

IPhO 2001

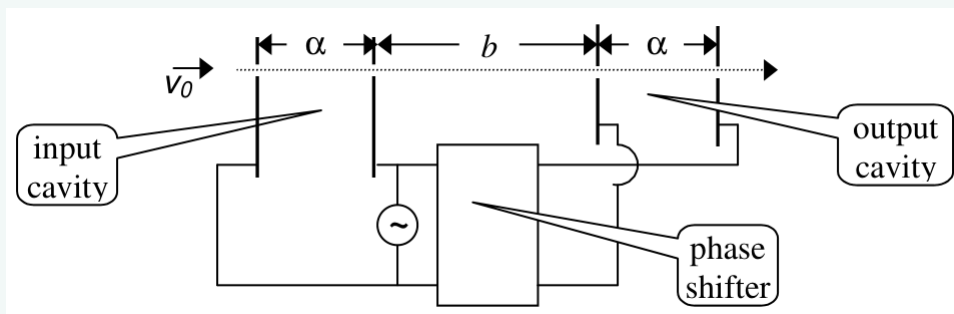
Theory problems and solutions · Antalya, Turkey · 3 theory problems with complete solutions

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Problem 1: Four short problems: 1a Klystron, 1b Intermolecular distance, 1c Simple sawtooth signal generator, 1d Atomic beam

1a) KLYSTRON

Klystrons are devices used for amplifying very high-frequency signals. A klystron basically consists of two identical pairs of parallel plates (cavities) separated by a distance b , as shown in the figure.



An electron beam with an initial speed v_0 traverses the entire system, passing through small holes in the plates. The high-frequency voltage to be amplified is applied to both pairs of plates with a certain phase difference (where period T corresponds to 2π phase) between them, producing horizontal, alternating electric fields in the cavities. The electrons entering the input cavity when the electric field is to the right are retarded and vice versa, so that the emerging electrons form bunches at a certain distance. If the output cavity is placed at the bunching point, the electric field in this cavity will absorb power from the beam provided that its phase is appropriately chosen. Let the voltage signal be a square wave with period $T = 1.0 \times 10^{-9}$ s, changing between $V = \pm 0.5$ volts. The initial velocity of the electrons is $v_0 = 2.0 \times 10^6$ m/s and the charge to mass ratio is $e/m = 1.76 \times 10^{11}$ C/kg. The distance α is so small that the transit time in the cavities can be neglected. Keeping 4 significant figures, calculate;

- the distance b , where the electrons bunch. Copy your result onto the **answer form**. [1.5 pts]
- the necessary phase difference to be provided by the phase shifter. Copy your result onto the **answer form**. [1.0 pts]

1b) INTERMOLECULAR DISTANCE

Let d_L and d_V represent the average distances between molecules of water in the liquid phase and in the vapor phase, respectively. Assume that both phases are at 100°C and atmospheric pressure, and the vapor behaves like an ideal gas. Using the following data, calculate the ratio d_V/d_L and copy your result onto the **answer form**. [2.5 pts]

Density of water in liquid phase: $\rho_L = 1.0 \times 10^3 \text{ kg/m}^3$,

Molar mass of water: $M = 1.8 \times 10^{-2} \text{ kg/mol}$

Atmospheric pressure: $P_a = 1.0 \times 10^5 \text{ N/m}^2$

Gas constant: $R = 8.3 \text{ J/mol} \cdot \text{K}$

Avagadro's number: $N_A = 6.0 \times 10^{23} / \text{mol}$

1c) SIMPLE SAWTOOTH SIGNAL GENERATOR

A sawtooth voltage waveform V_0 can be obtained across the capacitor C in Fig. 1. R is a variable resistor, V_i is an ideal battery, and SG is a spark gap consisting of two electrodes with an adjustable distance between them. When the voltage across the electrodes exceeds the firing voltage V_f , the air between the electrodes breaks down, hence the gap becomes a short circuit and remains so until the voltage across the gap becomes very small.

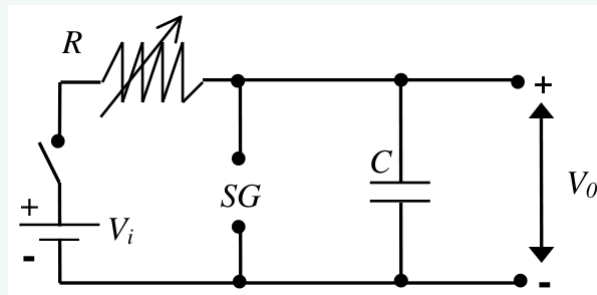


Figure 1

- Draw the voltage waveform V_0 versus time t , after the switch is closed. [0.5 pts]
- What condition must be satisfied in order to have an almost linearly varying sawtooth voltage waveform V_0 ? Copy your result onto the **answer form**. [0.2 pts]
- Provided that this condition is satisfied, derive a simplified expression for the period T of the waveform. Copy your result onto the **answer form**. [0.4 pts]
- What should you vary (R and/or SG) to change the period only? Copy your result onto the **answer form**. [0.2 pts]
- What should you vary (R and/or SG) to change the amplitude only? Copy your result onto the **answer form**. [0.2 pts]
- You are given an additional, adjustable DC voltage supply. Design and draw a new circuit indicating the terminals where you would obtain the voltage waveform V'_0 described in Fig. 2. [1.0 pts]

