

IPhO 2000

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Problem 1: Five short parts: bungee jumper, heat engine, age of the Earth from U-Pb dating, uniformly charged sphere, rotating copper ring in the Earth's magnetic field

A

A bungee jumper is attached to one end of a long elastic rope. The other end of the elastic rope is fixed to a high bridge. The jumper steps off the bridge and falls, from rest, towards the river below. He does not hit the water. The mass of the jumper is m , the unstretched length of the rope is L , the rope has a force constant (force to produce 1 m extension) of k and the gravitational field strength is g .

You may assume that

- the jumper can be regarded as a point mass m attached to the end of the rope,
- the mass of the rope is negligible compared to m ,
- the rope obeys Hooke's law,
- air resistance can be ignored throughout the fall of the jumper.

Obtain expressions for the following and insert on the answer sheet:

- the distance y dropped by the jumper before coming instantaneously to rest for the first time,
- the maximum speed v attained by the jumper during this drop,
- the time t taken during the drop before coming to rest for the first time.

B

A heat engine operates between two identical bodies at different temperatures T_A and T_B ($T_A > T_B$), with each body having mass m and constant specific heat capacity s . The bodies remain at constant pressure and undergo no change of phase.

- Showing full working, obtain an expression for the final temperature T_0 attained by the two bodies A and B, if the heat engine extracts from the system the maximum amount of mechanical work that is theoretically possible.

Write your expression for the final temperature T_0 on the answer sheet.

- Hence, obtain and write on the answer sheet an expression for this maximum amount of work available.

The heat engine operates between two tanks of water each of volume 2.50 m^3 . One tank is at 350 K and the other is at 300 K .

(c) Calculate the maximum amount of mechanical energy obtainable. Insert the value on the answer sheet.

Specific heat capacity of water $= 4.19 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$

Density of water $= 1.00 \times 10^3 \text{ kg m}^{-3}$

C

It is assumed that when the earth was formed the isotopes ^{238}U and ^{235}U were present but not their decay products. The decays of ^{238}U and ^{235}U are used to establish the age of the earth, T .

(a) The isotope ^{238}U decays with a half-life of 4.50×10^9 years. The decay products in the resulting radioactive series have half-lives short compared to this; to a first approximation their existence can be ignored. The decay series terminates in the stable lead isotope ^{206}Pb .

Obtain and insert on the answer sheet an expression for the number of ^{206}Pb atoms, denoted ^{206}n , produced by radioactive decay with time t , in terms of the present number of ^{238}U atoms, denoted ^{238}N , and the half-life time of ^{238}U . (You may find it helpful to work in units of 10^9 years.)

(b) Similarly, ^{235}U decays with a half-life of 0.710×10^9 years through a series of shorter half-life products to give the stable isotope ^{207}Pb .

Write down on the answer sheet an equation relating ^{207}n to ^{235}N and the half-life of ^{235}U .

(c) A uranium ore, mixed with a lead ore, is analysed with a mass spectrometer. The relative concentrations of the three lead isotopes ^{204}Pb , ^{206}Pb and ^{207}Pb are measured and the number of atoms are found to be in the ratios $1.00 : 29.6 : 22.6$ respectively. The isotope ^{204}Pb is used for reference as it is **not** of radioactive origin. Analysing a pure lead ore gives ratios of $1.00 : 17.9 : 15.5$.

Given that the ratio $^{238}N : ^{235}N$ is $137 : 1$, derive and insert on the answer sheet an equation involving T .

(d) Assume that T is much greater than the half lives of both uranium isotopes and hence obtain an approximate value for T .

(e) This approximate value is clearly not significantly greater than the longer half life, but can be used to obtain a much more accurate value for T . Hence, or otherwise, estimate a value for the age of the earth correct to within 2%.

D

Charge Q is uniformly distributed *in vacuo* throughout a spherical volume of radius R .

