

IPhO 2002

Theory problems and solutions · Bali, Indonesia · 3 theory problems with complete solutions

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Problem 1: Ground-Penetrating Radar

Ground-penetrating radar (GPR) is used to detect and locate underground objects near the surface by means of transmitting electromagnetic waves into the ground and receiving the waves reflected from those objects. The antenna and the detector are directly on the ground and they are located at the same point.

A linearly polarized electromagnetic plane wave of angular frequency ω propagating in the z direction is represented by the following expression for its field:

$$E = E_0 e^{-\alpha z} \cos(\omega t - \beta z), \quad (1)$$

where E_0 is constant, α is the attenuation coefficient and β is the wave number expressed respectively as follows

$$\alpha = \omega \left\{ \frac{\mu \varepsilon}{2} \left[\left(1 + \frac{\sigma^2}{\varepsilon^2 \omega^2} \right)^{1/2} - 1 \right] \right\}^{1/2}, \quad \beta = \omega \left\{ \frac{\mu \varepsilon}{2} \left[\left(1 + \frac{\sigma^2}{\varepsilon^2 \omega^2} \right)^{1/2} + 1 \right] \right\}^{1/2} \quad (2)$$

with μ , ε , and σ denoting the magnetic permeability, the electrical permittivity, and the electrical conductivity respectively.

The signal becomes undetected when the amplitude of the radar signal arriving at the object drops below $1/e$ ($\approx 37\%$) of its initial value. An electromagnetic wave of variable frequency (10 MHz – 1000 MHz) is usually used to allow adjustment of range and resolution of detection.

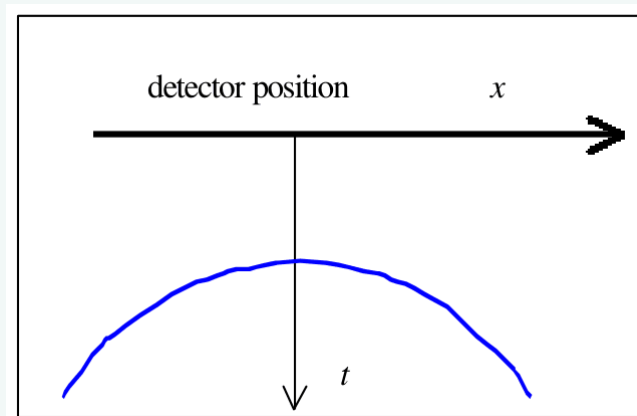
The performance of GPR depends on its resolution. The resolution is given by the minimum separation between the two adjacent reflectors to be detected. The minimum separation should give rise to a minimum phase difference of 180° between the two reflected waves at the detector.

Questions:

(Given : $\mu_0 = 4\pi \times 10^{-7}$ H/m and $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m)

1. Assume that the ground is non-magnetic ($\mu = \mu_0$) satisfying the condition $\left(\frac{\sigma}{\omega \varepsilon} \right)^2 \ll 1$.
Derive the expression of propagation speed v in terms of μ and ε , using equations (1) and (2) [1.0 pts].
2. Determine the maximum depth of detection of an object in the ground with conductivity of 1.0 mS/m and permittivity of $9\varepsilon_0$, satisfying the condition $\left(\frac{\sigma}{\omega \varepsilon} \right)^2 \ll 1$, ($S = \text{ohm}^{-1}$; use $\mu = \mu_0$). [2.0 pts]

3. Consider two parallel conducting rods buried horizontally in the ground. The rods are 4 meter deep. The ground is known to have conductivity of 1.0 mS/m and permittivity of $9\epsilon_0$. Suppose the GPR measurement is carried out at a position approximately above one of the rod. Assume point detector is used. Determine the minimum frequency required to get a lateral resolution of 50 cm **[3.5 pts]**.
4. To determine the depth of a buried rod d in the same ground, consider the measurements carried out along a line perpendicular to the rod. The result is described by the following figure:



Graph of traveltime t vs detector position x , $t_{\min} = 100$ ns.

Derive t as a function of x and determine d **[3.5 pts]**.

Official English text of the problem.

Answers

1. In the stated low-loss limit,

$$v \simeq \frac{1}{\sqrt{\mu\epsilon}}.$$

2. The maximum detection depth is

$$D_{\max} \simeq \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu_0}} = 15.9 \text{ m} \approx 16 \text{ m}.$$

3. With rod depth $h = 4$ m and lateral separation $s = 0.50$ m,

$$f_{\min} \simeq \frac{v}{4(\sqrt{h^2 + s^2} - h)} \approx 8.0 \times 10^8 \text{ Hz} = 800 \text{ MHz}.$$

4. Taking $x = 0$ directly above the rod,

$$t(x) = \frac{2}{v} \sqrt{d^2 + x^2} = \sqrt{t_{\min}^2 + \frac{4x^2}{v^2}}, \quad d = \frac{vt_{\min}}{2} \approx 5.0 \text{ m}.$$

Solution

1. Propagation speed

A point of constant phase satisfies

$$\omega t - \beta z = \text{constant}.$$

Differentiating gives the phase velocity:

$$v = \frac{dz}{dt} = \frac{\omega}{\beta}.$$

Define the small dimensionless parameter

$$q = \frac{\sigma}{\omega \varepsilon}, \quad q^2 \ll 1.$$

Using the binomial expansion,

$$\sqrt{1 + q^2} = 1 + \frac{q^2}{2} + O(q^4).$$

Equation (2) therefore gives

$$\beta = \omega \left[\frac{\mu \varepsilon}{2} \left(2 + \frac{q^2}{2} + O(q^4) \right) \right]^{1/2} = \omega \sqrt{\mu \varepsilon} \left(1 + \frac{q^2}{8} + O(q^4) \right).$$

Keeping the leading term,

$$\boxed{v \simeq \frac{1}{\sqrt{\mu \varepsilon}}}.$$

For non-magnetic ground, $\mu = \mu_0$. At this order, β is proportional to ω , so dispersion is negligible and the group velocity also equals this value.