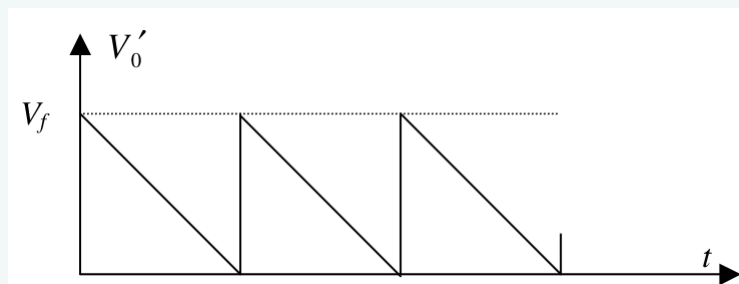
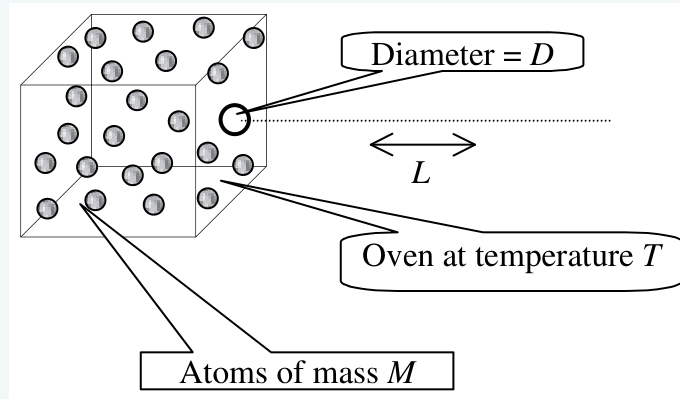


Figure 2

1d) ATOMIC BEAM

An atomic beam is prepared by heating a collection of atoms to a temperature T and allowing them to emerge horizontally through a small hole (of atomic dimensions) of diameter D in one side of the oven. Estimate the diameter of the beam after it has traveled a horizontal length L along its path. The mass of an atom is M . Copy your result onto the **answer form**. [2.5 pts]



Official English text of the problem.

Answers

- **1a(a):**

$$b = 2.270 \times 10^{-2} \text{ m} = 2.270 \text{ cm}.$$

- **1a(b):** The output field must **lag** the input field by

$$\Delta\phi_{\text{lag}} = 3.821 \text{ rad} = 218.9^\circ$$

modulo 2π . Equivalently, it may lead by $2.462 \text{ rad} = 141.1^\circ$. The output's retarding half-cycle must coincide with the arriving bunch.

- **1b:**

$$\frac{d_V}{d_L} = \left(\frac{\rho_L R T}{P_a M} \right)^{1/3} \simeq 12.$$

- **1c(a):** Repeated, concave-down exponential rises from approximately 0 to V_f , followed by nearly vertical drops to 0. Each charging segment is

$$V_0 = V_i \left(1 - e^{-\tau/(RC)} \right),$$

where τ is the time since the last discharge.

- **1c(b):**

$$V_f \ll V_i.$$

- **1c(c):**

$$T \simeq RC \frac{V_f}{V_i}.$$

- **1c(d):** Vary R **only**, keeping the spark gap fixed.
- **1c(e):** Vary **both** SG **and** R : change the gap to change V_f , and adjust R so that

$$RV_f = \text{constant}$$

in the linear-ramp approximation.

- **1c(f):** Set the additional supply to V_f , with its negative terminal connected to the original lower rail. Measure the output with its positive terminal at the additional supply's positive terminal and its negative terminal at the capacitor's upper terminal. Then

$$V'_0 = V_f - V_0,$$

giving a descending ramp from V_f to 0, followed by an abrupt return to V_f .

- **1d:** Using a representative thermal speed $v_{\text{th}} = \sqrt{3k_B T/M}$, the diffraction estimate is

$$d_{\text{beam}} \sim D + \frac{2hL}{D\sqrt{3Mk_B T}}.$$

Thus, when diffraction dominates the initial diameter,

$$d_{\text{beam}} \sim \frac{hL}{D\sqrt{Mk_B T}},$$

up to a numerical factor depending on the beam-width convention.

Solution

1a) Klystron

(a) Bunching distance

Let e denote the magnitude of the electron charge and let $U = 0.5$ V. Neglect space-charge effects and the transit time through either cavity, as intended in the problem.

Nonrelativistic energy conservation gives the two possible speeds after the input cavity:

$$\frac{1}{2}mv_s^2 = \frac{1}{2}mv_0^2 - eU, \quad \frac{1}{2}mv_f^2 = \frac{1}{2}mv_0^2 + eU,$$

where v_s and v_f are the slow and fast speeds. Hence

$$v_s = \sqrt{v_0^2 - 2(e/m)U} = 1.955505 \dots \times 10^6 \text{ m/s},$$

$$v_f = \sqrt{v_0^2 + 2(e/m)U} = 2.043526 \dots \times 10^6 \text{ m/s}.$$

Choose the input field to retard electrons during $0 \leq t < T/2$, and to accelerate them during $T/2 \leq t < T$.