(a) Derive expressions for the electric field strength at distance r from the centre of the sphere for $r \leq R$ and $r > R$.

(b) Obtain an expression for the total electric energy associated with this distribution of charge.

Insert your answers to (a) and (b) on the answer sheet.

E

A circular ring of thin copper wire is set rotating about a vertical diameter at a point within the Earth's magnetic field. The magnetic flux density of the Earth's magnetic field at this point is $44.5 \mu\text{T}$ directed at an angle of 64° below the horizontal. Given that the density of copper is $8.90 \times 10^3 \text{ kg m}^{-3}$ and its resistivity is $1.70 \times 10^{-8} \Omega \text{ m}$, calculate how long it will take for the angular velocity of the ring to halve. Show the steps of your working and insert the value of the time on the answer sheet. This time is much longer than the time for one revolution.

You may assume that the frictional effects of the supports and air are negligible, and for the purposes of this question you should ignore self-inductance effects, although these would not be negligible.

Official English text of the problem.

Answers

- A(a):

$$y = L + \frac{mg}{k} \left(1 + \sqrt{1 + \frac{2kL}{mg}} \right).$$

- A(b):

$$v = \sqrt{2gL + \frac{mg^2}{k}}.$$

- A(c):

$$t = \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}} \left[\pi - \arctan \sqrt{\frac{2kL}{mg}} \right].$$

- B(a):

$$T_0 = \sqrt{T_A T_B}.$$

- B(b):

$$W_{\max} = ms \left(T_A + T_B - 2\sqrt{T_A T_B} \right) = ms \left(\sqrt{T_A} - \sqrt{T_B} \right)^2.$$

- B(c):

$$W_{\max} = 2.02 \times 10^7 \text{ J} = 20.2 \text{ MJ}.$$

- C(a): Writing $h_{238} = 4.50 \times 10^9 \text{ yr}$,

$$^{206}_{\text{n}} = ^{238}_{\text{N}} \left(2^{t/h_{238}} - 1 \right).$$

- **C(b):** Writing $h_{235} = 0.710 \times 10^9$ yr,

$$^{207}_{\text{n}} = ^{235}_{\text{N}} \left(2^{t/h_{235}} - 1 \right).$$

- **C(c):**

$$\frac{2^{T/h_{235}} - 1}{2^{T/h_{238}} - 1} = 137 \frac{7.1}{11.7}, \quad T > 0.$$

- **C(d):**

$$T_{\text{approx}} = \frac{\ln(137 \times 7.1/11.7)}{(\ln 2)(h_{235}^{-1} - h_{238}^{-1})} = 5.38 \times 10^9 \text{ yr}.$$

- **C(e):**

$$T \simeq 4.56 \times 10^9 \text{ yr}.$$

- **D(a):** The radial component of the electric field is

$$E_r(r) = \frac{Q}{4\pi\epsilon_0} \begin{cases} \frac{r}{R^3}, & 0 \leq r \leq R, \\ \frac{1}{r^2}, & r > R. \end{cases}$$

The field is outward for $Q > 0$, inward for $Q < 0$, and zero at the centre.

- **D(b):**

$$U = \frac{3Q^2}{20\pi\epsilon_0 R}.$$

- **E:**

$$t_{1/2} \simeq 1.10 \times 10^6 \text{ s} \simeq 12.8 \text{ days}.$$

Solution

A(a)

Let z be the downward distance from the bridge. The rope is slack until $z = L$; thereafter its extension is $x = z - L$, and its tension is kx .