2. Maximum detection depth

For the attenuation coefficient, retain the leading nonzero term of the same expansion:

$$\alpha \simeq \omega \left[\frac{\mu \varepsilon}{2} \frac{q^2}{2} \right]^{1/2} = \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon}}.$$

Thus, to this order, the attenuation coefficient is independent of frequency.

For an object directly below the antenna at depth D , equation (1) gives the incident amplitude ratio

$$\frac{E_{\text{amp}}(D)}{E_0} = e^{-\alpha D}.$$

The problem specifies the threshold for the signal **arriving at the object**, so the propagation distance entering this condition is D . At the limiting depth,

$$e^{-\alpha D_{\text{max}}} = \frac{1}{e},$$

and hence

$$D_{\text{max}} = \frac{1}{\alpha} \simeq \frac{2}{\sigma} \sqrt{\frac{\varepsilon}{\mu_0}}.$$

Using

$$\sigma = 1.0 \times 10^{-3} \text{ S/m}, \quad \varepsilon = 9(8.85 \times 10^{-12}) = 7.965 \times 10^{-11} \text{ F/m},$$

we obtain

$$\alpha \simeq \frac{1.0 \times 10^{-3}}{2} \sqrt{\frac{4\pi \times 10^{-7}}{7.965 \times 10^{-11}}} = 6.28 \times 10^{-2} \text{ m}^{-1}.$$

Therefore,

$$D_{\max} = 15.9 \text{ m} \approx 16 \text{ m}.$$

3. Minimum frequency for a lateral resolution of 50 cm

Work in the vertical plane perpendicular to the rods. Let their common depth be h , their lateral separation be s , and take the antenna–detector position directly above the first rod.

The one-way distances to the two rods are

$$r_1 = h, \quad r_2 = \sqrt{h^2 + s^2}.$$

Since transmitter and detector are at the same point, the reflected signals travel distances $2r_1$ and $2r_2$. Their path difference is therefore

$$\Delta L = 2 \left(\sqrt{h^2 + s^2} - h \right).$$

Treating the conducting rods as equivalent reflectors, their common reflection phase cancels. The phase difference at the detector is

$$\Delta\phi = \beta\Delta L = \frac{4\pi f}{v} \left(\sqrt{h^2 + s^2} - h \right).$$

The specified resolution criterion is $\Delta\phi \geq \pi$. The minimum frequency occurs at equality:

$$f_{\min} = \frac{v}{4 \left(\sqrt{h^2 + s^2} - h \right)}.$$

The wave speed in this ground is

$$v \simeq \frac{1}{\sqrt{\mu_0(9\varepsilon_0)}} = 9.995 \times 10^7 \text{ m/s}.$$

For $h = 4 \text{ m}$ and $s = 0.50 \text{ m}$,

$$\sqrt{h^2 + s^2} - h = \sqrt{4^2 + 0.50^2} - 4 = 0.0311289 \text{ m}.$$

Thus,

$$f_{\min} \simeq \frac{9.995 \times 10^7}{4(0.0311289)} = 8.027 \times 10^8 \text{ Hz},$$

or, to sensible precision,

$$f_{\min} \approx 8.0 \times 10^8 \text{ Hz} = 800 \text{ MHz}.$$

The small-separation approximation gives the same rounded result:

$$\sqrt{h^2 + s^2} - h \simeq \frac{s^2}{2h} \implies f_{\min} \simeq \frac{vh}{2s^2}.$$

The required frequency lies within the stated 10–1000 MHz range. Moreover, at the calculated frequency,

$$\left(\frac{\sigma}{2\pi f_{\min}\epsilon}\right)^2 \simeq 6.2 \times 10^{-6} \ll 1,$$

so the low-loss approximation is consistent. Both rods also lie within the detection range found in part 2.

4. Travel-time curve and rod depth

Choose the horizontal coordinate $x = 0$ directly above the buried rod. In homogeneous ground, the one-way distance from the antenna to the rod is

$$r(x) = \sqrt{d^2 + x^2}.$$

The measured travel time includes the outward and return journeys:

$$t(x) = \frac{2}{v} \sqrt{d^2 + x^2}.$$

The shortest path occurs at $x = 0$, giving

$$t_{\min} = \frac{2d}{v}.$$

Consequently,

$$d = \frac{vt_{\min}}{2} = \frac{(9.995 \times 10^7)(100 \times 10^{-9})}{2} \text{ m} = 4.998 \text{ m}.$$

Thus,

$$d \approx 5.0 \text{ m}.$$

Substituting the minimum travel time into the curve gives

$$t(x) = \sqrt{t_{\min}^2 + \frac{4x^2}{v^2}} \simeq (100 \text{ ns}) \sqrt{1 + \left(\frac{x}{5.0 \text{ m}}\right)^2}.$$

This is the positive-time branch of the hyperbola

$$\frac{t^2}{t_{\min}^2} - \frac{x^2}{d^2} = 1.$$

It is symmetric about $x = 0$, has its minimum travel time of 100 ns directly above the rod, and increases with $|x|$. For $|x| \gg d$,

$$t(x) \sim \frac{2|x|}{v}.$$

Because the figure's time axis points downward, larger travel times appear lower on the page, as shown.

The solution was checked against the official solution.