A slow electron entering at time t_0 and a fast electron entering at $t_0 + T/2$ reach the output together when

$$t_0 + \frac{b}{v_s} = t_0 + \frac{T}{2} + \frac{b}{v_f}.$$

Therefore,

$$b \left(\frac{1}{v_s} - \frac{1}{v_f} \right) = \frac{T}{2},$$

so that

$$b = \frac{T v_s v_f}{2(v_f - v_s)}.$$

For a square-wave modulation, this condition makes the entire slow half-cycle packet overlap in arrival time with the following fast half-cycle packet. It is the first maximum-bunching distance; the bunch has finite duration rather than being an instantaneous pulse.

Substitution, retaining extra digits until the final step, gives

$$b = 0.022699764 \dots \text{ m.}$$

Thus, to four significant figures,

$$b = 0.02270 \text{ m} = 2.270 \text{ cm}.$$

(b) Phase difference

The slow and fast flight times across the drift region are

$$\tau_s = \frac{b}{v_s} = 11.6081337 \dots T, \quad \tau_f = \frac{b}{v_f} = 11.1081337 \dots T.$$

The slow packet reaches the output during

$$[\tau_s, \tau_s + T/2),$$

while the following fast packet reaches it during

$$[T/2 + \tau_f, T + \tau_f).$$

These intervals coincide because $\tau_s - \tau_f = T/2$.

To extract energy from the beam, the output electric field must point to the right throughout this occupied half-cycle, thereby retarding the electrons. Thus its waveform must reproduce the input waveform delayed by τ_s :

$$E_{\text{out}}(t) = E_{\text{in}}(t - \tau_s).$$

Only the fractional part of τ_s/T matters:

$$\frac{\tau_s \bmod T}{T} = 0.6081337 \dots$$

The required output phase lag is therefore

$$\Delta\phi_{\text{lag}} = 2\pi(0.6081337 \dots) = 3.8210167 \dots \text{ rad.}$$

Hence

$$\Delta\phi_{\text{lag}} = 3.821 \text{ rad} = 218.9^\circ.$$

If phase is instead written as an additive argument, $E_{\text{out}} \propto s(\omega t + \phi)$, the same setting is

$$\phi = -3.821 \text{ rad} \equiv 2.462 \text{ rad} \pmod{2\pi}.$$

1b) Intermolecular distance

For molecular number density n , the characteristic intermolecular spacing is estimated by assigning a volume d^3 to each molecule:

$$d \sim n^{-1/3}.$$

The liquid's number density is

$$n_L = \frac{\rho_L N_A}{M},$$

so

$$d_L = \left(\frac{M}{\rho_L N_A} \right)^{1/3}.$$

For the ideal vapor,

$$P_a V = \frac{N}{N_A} RT,$$

and therefore

$$n_V = \frac{N}{V} = \frac{P_a N_A}{RT}, \quad d_V = \left(\frac{RT}{P_a N_A} \right)^{1/3}.$$

Taking the ratio cancels Avogadro's number:

$$\frac{d_V}{d_L} = \left(\frac{\rho_L RT}{P_a M} \right)^{1/3}.$$

Using $T \simeq 373 \text{ K}$,

$$\frac{d_V}{d_L} = \left[\frac{(1.0 \times 10^3)(8.3)(373)}{(1.0 \times 10^5)(1.8 \times 10^{-2})} \right]^{1/3} = 11.9813 \dots$$

Thus, consistently with the supplied data,

$$\boxed{\frac{d_V}{d_L} \simeq 12}.$$

The corresponding characteristic distances are approximately

$$d_L = 3.1 \times 10^{-10} \text{ m}, \quad d_V = 3.7 \times 10^{-9} \text{ m}.$$

1c) Simple sawtooth signal generator

Assume the capacitor is initially uncharged, the residual voltage after a spark is negligible, and discharge takes much less time than charging.

(a) Voltage waveform

While the spark gap is nonconducting, Kirchhoff's voltage law gives

$$V_i = IR + V_0, \quad I = C \frac{dV_0}{dt}.$$

Consequently,

$$RC \frac{dV_0}{dt} = V_i - V_0.$$

Starting from $V_0 = 0$, the charging solution is

$$V_0(\tau) = V_i \left(1 - e^{-\tau/(RC)} \right).$$

When V_0 reaches V_f , the spark gap conducts and rapidly discharges the capacitor. The gap then becomes nonconducting, and charging begins again.

The requested sketch therefore has:

- An initial point at $(t, V_0) = (0, 0)$.
- A rising, concave-down exponential segment ending at V_f .
- A nearly vertical drop from V_f to 0.
- Identical repetitions at regular intervals.

If continued without firing, each charging segment would approach the asymptote V_i . Sustained oscillation requires $V_i > V_f$; otherwise the capacitor cannot reach the firing threshold in finite time.

