \sin \theta = n \frac{r}{R}.$$

For glass with $n > 1$, the emerging ray bends towards the axis. Its angle from the upward vertical has magnitude

$$\alpha = \theta' - \theta.$$

Since $r/R \leq \delta/R \ll 1$, expansion at fixed n gives

$$\alpha = (n - 1) \frac{r}{R} + O\left(\frac{r^3}{R^3}\right).$$

Both before entering and after leaving the hemisphere, light has momentum equal to its energy divided by c . Absorption and reflection are negligible, so a ray bundle carrying power dP has incoming vertical momentum flux dP/c and outgoing vertical momentum flux $(dP/c) \cos \alpha$.

By momentum conservation, its upward force on the hemisphere is

$$dF_z = \frac{dP}{c} (1 - \cos \alpha).$$

Using the given small-angle approximation,

$$1 - \cos \alpha \simeq \frac{\alpha^2}{2} \simeq \frac{(n - 1)^2 r^2}{2R^2}.$$

The transverse force components cancel by axial symmetry.

The incident intensity is uniform:

$$I = \frac{P}{\pi \delta^2}.$$

An annulus of radius r and width dr therefore carries power

$$dP = I 2\pi r dr.$$

Integrating the vertical force over the beam,

$$\begin{aligned} F_z &\simeq \int_0^\delta \frac{I 2\pi r dr}{c} \frac{(n - 1)^2 r^2}{2R^2} \\ &= \frac{\pi I (n - 1)^2}{c R^2} \int_0^\delta r^3 dr \\ &= \frac{P (n - 1)^2 \delta^2}{4c R^2}. \end{aligned}$$

This retains the contribution of order $(\delta/R)^2$; the next correction to $F_z/(P/c)$ is of order $(\delta/R)^4$.

Balancing this upward force against the weight,

$$mg = \frac{P (n - 1)^2 \delta^2}{4c R^2},$$

yields

$$P = \frac{4mgcR^2}{(n-1)^2\delta^2}.$$

The solution was checked against the official solution.