Problem 2: Sensing Electrical Signals

Some seawater animals have the ability to detect other creatures at some distance away due to electric currents produced by the creatures during the breathing processes or other processes involving muscular contraction. Some predators use this electrical signal to locate their preys, even when buried under the sands.

The physical mechanism underlying the current generation at the prey and its detection by the predator can be modeled as described by Figure II-1. The current generated by the prey flows between two spheres with positive and negative potential in the prey's body. The distance between the centers of the two spheres is l_s , each having a radius of r_s , which is much smaller than l_s . The seawater resistivity is ρ . Assume that the resistivity of the prey's body is the same as that of the surrounding seawater, implying that the boundary surrounding the prey in the figure can be ignored.

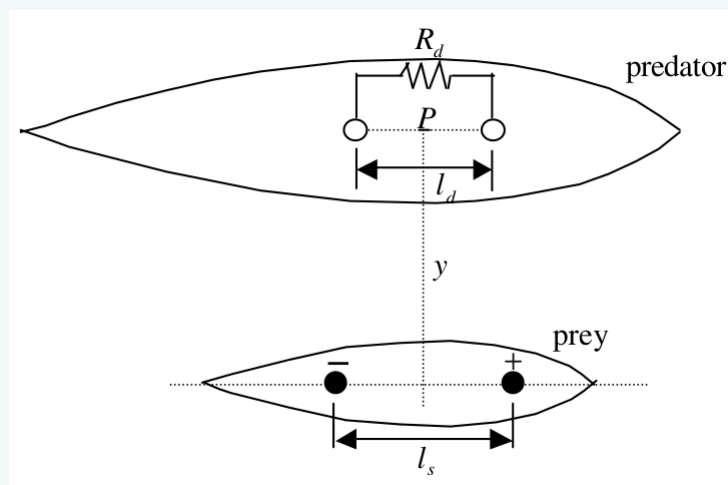


Figure II-1. A model describing the detection of electric power coming from a prey by its predator.

In order to describe the detection of electric power by the predator coming from the prey, the detector is modeled similarly by two spheres on the predator's body and in contact with the surrounding seawater, lying parallel to the pair in the prey's body. They are separated by a distance of l_d , each having a radius of r_d which is much smaller than l_d . In this case, the center of the detector is located at a distance y right above the source and the line connecting the two spheres is parallel to the electric field as shown in Figure II-1. Both l_s and l_d are also much smaller than y . The electric field strength along the line connecting the two spheres is assumed to be constant. Therefore the detector forms a closed circuit system connecting the prey, the surrounding seawater and the predator as described in Figure II-2.

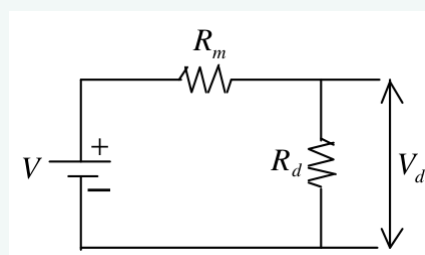


Figure II-2. The equivalent closed circuit system involving the sensing predator, the prey and the surrounding seawater.

In the figure, V is the voltage difference between the detector's spheres due to the electric field induced by the prey, R_m is the inner resistance due to the surrounding sea water. Further, V_d and R_d are respectively the voltage difference between the detecting spheres and the resistance of the detecting element within the predator.

Questions:

1. Determine the current density vector $\vec{j}$ (current per unit area) caused by a point current source I_s at a distance r in an infinite medium. **[1.5 pts]**
2. Based on the law $\vec{E} = \rho \vec{j}$, determine the electric field strength $\vec{E}_p$ at the middle of the detecting spheres (at point P) for a given current I_s that flows between two spheres in the prey's body **[2.0 pts]**.
3. Determine for the same current I_s , the voltage difference between the source spheres (V_s) in the prey **[1.5 pts]**. Determine the resistance between the two source spheres (R_s) **[0.5 pts]** and the power produced by the source (P_s) **[0.5 pts]**.
4. Determine R_m **[0.5 pts]**, V_d **[1.0 pts]** in Figure II-2 and calculate also the power transferred from the source to the detector (P_d) **[0.5 pts]**.
5. Determine the optimum value of R_d leading to maximum detected power **[1.5 pts]** and determine also the maximum power **[0.5 pts]**.

Official English text of the problem.

Answers

Let $\hat{\mathbf{x}}$ point from the prey's negative sphere toward its positive sphere. Voltages below are positive magnitudes. Results retain the leading terms under the stated assumptions

$$r_s \ll l_s, \quad r_d \ll l_d, \quad l_s, l_d \ll y.$$

The problem supplies no numerical parameter values, so the answers are symbolic.

1. With $\hat{\mathbf{r}}$ directed away from the point source,

$$\vec{j}(r) = \frac{I_s}{4\pi r^2} \hat{\mathbf{r}}.$$

2. The field points from the positive-source side toward the negative-source side:

$$\vec{E}_p \simeq -\frac{\rho I_s l_s}{4\pi y^3} \hat{\mathbf{x}}.$$

3. The source voltage, resistance, and power are

$$V_s \simeq \frac{\rho I_s}{2\pi r_s}, \quad R_s \simeq \frac{\rho}{2\pi r_s}, \quad P_s \simeq \frac{\rho I_s^2}{2\pi r_s}.$$

4. The seawater resistance and open-circuit detector voltage are

$$R_m \simeq \frac{\rho}{2\pi r_d}, \quad V \simeq \frac{\rho I_s l_s l_d}{4\pi y^3}.$$

Consequently,

$$V_d = \frac{R_d}{R_m + R_d} V, \quad P_d = \frac{V^2 R_d}{(R_m + R_d)^2}.$$

5. Maximum detected power occurs at

$$R_{d,\text{opt}} = R_m \simeq \frac{\rho}{2\pi r_d}.$$

Its value is

$$P_{d,\text{max}} = \frac{V^2}{4R_m} \simeq \frac{\rho I_s^2 l_s^2 l_d^2 r_d}{32\pi y^6} \simeq P_s \frac{r_s r_d l_s^2 l_d^2}{16y^6}.$$

Solution

1. Current density of a point current source

In a homogeneous, isotropic, infinite medium, spherical symmetry requires the current density to be radial and to have the same magnitude everywhere on a sphere centered on the source.