(b) Condition for an almost linear ramp

The charging slope is

$$\frac{dV_0}{dt} = \frac{V_i - V_0}{RC}.$$

It remains nearly constant throughout $0 \leq V_0 \leq V_f$ if

$$V_f \ll V_i.$$

Then

$$\frac{dV_0}{dt} \simeq \frac{V_i}{RC}, \quad V_0(\tau) \simeq \frac{V_i}{RC} \tau.$$

(c) Period

Neglecting the discharge duration, the exact charging time to the threshold follows from

$$V_f = V_i \left(1 - e^{-T/(RC)} \right).$$

Thus,

$$T = RC \ln \left(\frac{V_i}{V_i - V_f} \right).$$

For $V_f/V_i \ll 1$, use $-\ln(1 - x) \simeq x$:

$$T \simeq RC \frac{V_f}{V_i}.$$

Equivalently, the nearly constant charging current V_i/R must deliver the charge CV_f , giving the same result.

(d) Changing the period only

The peak voltage is fixed by the spark-gap threshold V_f , while the charging time is proportional to R .

Therefore, vary R **only**, keeping SG fixed. Increasing R increases the period without changing the peak-to-peak amplitude.

(e) Changing the amplitude only

The peak-to-peak amplitude is approximately V_f . Change the electrode separation of SG to change this threshold.

Because

$$T \simeq \frac{C}{V_i} R V_f,$$

keeping the period constant requires

$$\boxed{R_{\text{new}} V_{f,\text{new}} = R_{\text{old}} V_{f,\text{old}}}.$$

Thus **both SG and R must be adjusted**. For example, increasing the firing voltage requires a compensating decrease in resistance.

Without the linear approximation, the corresponding exact condition is

$$R \ln \left(\frac{V_i}{V_i - V_f} \right) = \text{constant}.$$

(f) Circuit for the descending sawtooth

Define the original circuit's lower rail as node G , and the capacitor's upper terminal as node A . Then

$$V(A) - V(G) = V_0.$$

Make the following connections:

1. Retain the original oscillator circuit.
2. Connect the additional DC supply's negative terminal to G .
3. Adjust this supply to V_f , and call its positive terminal B .
4. Take the new output between B and A , with **positive at B and negative at A** .

For a high-impedance output measurement,

$$V'_0 = V(B) - V(A) = V_f - V_0.$$

During each nearly linear charging interval,

$$V'_0 \simeq V_f - \frac{V_i}{RC} \tau = V_f \left(1 - \frac{\tau}{T} \right).$$

Thus the output starts at V_f , falls almost linearly to 0, and jumps back to V_f when the spark discharges the capacitor. These are precisely the intercepts, slopes, and jumps shown in Fig. 2.

1d) Atomic beam

The atomic-scale opening produces quantum diffraction. Let h be Planck's constant and k_B Boltzmann's constant.

A representative thermal speed follows from equipartition:

$$\frac{1}{2} M v_{\text{th}}^2 = \frac{3}{2} k_B T,$$

giving

$$v_{\text{th}} = \sqrt{\frac{3k_B T}{M}}, \quad p_{\text{th}} = M v_{\text{th}} = \sqrt{3M k_B T}.$$

This uses the rms thermal speed as a representative speed; an estimate does not require resolving the full distribution of emerging velocities.

The corresponding de Broglie wavelength is

$$\lambda = \frac{h}{p_{\text{th}}} = \frac{h}{\sqrt{3Mk_B T}}.$$

For an opening of width D , the characteristic diffraction half-angle satisfies

$$\theta \sim \frac{\lambda}{D}.$$

This follows because an angular deflection θ produces a path difference of order $D\theta$ across the opening; appreciable diffraction occurs when that difference is comparable with λ .

For small angles, propagation over distance L increases the beam radius by approximately $L\theta$. A simple diameter estimate is therefore

$$d_{\text{beam}} \sim D + 2L\theta,$$

or

$$d_{\text{beam}} \sim D + \frac{2hL}{D\sqrt{3Mk_B T}}.$$

The initial aperture diameter matters when spreading is small. When diffraction dominates,

$$d_{\text{beam}} \sim \frac{hL}{D\sqrt{Mk_B T}}$$

up to numerical factors of order unity.

For a more specific convention, the first diffraction minimum of an ideal circular aperture occurs at $\theta \simeq 1.22\lambda/D$. Its far-field central-disk diameter would therefore be

$$d_{\text{first minimum}} \simeq \frac{2.44hL}{D\sqrt{3Mk_B T}}.$$

The small-angle estimates require $\lambda/D \ll 1$. If this ratio is not small, the emerging wave is broadly divergent.

The solution was checked against the official solution.

Problem 2: Binary Star System

a) It is well known that most stars form binary systems. One type of binary system consists of an ordinary star with mass m_0 and radius R , and a more massive, compact neutron star with mass M , rotating around each other. In all the following ignore the motion of the earth. Observations of such a binary system reveal the following information:

- The maximum angular displacement of the ordinary star is $\Delta\theta$, whereas that of the neutron star is $\Delta\phi$ (see Fig. 1).
- The time it takes for these maximum displacements is τ .

- The radiation characteristics of the ordinary star indicate that its surface temperature is T and the radiated energy incident on a unit area on earth's surface per unit time is P .
- The calcium line in this radiation differs from its normal wavelength λ_0 by an amount $\Delta\lambda$, due only to the gravitational field of the ordinary star. (For this calculation the photon can be considered to have an effective mass of $h/c\lambda$.)