There is no dissipation, so mechanical energy is conserved. At the lowest point, $z = y$, the jumper is instantaneously at rest. Thus

$$mgy = \frac{1}{2}k(y - L)^2.$$

Equivalently, with $x = y - L$,

$$x^2 - \frac{2mg}{k}x - \frac{2mgL}{k} = 0.$$

The physically admissible root has positive extension:

$$x = \frac{mg}{k} + \sqrt{\left(\frac{mg}{k}\right)^2 + \frac{2mgL}{k}}.$$

Therefore

$$y = L + \frac{mg}{k} \left(1 + \sqrt{1 + \frac{2kL}{mg}} \right).$$

A(b)

The speed increases while the downward acceleration is positive. Once the rope is taut,

$$m\ddot{z} = mg - k(z - L).$$

The speed is therefore greatest when the acceleration first becomes zero:

$$z = L + \frac{mg}{k}.$$

Energy conservation at any point with the rope taut gives

$$\frac{1}{2}mv^2 = mgz - \frac{1}{2}k(z - L)^2.$$

At the position of maximum speed,

$$\frac{1}{2}mv^2 = mg\left(L + \frac{mg}{k}\right) - \frac{1}{2}k\left(\frac{mg}{k}\right)^2 = mgL + \frac{m^2g^2}{2k}.$$

Hence

$$v = \sqrt{2gL + \frac{mg^2}{k}}.$$

A(c)

During the initial free fall through distance L , the elapsed time and final speed are

$$t_1 = \sqrt{\frac{2L}{g}}, \quad v_1 = \sqrt{2gL}.$$

Set $\tau = 0$ when the rope first becomes taut, and define

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad x_{\text{eq}} = \frac{mg}{k}.$$

The subsequent equation of motion is

$$\ddot{x} + \omega_0^2 x = g.$$

With $x(0) = 0$ and $\dot{x}(0) = v_1$, its solution is

$$x(\tau) = x_{\text{eq}}(1 - \cos \omega_0 \tau) + \frac{v_1}{\omega_0} \sin \omega_0 \tau,$$

so

$$\dot{x}(\tau) = x_{\text{eq}}\omega_0 \sin \omega_0 \tau + v_1 \cos \omega_0 \tau.$$

The first zero of this velocity occurs for $\pi/2 < \omega_0 \tau < \pi$. Consequently,

$$\omega_0 \tau = \pi - \arctan \left(\frac{v_1}{x_{\text{eq}} \omega_0} \right) = \pi - \arctan \sqrt{\frac{2kL}{mg}}.$$

Adding the free-fall time gives

$$t = \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}} \left[\pi - \arctan \sqrt{\frac{2kL}{mg}} \right].$$

B(a)

Maximum work is obtained by operating the engine reversibly between the bodies' instantaneous temperatures, until they reach a common temperature. All temperatures below are absolute temperatures.

Each body has constant heat capacity ms at constant pressure. Its entropy change is therefore

$$\Delta S = ms \int \frac{dT}{T}.$$

The engine operates cyclically, so its net entropy change is zero. For reversible operation, the combined entropy change of the bodies is also zero:

$$ms \int_{T_A}^{T_0} \frac{dT}{T} + ms \int_{T_B}^{T_0} \frac{dT}{T} = 0.$$

Thus

$$\ln \frac{T_0}{T_A} + \ln \frac{T_0}{T_B} = 0,$$

which yields

$$T_0 = \sqrt{T_A T_B}.$$

To see why this gives the maximum, an irreversible process would instead produce positive total entropy, requiring

$$T_0 \geq \sqrt{T_A T_B}.$$

A higher final temperature leaves more thermal energy in the bodies and consequently permits less work to be extracted.