Conservation of current gives

$$I_s = \oint \vec{j} \cdot d\vec{A} = 4\pi r^2 j(r).$$

Therefore,

$$\vec{j}(r) = \frac{I_s}{4\pi r^2} \hat{\mathbf{r}}.$$

For a positive source the current is outward; a sink carrying the same current has the opposite sign. The current density has SI units A m^{-2} .

2. Electric field at P

Place the midpoint of the prey's two spheres at the origin. The positive sphere is at $x = l_s/2$, the negative sphere at $x = -l_s/2$, and P is at $(0, y)$.

Since the spheres are small compared with their separation and their distance from P, replace them by a point source $+I_s$ and a point sink $-I_s$. Their common distance from P is

$$R = \sqrt{y^2 + \frac{l_s^2}{4}}.$$

By superposition,

$$\begin{aligned} \vec{j}(P) &= \frac{I_s}{4\pi R^3} \left[\left(-\frac{l_s}{2} \hat{\mathbf{x}} + y \hat{\mathbf{y}} \right) - \left(\frac{l_s}{2} \hat{\mathbf{x}} + y \hat{\mathbf{y}} \right) \right] \\ &= -\frac{I_s l_s}{4\pi R^3} \hat{\mathbf{x}}. \end{aligned}$$

The vertical components cancel, while the horizontal components add.

Using the local form of Ohm's law, $\vec{E} = \rho \vec{j}$,

$$\vec{E}_p = -\frac{\rho I_s l_s}{4\pi(y^2 + l_s^2/4)^{3/2}} \hat{\mathbf{x}}.$$

This expression is exact within the point-source approximation. With $l_s \ll y$,

$$\boxed{\vec{E}_p \simeq -\frac{\rho I_s l_s}{4\pi y^3} \hat{\mathbf{x}}}.$$

Thus P lies on the equatorial line of a current dipole, whose field decreases as y^{-3} .

3. Source voltage, resistance, and power

For a single point source, the radial electric field is

$$E_r(r) = \frac{\rho I_s}{4\pi r^2}.$$

Taking the potential at infinity as zero,

$$\phi(r) = \int_r^\infty E_r(r') dr' = \frac{\rho I_s}{4\pi r}.$$

A sink of current I_s produces the negative of this potential.

At either source sphere, its own contribution to the surface potential has magnitude

$$\frac{\rho I_s}{4\pi r_s}.$$

The contribution from the other sphere is of order $\rho I_s/(4\pi l_s)$, smaller by r_s/l_s , and is neglected at leading order. Hence

$$\phi_+ \simeq \frac{\rho I_s}{4\pi r_s}, \quad \phi_- \simeq -\frac{\rho I_s}{4\pi r_s}.$$

The voltage difference is therefore

$$\boxed{V_s = \phi_+ - \phi_- \simeq \frac{\rho I_s}{2\pi r_s}}.$$

For completeness, retaining the mutual potential of the two monopoles would give

$$V_s \simeq \frac{\rho I_s}{2\pi} \left(\frac{1}{r_s} - \frac{1}{l_s} \right),$$

whose second term is negligible under the stated assumption.

The resistance seen between the source spheres follows from Ohm's law:

$$\boxed{R_s = \frac{V_s}{I_s} \simeq \frac{\rho}{2\pi r_s}}.$$

The electrical power supplied by the prey is

$$\boxed{P_s = I_s V_s = I_s^2 R_s \simeq \frac{\rho I_s^2}{2\pi r_s}}.$$

4. Seawater resistance, detector voltage, and detected power

A current through the detector enters the surrounding water at one detector sphere and leaves it at the other. The resistance of this path is obtained by applying part 3 to spheres of radius r_d , separated by l_d . Since $r_d \ll l_d$,

$$R_m \simeq \frac{\rho}{2\pi r_d}.$$

Physically, each small spherical contact contributes a spreading resistance $\rho/(4\pi r_d)$.

The prey's electric field is approximately uniform along the detector because $l_d \ll y$. The right detector sphere is at higher potential, so the open-circuit voltage is

$$\begin{aligned} V &= \phi_{\text{right}} - \phi_{\text{left}} = - \int_{\text{left}}^{\text{right}} \vec{E}_p \cdot d\vec{l} \\ &\simeq |\vec{E}_p| l_d = \frac{\rho I_s l_s l_d}{4\pi y^3}. \end{aligned}$$

In the equivalent circuit of Figure II-2, Kirchhoff's voltage law gives

$$I_d = \frac{V}{R_m + R_d}.$$

Therefore,

$$V_d = I_d R_d = \frac{R_d}{R_m + R_d} V \simeq \frac{\rho I_s l_s l_d}{4\pi y^3} \frac{R_d}{R_d + \rho/(2\pi r_d)}.$$

The power dissipated in the detecting element is

$$P_d = I_d^2 R_d = \frac{V_d^2}{R_d} = \frac{V^2 R_d}{(R_m + R_d)^2}.$$

In terms of the given parameters,

$$P_d \simeq \frac{\rho^2 I_s^2 l_s^2 l_d^2}{16\pi^2 y^6} \frac{R_d}{[R_d + \rho/(2\pi r_d)]^2}.$$

5. Optimum detector resistance and maximum power

For the given source current and geometry, V and R_m are fixed. Differentiating the detected power with respect to R_d ,

$$\frac{dP_d}{dR_d} = V^2 \frac{R_m - R_d}{(R_m + R_d)^3}.$$

The derivative is positive for $R_d < R_m$ and negative for $R_d > R_m$. Thus the maximum occurs when

$$R_{d,\text{opt}} = R_m \simeq \frac{\rho}{2\pi r_d}.$$

This is the maximum-power-transfer condition.