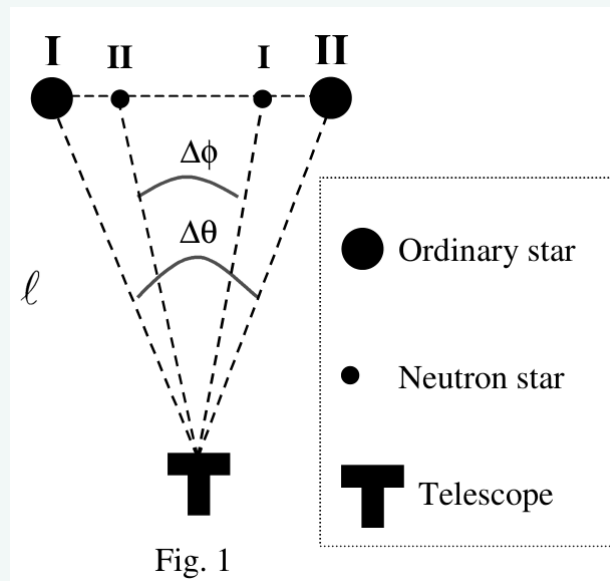


Fig. 1

Find an expression for the distance ℓ from earth to this system, only in terms of the observed quantities and universal constants. Copy your result onto the **answer form**. [7 pts]

b) Assume that $M \gg m_0$, so that the ordinary star is basically rotating around the neutron star in a circular orbit of radius r_0 . Assume that the ordinary star starts emitting gas toward the neutron star with a speed v_0 , relative to the ordinary star (see Fig. 2). Assuming that the neutron star is the dominant gravitational force in this problem and neglecting the orbital changes of the ordinary star find the distance of closest approach r_f shown in Fig. 2. Copy your result onto the **answer form**. [3pts]

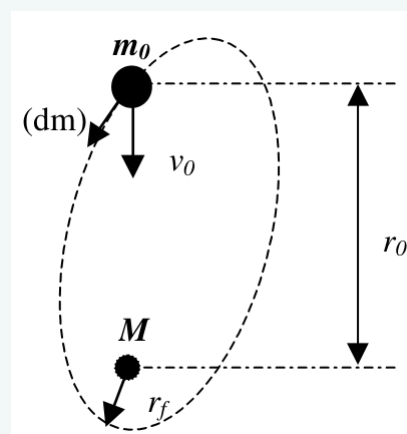


Fig. 2

Official English text of the problem.

Answers

- a) With the angular displacements in radians,

$$\ell = \frac{2\sqrt{2}c\tau}{\pi(\Delta\theta + \Delta\phi)} \left(\frac{\Delta\lambda}{\lambda_0\Delta\phi} \right)^{1/2} \left(\frac{P}{\sigma T^4} \right)^{1/4}$$

where σ is the Stefan–Boltzmann constant. The orbital interpretation is specified below.

- b)

$$r_f = \frac{r_0}{1 + v_0 \sqrt{\frac{r_0}{GM}}}.$$

Solution

a)

We use Newtonian orbital dynamics, the small-angle approximation, blackbody radiation from the ordinary star, and weak-field gravitational redshift. The system's dimensions are negligible compared with its distance from earth.

Interpret Fig. 1 as a circular binary orbit whose observed angular excursions span opposite extrema. Thus τ is half the orbital period:

$$\mathcal{T} = 2\tau.$$

Let a_0 and a_N be the ordinary and neutron stars' orbital radii about their common centre of mass. Since the angular excursions are full diameters,

$$a_0 = \frac{\ell\Delta\theta}{2}, \quad a_N = \frac{\ell\Delta\phi}{2}.$$

The separation is therefore

$$a = a_0 + a_N = \frac{\ell(\Delta\theta + \Delta\phi)}{2}.$$

The centre-of-mass relation gives

$$m_0 a_0 = M a_N, \quad m_0 = (m_0 + M) \frac{a_N}{a}.$$

Kepler's third law for the relative orbit is

$$G(m_0 + M) = \frac{4\pi^2 a^3}{\mathcal{T}^2}.$$

Combining these equations,

$$Gm_0 = \frac{4\pi^2 a^2 a_N}{\mathcal{T}^2} = \frac{\pi^2 \ell^3}{8\tau^2} \Delta\phi (\Delta\theta + \Delta\phi)^2. \quad (1)$$

Next, the ordinary star's luminosity is

$$L = 4\pi R^2 \sigma T^4.$$

Assuming isotropic emission and negligible absorption between the star and earth,

$$P = \frac{L}{4\pi\ell^2} = \sigma T^4 \frac{R^2}{\ell^2},$$

so

$$R = \ell \left(\frac{P}{\sigma T^4} \right)^{1/2}. \quad (2)$$

Finally, consider the photon's gravitational energy loss. Its emitted energy and approximate effective mass are

$$E_0 = \frac{hc}{\lambda_0}, \quad m_\gamma \simeq \frac{E_0}{c^2}.$$

On travelling from the stellar surface to earth, effectively at infinity relative to R , it loses energy

$$E_0 - E_{\text{obs}} \simeq \frac{Gm_0 m_\gamma}{R}.$$

Consequently,

$$\frac{E_0 - E_{\text{obs}}}{E_0} \simeq \frac{Gm_0}{Rc^2}.$$

For $\Delta\lambda \ll \lambda_0$,

$$\frac{E_0 - E_{\text{obs}}}{E_0} = 1 - \frac{\lambda_0}{\lambda_0 + \Delta\lambda} \simeq \frac{\Delta\lambda}{\lambda_0}.$$

Thus the positive gravitational redshift determines

$$Gm_0 = c^2 R \frac{\Delta\lambda}{\lambda_0} = c^2 \ell \left(\frac{P}{\sigma T^4} \right)^{1/2} \frac{\Delta\lambda}{\lambda_0}. \quad (3)$$

Equating (1) and (3) and cancelling one factor of ℓ ,

$$\ell^2 = \frac{8c^2\tau^2}{\pi^2\Delta\phi(\Delta\theta + \Delta\phi)^2} \frac{\Delta\lambda}{\lambda_0} \left(\frac{P}{\sigma T^4} \right)^{1/2}.$$

Taking the positive square root,

$$\ell = \frac{2\sqrt{2}c\tau}{\pi(\Delta\theta + \Delta\phi)} \left(\frac{\Delta\lambda}{\lambda_0\Delta\phi} \right)^{1/2} \left(\frac{P}{\sigma T^4} \right)^{1/4}.$$

The gravitational constant cancels because both the orbital dynamics and the redshift measure Gm_0 . All factors except $c\tau$ are dimensionless.

b)

Since $M \gg m_0$, treat the neutron star as stationary. The ordinary star's circular orbital speed follows from

$$\frac{v_c^2}{r_0} = \frac{GM}{r_0^2}, \quad v_c = \sqrt{\frac{GM}{r_0}}.$$

A gas element inherits this tangential speed and receives an additional inward radial speed v_0 . Its initial velocity components are therefore

$$v_r = -v_0, \quad v_t = v_c.$$

Neglecting the ordinary star's gravity, the gas moves in the neutron star's central gravitational field. Its specific angular momentum j and specific mechanical energy ε are conserved:

$$j = r_0 v_c = \sqrt{GM r_0},$$

$$\varepsilon = \frac{1}{2}(v_c^2 + v_0^2) - \frac{GM}{r_0} = \frac{1}{2}v_0^2 - \frac{GM}{2r_0}.$$

At closest approach, the radial velocity vanishes. If the tangential speed there is v_f , angular-momentum conservation gives

$$r_f v_f = r_0 v_c, \quad v_f = \frac{r_0 v_c}{r_f}.$$

Energy conservation then gives

$$\frac{1}{2}v_0^2 - \frac{1}{2}v_c^2 = \frac{1}{2}v_c^2 \frac{r_0^2}{r_f^2} - v_c^2 \frac{r_0}{r_f}.$$

Rearranging,

$$v_0^2 = v_c^2 \left(\frac{r_0}{r_f} - 1 \right)^2.$$

The gas initially moves inward, so its closest approach satisfies $r_f \leq r_0$. We therefore choose

$$\frac{r_0}{r_f} - 1 = \frac{v_0}{v_c},$$

which yields

$$r_f = \frac{r_0}{1 + v_0/v_c} = \frac{r_0}{1 + v_0 \sqrt{\frac{r_0}{GM}}}.$$

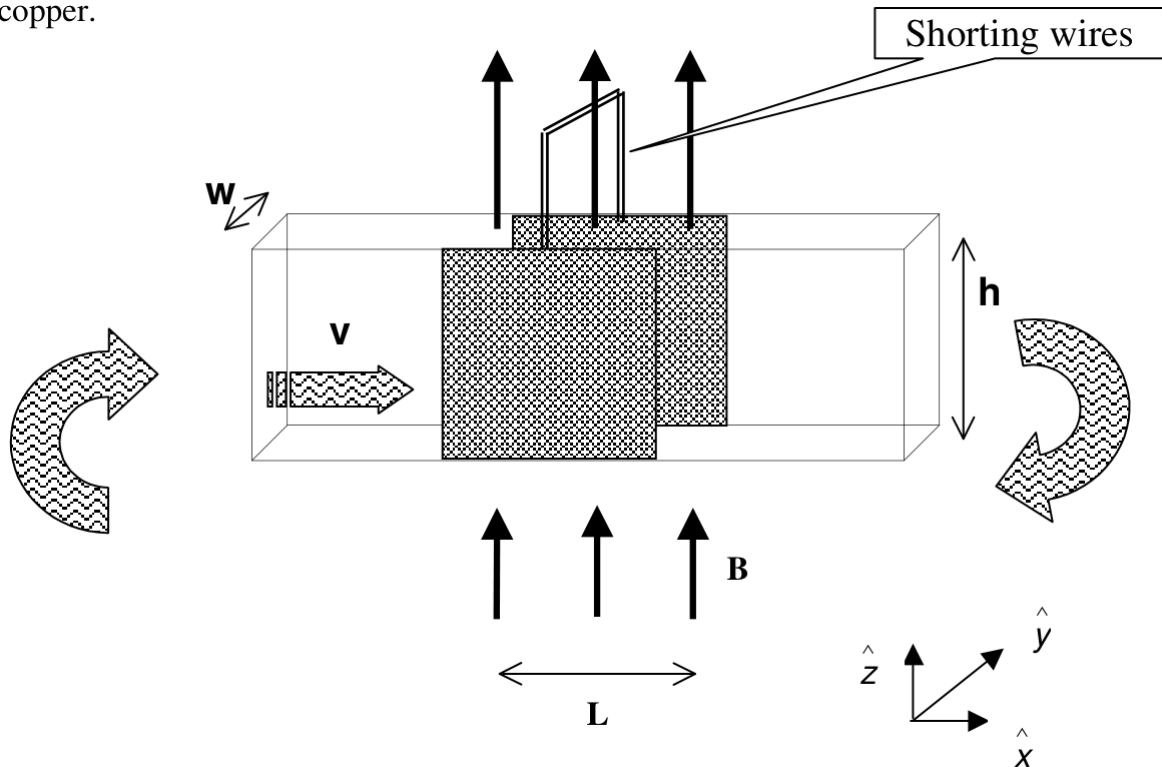
For $v_0 = 0$, this correctly gives the original circular orbit, $r_f = r_0$. The gas trajectory is elliptical as drawn when $v_0 < v_c$, parabolic when $v_0 = v_c$, and hyperbolic when $v_0 > v_c$; the closest-approach formula holds in all three cases.