At this resistance, $V_d = V/2$, giving

$$P_{d,\text{max}} = \frac{V^2}{4R_m}.$$

Substitution yields

$$P_{d,\max} \simeq \left(\frac{\rho I_s l_s l_d}{4\pi y^3} \right)^2 \frac{2\pi r_d}{4\rho}$$

$$= \boxed{\frac{\rho I_s^2 l_s^2 l_d^2 r_d}{32\pi y^6}}.$$

Using the source power from part 3, this can also be written as

$$\boxed{\frac{P_{d,\max}}{P_s} \simeq \frac{r_s r_d l_s^2 l_d^2}{16y^6}}.$$

The y^{-6} dependence results from squaring the dipole field, which decreases as y^{-3} .

The solution was checked against the official solution.

Problem 3: A Heavy Vehicle Moving on An Inclined Road

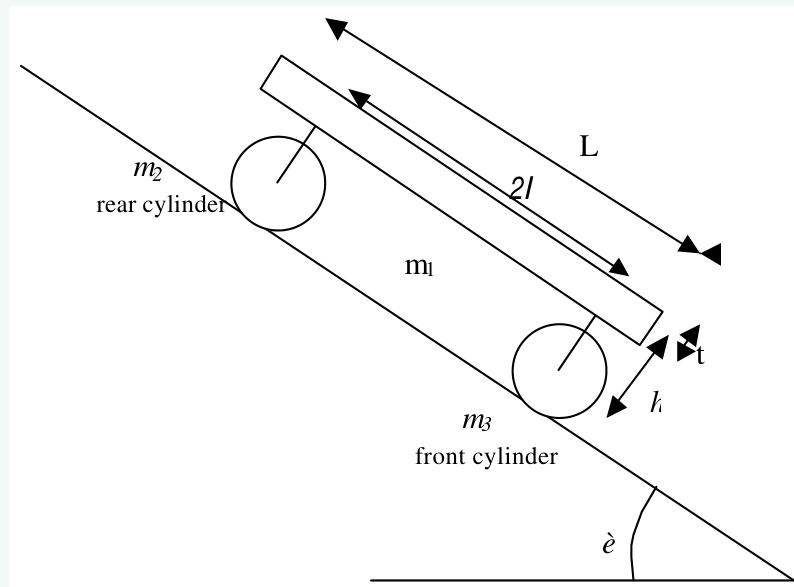


Figure III-1: A simplified model of a heavy vehicle moving on an inclined road.

The above figure is a simplified model of a heavy vehicle (road roller) with one rear and one front cylinder as its wheels on an inclined road with inclination angle of θ (labelled θ in the figure) as shown in Figure III-1. Each of the two cylinders has a total mass M ($m_2 = m_3 = M$) and consists of a cylindrical shell of outer radius R_o , inner radius $R_i = 0.8R_o$ and eight number of spokes with total mass $0.2M$. The mass of the undercarriage supporting the vehicle's body is negligible. The cylinder can be modeled as shown in Figure III-2. The vehicle is moving down the road under the influence of gravitational and frictional forces. The front and rear cylinder are positioned symmetrically with respect to the vehicle.

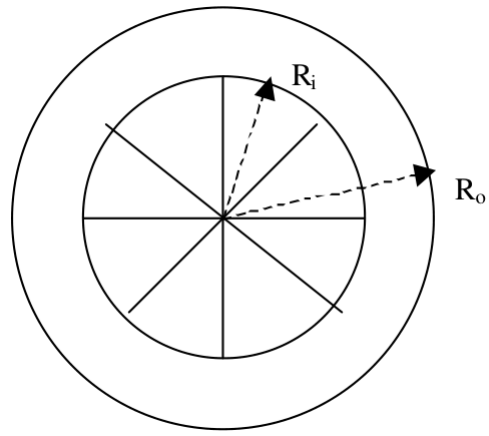


Figure III-2: A simplified model of the cylinders.

The static and kinetic friction coefficients between the cylinder and the road are μ_s and μ_k respectively. The body of the vehicle has a mass of $5M$, length of L and thickness of t . The distance between the front and the rear cylinder is $2l$ while the distance from the center of cylinder to the base of the vehicle's body is h . Assume that the rolling friction between the cylinder and its axis is negligible.

Questions:

1. Calculate the moment of inertia of either cylinder **[1.5 pts]**.
2. Draw all forces that act on the body, the front cylinder, and the rear one. Write down equations of motion for each part of them **[2.5 pts]**.
3. The vehicle is assumed to move from rest, then freely move under gravitational influence. State all the possible types of motion of the system and derive their accelerations in terms of the given physical quantities **[4.0 pts]**.
4. Assume that after the vehicle travels a distance d by pure rolling from rest the vehicle enters a section of the road with all the friction coefficients drop to smaller constant values μ'_s and μ'_k such that the two cylinders start to slide. Calculate the linear and angular velocities of each cylinder after the vehicle has traveled a total distance of s meters. Here we assume that d and s is much larger than the dimension of vehicle **[2.0 pts]**

Official English text of the problem.