The solution was checked against the official solution.

Problem 3: Magnetohydrodynamic (MHD) Generator

A horizontal rectangular plastic pipe of width w and height h , which closes upon itself, is filled with mercury of resistivity ρ . An overpressure P is produced by a turbine which drives this fluid with a constant speed v_0 . The two opposite vertical walls of a section of the pipe with length L are made of copper.

copper.



The motion of a real fluid is very complex. To simplify the situation we assume the following:

- Although the fluid is viscous, its speed is uniform over the entire cross section.
- The speed of the fluid is always proportional to the net external force acting upon it.
- The fluid is incompressible.

These walls are electrically shorted externally and a uniform, magnetic field $\mathbf{B}$ is applied vertically upward only in this section. The set up is illustrated in the figure above, with the unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ to be used in the solution.

- Find the force acting on the fluid due to the magnetic field (in terms of L , B , h , w , ρ and the velocity v) [2.0 pts]
- Derive an expression for the new speed v of the fluid (in terms of v_0 , P , L , B and ρ) after the magnetic field is applied. [3.0 pts]
- Derive an expression for the additional power that must be supplied by the turbine to increase the speed to its original value v_0 . Copy your result onto the **answer form**. [2.0 pts]
- Now the magnetic field is turned off and mercury is replaced by water flowing with speed v_0 . An electromagnetic wave with a single frequency is sent along the section with length L in the direction of the flow. The refractive index of water is n , and $v_0 \ll c$. Derive an expression for the contribution of the fluid's motion to the phase difference between the waves entering and leaving section L . Copy your result onto the **answer form**. [3.0 pts]

Official English text of the problem.

Answers

- **a)** The magnetic force opposes the flow:

$$\mathbf{F}_B = -\frac{LB^2hw}{\rho} v \hat{x}.$$

- **b)**

$$v = \frac{v_0}{1 + \frac{LB^2v_0}{\rho P}}.$$

- **c)** At the restored speed v_0 , the extra power required because of the magnetic field, above the original field-free power, is

$$\Delta\mathcal{P}_{\text{mag}} = \frac{LB^2hw}{\rho} v_0^2.$$

If the reference is instead the slowed operating state in (b), the total turbine-power increase is

$$\Delta\mathcal{P}_{(b) \rightarrow v_0} = \frac{LB^2hw}{\rho} v_0^2 + Pwh(v_0 - v).$$

- **d)** Let $\omega = 2\pi f$ be the laboratory angular frequency and define the phase lag across the section as $\Phi = kL$. To first order in v_0/c ,

$$\delta\Phi \equiv \Phi(v_0) - \Phi(0) = -\frac{\omega Lv_0}{c^2}(n^2 - 1) = -\frac{2\pi fLv_0}{c^2}(n^2 - 1).$$

Thus the flow reduces the phase lag. The outgoing wave's corresponding phase advance has the same magnitude and positive sign.

Solution

a) Magnetic force on the fluid

The fluid velocity and applied magnetic field are

$$\mathbf{v} = v\hat{x}, \quad \mathbf{B} = B\hat{z}.$$

Consequently,

$$\mathbf{v} \times \mathbf{B} = -vB\hat{y}.$$

The magnetic force on moving charge carriers therefore drives conventional current through the mercury in the $-\hat{y}$ direction, between the copper walls.

The walls are separated by w , so the motional electromotive force has magnitude

$$\mathcal{E} = vBw.$$

The current travels a distance w through mercury with conducting cross-sectional area Lh . Its resistance is therefore

$$R = \rho \frac{w}{Lh}.$$

Neglecting the resistance of the copper walls and shorting wires, the current is

$$I = \frac{\mathcal{E}}{R} = \frac{vBLh}{\rho}.$$

The force on this current in the mercury is

$$\mathbf{F}_B = I(-w\hat{y}) \times (B\hat{z}) = -IwB\hat{x}.$$

Hence

$$\mathbf{F}_B = -\frac{LB^2hw}{\rho} v \hat{x}.$$

Its direction is opposite to the fluid velocity, as expected for magnetic braking.

b) Speed after applying the magnetic field

Let the pipe cross-sectional area be

$$A = wh.$$

The turbine's pressure rise supplies the driving force

$$F_P = PA.$$

The assumed proportionality between speed and applied driving force is equivalent, for steady motion, to a linear viscous resistance κv . Before the magnetic field is applied,

$$PA = \kappa v_0,$$

so

$$\kappa = \frac{PA}{v_0}.$$

With the field applied and the same pressure rise P , the steady force balance is

$$PA - \kappa v - \frac{LB^2A}{\rho} v = 0.$$

Substitution for κ gives

$$PA = \left(\frac{PA}{v_0} + \frac{LB^2A}{\rho} \right) v.$$

Canceling A and solving,

$$v = \frac{v_0}{1 + \frac{LB^2v_0}{\rho P}}.$$

The result approaches v_0 as $B \rightarrow 0$, and the field always reduces the speed.

c) Additional turbine power

At the restored speed v_0 , the viscous resistance returns to its original value,

$$F_{\text{visc}} = \kappa v_0 = PA.$$

The turbine must also overcome the magnetic force

$$|F_B(v_0)| = \frac{LB^2A}{\rho}v_0.$$

Thus the required pressure rise is

$$P_{\text{restored}} = P + \frac{LB^2v_0}{\rho}.$$

For incompressible flow, mechanical power supplied by a pressure rise is pressure rise times volume flow rate:

$$\mathcal{P} = P_{\text{rise}}Av.$$

Therefore the total power at the restored speed is

$$\mathcal{P}_{\text{restored}} = \left(P + \frac{LB^2v_0}{\rho}\right)Av_0 = PAv_0 + \frac{LB^2A}{\rho}v_0^2.$$

The original field-free power was $\mathcal{P}_0 = PAv_0$. The additional power needed at the same speed because of magnetic braking is consequently

$$\Delta\mathcal{P}_{\text{mag}} = \mathcal{P}_{\text{restored}} - \mathcal{P}_0 = \frac{LB^2hw}{\rho}v_0^2.$$

This equals the Joule heating in the mercury:

$$I^2R = \left(\frac{v_0BLh}{\rho}\right)^2 \frac{\rho w}{Lh} = \frac{LB^2hw}{\rho}v_0^2,$$

which checks the energy balance.

There is a distinction if "additional power" is measured from the slowed state in (b). In that state, the turbine supplies only

$$\mathcal{P}_{\text{slowed}} = PAv.$$

The increase from that operating state is

$$\begin{aligned} \mathcal{P}_{\text{restored}} - \mathcal{P}_{\text{slowed}} &= \frac{LB^2hw}{\rho}v_0^2 + Pwh(v_0 - v) \\ &= \frac{LB^2hw}{\rho}v_0^2 \left(1 + \frac{1}{1 + \frac{LB^2v_0}{\rho P}}\right). \end{aligned}$$

The second term accounts for restoring the original viscous dissipation as the speed increases.

d) Contribution of the water's motion to the phase difference

Treat the supplied refractive index n as constant, neglecting dispersion. In the water's rest frame, the wave's phase velocity is

$$u' = \frac{c}{n}.$$

The water moves along the propagation direction at speed v_0 . Relativistic velocity addition gives the laboratory phase velocity

$$u = \frac{u' + v_0}{1 + \frac{u'v_0}{c^2}} = \frac{\frac{c}{n} + v_0}{1 + \frac{v_0}{nc}}.$$

Expanding to first order in v_0/c ,

$$u \simeq \left(\frac{c}{n} + v_0 \right) \left(1 - \frac{v_0}{nc} \right) = \frac{c}{n} + v_0 \left(1 - \frac{1}{n^2} \right).$$

The coefficient $1 - 1/n^2$ is the Fresnel drag coefficient.

For a laboratory monochromatic wave written as

$$E(x, t) = E_0 \cos(\omega t - kx),$$

the entrance-to-exit phase lag is

$$\Phi = kL = \frac{\omega L}{u}.$$

The source fixes the laboratory frequency ω .

Expanding the reciprocal velocity,

$$\frac{1}{u} \simeq \frac{n}{c} - \frac{(n^2 - 1)v_0}{c^2}.$$

Thus

$$\Phi(v_0) \simeq \frac{n\omega L}{c} - \frac{\omega L(n^2 - 1)v_0}{c^2}.$$

For stationary water,

$$\Phi(0) = \frac{n\omega L}{c}.$$

Subtracting,

$$\boxed{\delta\Phi = -\frac{\omega L(n^2 - 1)v_0}{c^2}.$$

Equivalently, with laboratory vacuum wavelength $\lambda_0 = c/f$,

$$\boxed{\delta\Phi = -\frac{2\pi L v_0}{\lambda_0 c} (n^2 - 1).}$$

The negative sign denotes a reduction of the phase lag for propagation along the flow. If phase difference is defined as exit phase minus entrance phase, its change has the opposite sign.

The solution was checked against the official solution.