Answers

Write θ for the inclination angle and define

$$R = R_o, \quad b = h + \frac{t}{2}, \quad \kappa = \frac{262}{375}, \quad q = \frac{5b}{7l}.$$

Subscripts r, f denote rear and front. Linear acceleration is positive downhill; angular acceleration is positive in the direction of downhill rolling.

1. Moment of inertia of either cylinder

$$I = \kappa MR^2 = \frac{262}{375} MR^2 \simeq 0.699 MR^2.$$

Its SI unit is kg m^2 .

2. Equations of motion

Let the axle force on cylinder i have components P_i downhill and Q_i toward the road. Let N_i be its road-normal force and f_i its uphill friction force. These are signed reference components.

For the body and massless undercarriage,

$$\begin{aligned} 5Ma &= 5Mg \sin \theta - P_r - P_f, \\ Q_r + Q_f &= 5Mg \cos \theta, \\ l(Q_f - Q_r) &= b(P_r + P_f). \end{aligned}$$

For each cylinder,

$$Ma = Mg \sin \theta + P_i - f_i, \quad N_i = Mg \cos \theta + Q_i, \quad I\alpha_i = Rf_i.$$

Consequently,

$$\begin{aligned} 7Ma &= 7Mg \sin \theta - f_r - f_f, \\ N_{r,f}(a) &= \frac{7M}{2} [g \cos \theta \mp q(g \sin \theta - a)]. \end{aligned}$$

3. Rolling and sliding regimes

For compactness, define

$$\alpha_i^{\text{sl}}(a) = \frac{\mu_k N_i(a)}{\kappa MR}.$$

Rear cylinder	Front cylinder	Common linear acceleration a	α_r	α_f
Rolls	Rolls	$a_0 = \frac{7g \sin \theta}{7 + 2\kappa} \simeq 0.8336g \sin \theta$	a_0/R	a_0/R
Slides	Rolls	$a_{\text{SR}} = \frac{7g[(2 + q\mu_k) \sin \theta - \mu_k \cos \theta]}{14 + 2\kappa + 7q\mu_k}$	$\alpha_r^{\text{sl}}(a_{\text{SR}})$	a_{SR}/R
Rolls	Slides	$a_{\text{RS}} = \frac{7g[(2 - q\mu_k) \sin \theta - \mu_k \cos \theta]}{14 + 2\kappa - 7q\mu_k}$	a_{RS}/R	$\alpha_f^{\text{sl}}(a_{\text{RS}})$
Slides	Slides	$a_{\text{SS}} = g(\sin \theta - \mu_k \cos \theta)$	$\alpha_r^{\text{sl}}(a_{\text{SS}})$	$\alpha_f^{\text{sl}}(a_{\text{SS}})$

These are the four contact-mode equations. Each requires $N_r, N_f \geq 0$, together with

$$\kappa Ma \leq \mu_s N_i \quad \text{for a rolling cylinder,}$$

$$a > R\alpha_i^{\text{sl}} \quad \text{for a cylinder sliding downhill from rest.}$$

Equalities describe boundaries between modes.

Both cylinders roll from rest if

$$\tan \theta \leq \frac{(7 + 2\kappa)\mu_s}{2\kappa(1 + q\mu_s)}.$$

The rear cylinder reaches its static-friction limit first. Thus loss of grip from the both-rolling branch leads to rear-only sliding or directly to both sliding; front-only sliding is the alternative conditional mixed contact mode, not the first loss of grip of that branch.

4. Velocities after entering the lower-friction section

For $s \geq d$, put

$$v_d = \sqrt{2a_0d}, \quad a_1 = g(\sin \theta - \mu'_k \cos \theta),$$

$$v = \sqrt{v_d^2 + 2a_1(s - d)}, \quad \Delta\tau = \frac{2(s - d)}{v + v_d}.$$

Then

$$v_r = v_f = v,$$

$$\omega_{r,f} = \frac{v_d}{R} + \frac{7\mu'_k g \cos \theta}{2\kappa R} (1 \mp q\mu'_k) \Delta\tau.$$

The minus sign is for the rear cylinder. Linear velocities have units m s^{-1} , and angular velocities have units rad s^{-1} .

Solution

1. Moment of inertia

Treat the shell as uniform and each spoke as a uniform thin radial rod extending from the axle to R_i .

The shell has mass $0.8M$, so its moment of inertia about the cylinder axis is

$$I_{\text{shell}} = \frac{1}{2}(0.8M)(R^2 + R_i^2).$$

Each spoke has mass $0.2M/8$. A uniform rod of length R_i , rotating about an axis through one end perpendicular to the rod, has moment of inertia $mR_i^2/3$. Hence

$$I_{\text{spokes}} = 8 \left[\frac{1}{3} \left(\frac{0.2M}{8} \right) R_i^2 \right] = \frac{0.2M}{3} R_i^2.$$

Using $R_i = 0.8R$,

$$I = MR^2 \left[\frac{0.8}{2}(1 + 0.8^2) + \frac{0.2}{3}(0.8^2) \right]$$

$$= MR^2 \left(\frac{82}{125} + \frac{16}{375} \right).$$

Therefore

$$I = \frac{262}{375} MR^2 = \kappa MR^2.$$

2. Force diagrams and equations of motion

Choose the x -axis downhill along the road and the y -axis perpendicular to the road, pointing away from it.

We use the intended rigid-vehicle model: both cylinders remain in contact with the straight road, their centers have the same acceleration a , and the body remains parallel to the road. Its angular acceleration is therefore zero. The body's center of mass is midway between the axles and a perpendicular distance

$$b = h + \frac{t}{2}$$

above the axle line.

The force diagrams should show the following.

- **Body, including the massless undercarriage:** weight $5Mg$, vertically downward at its center of mass; at each axle position, a force with components $-P_i$ along the road and $+Q_i$ perpendicular to it.
- **Rear cylinder:** weight Mg , vertically downward through its center; axle-force components P_r downhill and Q_r toward the road; road reaction N_r , perpendicular to the road at contact; friction f_r , uphill at contact.
- **Front cylinder:** the corresponding forces Mg, P_f, Q_f, N_f, f_f .

The axle-force arrows are reference directions: an individual signed component may be negative.

Friction points uphill in the downhill-sliding modes considered here and also supplies the torque for downhill rolling.

Newton's second law for the body gives

$$5Ma = 5Mg \sin \theta - P_r - P_f$$

and

$$Q_r + Q_f = 5Mg \cos \theta.$$

Taking moments about the body's center of mass, the two axle positions have road-parallel coordinates $-l, +l$ and perpendicular coordinate $-b$. Zero angular acceleration gives

$$l(Q_f - Q_r) - b(P_r + P_f) = 0.$$

Thus

$$l(Q_f - Q_r) = b(P_r + P_f).$$

For cylinder i , translational motion gives

$$\begin{aligned} Ma &= Mg \sin \theta + P_i - f_i, \\ N_i - Mg \cos \theta - Q_i &= 0. \end{aligned}$$

Only the road friction produces torque about the cylinder axis, because bearing friction is neglected:

$$I\alpha_i = Rf_i.$$

Adding the three downhill equations eliminates the internal axle forces:

$$7Ma = 7Mg \sin \theta - (f_r + f_f).$$

The normal reactions satisfy

$$N_r + N_f = 7Mg \cos \theta.$$

Also,

$$N_f - N_r = Q_f - Q_r = \frac{b}{l}(P_r + P_f).$$

Using the body's downhill equation,

$$N_f - N_r = \frac{5Mb}{l}(g \sin \theta - a) = 7Mq(g \sin \theta - a).$$

Therefore

$$\boxed{N_r(a) = \frac{7M}{2}[g \cos \theta - q(g \sin \theta - a)],}$$

$$\boxed{N_f(a) = \frac{7M}{2}[g \cos \theta + q(g \sin \theta - a)].}$$

These equations include the transfer of load toward the front. Equivalently,

$$N_f - N_r = q(f_r + f_f).$$

The body length L does not enter: the body has no angular acceleration, and its center of mass is fixed by symmetry.

3. Possible motions and their accelerations

Assume the usual ordering

$$0 \leq \mu_k \leq \mu_s.$$

For a cylinder rolling without slipping,

$$a = R\alpha_i.$$

Its rotational equation then requires

$$\boxed{f_i = \frac{I}{R^2}a = \kappa Ma.}$$

This force must satisfy the static-friction bound

$$\kappa Ma \leq \mu_s N_i.$$

For a cylinder whose contact point slides downhill,

$$f_i = \mu_k N_i, \quad \boxed{\alpha_i = \frac{\mu_k N_i}{\kappa M R}.}$$

To check the direction of sliding, define the downhill velocity of the cylinder's contact point relative to the road:

$$u_i = v - R\omega_i.$$

Starting from rest, a constant-acceleration sliding solution has

$$u_i = (a - R\alpha_i)\tau.$$

Thus uphill kinetic friction is consistent with downhill sliding only if

$$\boxed{a > R\alpha_i.}$$

All formulas below also require nonnegative normal reactions. A vanishing reaction marks possible loss of contact; beyond that boundary the assumed two-contact, non-pitching model no longer applies.

Both cylinders roll without slipping.

Here

$$f_r = f_f = \kappa Ma.$$

The combined downhill equation becomes

$$7Ma = 7Mg \sin \theta - 2\kappa Ma.$$

Hence

$$a_0 = \frac{7g \sin \theta}{7 + 2\kappa} = \frac{2625}{3149} g \sin \theta \simeq 0.8336 g \sin \theta,$$

and

$$\alpha_r = \alpha_f = \frac{a_0}{R}.$$

The required friction at either contact is

$$f_0 = \frac{7\kappa Mg \sin \theta}{7 + 2\kappa}.$$

Writing

$$\eta = \frac{2\kappa}{7 + 2\kappa},$$

the reactions are

$$N_{r,f}^{(0)} = \frac{7Mg}{2} (\cos \theta \mp q\eta \sin \theta).$$

Since $N_r^{(0)} \leq N_f^{(0)}$, the rear contact imposes the stricter static-friction condition:

$$f_0 \leq \mu_s N_r^{(0)}.$$

Rearrangement gives

$$\eta(1 + q\mu_s) \tan \theta \leq \mu_s,$$

or

$$\tan \theta \leq \frac{(7 + 2\kappa)\mu_s}{2\kappa(1 + q\mu_s)}.$$

As an independent check, pure rolling has kinetic energy

$$K = \frac{1}{2}(7M)v^2 + 2\left(\frac{1}{2}I\frac{v^2}{R^2}\right) = \frac{1}{2}(7 + 2\kappa)Mv^2.$$

Equating its increase to the loss of gravitational potential energy gives the same a_0 .

Rear cylinder slides; front cylinder rolls.

Now

$$f_r = \mu_k N_r(a), \quad f_f = \kappa Ma.$$

Therefore

$$(7 + \kappa)Ma = 7Mg \sin \theta - \mu_k N_r(a).$$

Substituting the normal reaction,

$$(7 + \kappa)a = 7g \sin \theta - \frac{7\mu_k}{2}[g \cos \theta - q(g \sin \theta - a)].$$

Collecting the terms containing a ,

$$a_{\text{SR}} = \frac{7g[(2 + q\mu_k) \sin \theta - \mu_k \cos \theta]}{14 + 2\kappa + 7q\mu_k}.$$

The angular accelerations are

$$\alpha_r = \frac{\mu_k N_r(a_{\text{SR}})}{\kappa MR}, \quad \alpha_f = \frac{a_{\text{SR}}}{R}.$$

The consistency conditions are

$$a_{\text{SR}} > \frac{\mu_k N_r(a_{\text{SR}})}{\kappa M}, \quad \kappa M a_{\text{SR}} \leq \mu_s N_f(a_{\text{SR}}).$$

The first makes the rear contact slide downhill; the second allows the front to remain rolling.

This is the mixed branch to test when the rear loses static grip. If the front cannot supply its required static friction, both cylinders must slide.

Rear cylinder rolls; front cylinder slides.

For completeness, the other mixed contact combination has

$$f_r = \kappa Ma, \quad f_f = \mu_k N_f(a).$$

Thus

$$(7 + \kappa)Ma = 7Mg \sin \theta - \mu_k N_f(a),$$

which yields

$$a_{\text{RS}} = \frac{7g[(2 - q\mu_k) \sin \theta - \mu_k \cos \theta]}{14 + 2\kappa - 7q\mu_k}.$$

Its angular accelerations are

$$\alpha_r = \frac{a_{\text{RS}}}{R}, \quad \alpha_f = \frac{\mu_k N_f(a_{\text{RS}})}{\kappa MR}.$$

Its consistency conditions are

$$\kappa M a_{\text{RS}} \leq \mu_s N_r(a_{\text{RS}}), \quad a_{\text{RS}} > \frac{\mu_k N_f(a_{\text{RS}})}{\kappa M}.$$

This is a conditional sliding-state solution. It does not describe the front being the first cylinder to lose grip from the both-rolling branch, because that branch requires equal friction at the two contacts while the front has the larger normal reaction.

Both cylinders slide.

In this case,

$$f_r + f_f = \mu_k(N_r + N_f) = 7\mu_k Mg \cos \theta.$$

Hence

$$a_{SS} = g(\sin \theta - \mu_k \cos \theta).$$

Since

$$g \sin \theta - a_{SS} = \mu_k g \cos \theta,$$

the normal reactions reduce to

$$N_{r,f}^{SS} = \frac{7Mg \cos \theta}{2}(1 \mp q\mu_k).$$

Therefore

$$\alpha_{r,f} = \frac{7\mu_k g \cos \theta}{2\kappa R}(1 \mp q\mu_k).$$

The front has the larger angular acceleration. Consequently, the stricter condition for both contact points to slide downhill from rest is

$$a_{SS} > R\alpha_f.$$

For $0 \leq \theta < \pi/2$, this is

$$\tan \theta > \mu_k \left[1 + \frac{7}{2\kappa}(1 + q\mu_k) \right].$$

Maintaining both contacts also requires

$$q\mu_k \leq 1.$$

If $\mu_k = 0$, the cylinders undergo pure translational sliding from rest:

$$a = g \sin \theta, \quad \alpha_r = \alpha_f = 0.$$

The distinction between the onset of sliding and its continuation matters: static friction determines whether a contact loses grip, whereas the kinetic conditions above determine whether an assumed sliding state is consistent. A kinetic solution should not replace an available no-slip solution merely because $\mu_k < \mu_s$.

4. Motion on the lower-friction section

During the initial pure-rolling distance d , the acceleration is a_0 . Starting from rest,

$$v_d = \sqrt{2a_0 d}, \quad \omega_{r,d} = \omega_{f,d} = \frac{v_d}{R}.$$

We neglect the short transition during which only one cylinder has reached the new surface, using the stated large-distance approximation. There is no impulsive change in either linear or angular velocity.

Once both cylinders slide on the new surface, replace μ_k by μ'_k in the preceding both-sliding results:

$$\begin{aligned} a_1 &= g(\sin \theta - \mu'_k \cos \theta), \\ N'_{r,f} &= \frac{7Mg \cos \theta}{2}(1 \mp q\mu'_k), \\ \alpha'_{r,f} &= \frac{7\mu'_k g \cos \theta}{2\kappa R}(1 \mp q\mu'_k). \end{aligned}$$

These quantities are constant.

Over the remaining distance $s - d$,

$$v^2 = v_d^2 + 2a_1(s - d),$$

so both cylinder centers have velocity

$$v_r = v_f = \sqrt{\frac{14gd \sin \theta}{7 + 2\kappa} + 2g(\sin \theta - \mu'_k \cos \theta)(s - d)}.$$

The elapsed time on this section follows from the average velocity:

$$s - d = \frac{v + v_d}{2} \Delta\tau,$$

hence

$$\Delta\tau = \frac{2(s - d)}{v + v_d}.$$

Integrating the constant angular accelerations gives

$$\begin{aligned} \omega_r &= \frac{v_d}{R} + \frac{7\mu'_k g \cos \theta}{2\kappa R} (1 - q\mu'_k) \frac{2(s - d)}{v + v_d}, \\ \omega_f &= \frac{v_d}{R} + \frac{7\mu'_k g \cos \theta}{2\kappa R} (1 + q\mu'_k) \frac{2(s - d)}{v + v_d}. \end{aligned}$$

The premise that both cylinders begin sliding requires a consistent downhill-sliding regime. Indeed, immediately after entry,

$$v - R\omega_i = (a_1 - R\alpha'_i) \Delta\tau,$$

so the condition is $a_1 > R\alpha'_f$, together with nonnegative reactions. The reduced static coefficient μ'_s determines loss of grip, but does not enter the subsequent velocities once both contacts are sliding.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.