B(b)

The heat extracted from the initially hotter body is

$$Q_H = ms(T_A - T_0),$$

and the heat delivered to the initially colder body is

$$Q_C = ms(T_0 - T_B).$$

Energy conservation for the cyclic engine gives

$$W = Q_H - Q_C = ms(T_A + T_B - 2T_0).$$

Substituting the reversible final temperature,

$$W_{\text{max}} = ms \left(T_A + T_B - 2\sqrt{T_A T_B} \right) = ms \left(\sqrt{T_A} - \sqrt{T_B} \right)^2.$$

B(c)

The mass of water in each tank is

$$m = \rho V = (1.00 \times 10^3)(2.50) = 2.50 \times 10^3 \text{ kg}.$$

The reversible final temperature is

$$T_0 = \sqrt{350 \times 300} = 324.037 \dots \text{ K}.$$

Using this value without premature rounding,

$$\begin{aligned} W_{\max} &= (2.50 \times 10^3)(4.19 \times 10^3) \left[350 + 300 - 2\sqrt{350 \times 300} \right] \\ &= 2.0174 \dots \times 10^7 \text{ J}. \end{aligned}$$

Therefore

$$W_{\max} = 2.02 \times 10^7 \text{ J} = 20.2 \text{ MJ}.$$

C(a)

For a radioactive parent isotope with half-life h ,

$$N(t) = N(0)e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{h}.$$

Hence the initial number of parent atoms can be expressed in terms of the present number:

$$N(0) = N(t)e^{\lambda t} = N(t)2^{t/h}.$$

Neglecting the lifetimes of the intermediate products, each decayed uranium atom produces one terminal lead atom. Since there were initially no decay products,

$$n(t) = N(0) - N(t) = N(t) \left(2^{t/h} - 1 \right).$$

For the ^{238}U series,

$$^{206}_{\text{Pb}} n = ^{238}\text{U} N \left(2^{t/h_{238}} - 1 \right), \quad h_{238} = 4.50 \times 10^9 \text{ yr}.$$

C(b)

Applying the same reasoning to the ^{235}U series,

$$^{207}_{\text{Pb}} n = ^{235}\text{U} N \left(2^{t/h_{235}} - 1 \right), \quad h_{235} = 0.710 \times 10^9 \text{ yr}.$$

C(c)

The pure lead ore supplies the background isotope ratios. These background amounts must be subtracted from the mixed sample to find the lead produced by uranium decay.

Let N_{204} be the number of ^{204}Pb atoms in the mixed sample. Since this isotope is not radiogenic,

$$^{206}_{\text{Pb}} n = (29.6 - 17.9)N_{204} = 11.7N_{204},$$

and

$$^{207}_{\text{Pb}} n = (22.6 - 15.5)N_{204} = 7.1N_{204}.$$

Therefore

$$\frac{{}^{207}\text{n}}{{}^{206}\text{n}} = \frac{7.1}{11.7}.$$

Using the decay equations and the present uranium ratio,

$$\frac{7.1}{11.7} = \frac{{}^{235}\text{N}}{{}^{238}\text{N}} \frac{2^{T/h_{235}} - 1}{2^{T/h_{238}} - 1} = \frac{1}{137} \frac{2^{T/h_{235}} - 1}{2^{T/h_{238}} - 1}.$$

Thus the required equation is

$$\boxed{\frac{2^{T/h_{235}} - 1}{2^{T/h_{238}} - 1} = 137 \frac{7.1}{11.7} = 83.13675\dots}, \quad T > 0.$$

C(d)

Under the stated approximation $T \gg h_{238}, h_{235}$,

$$2^{T/h} - 1 \simeq 2^{T/h}.$$

The dating equation becomes

$$2^{T(1/h_{235} - 1/h_{238})} \simeq 137 \frac{7.1}{11.7}.$$

Taking logarithms,

$$T_{\text{approx}} = \frac{\ln(137 \times 7.1/11.7)}{(\ln 2)(h_{235}^{-1} - h_{238}^{-1})}.$$

Numerically,

$$T_{\text{approx}} = \frac{\ln(83.13675\dots)}{(\ln 2)(1/0.710 - 1/4.50)} \times 10^9 \text{ yr},$$

so

$$\boxed{T_{\text{approx}} = 5.38 \times 10^9 \text{ yr}}.$$

This is only about $1.19h_{238}$, so neglecting the subtraction of 1 in the ${}^{238}\text{U}$ expression is not justified for an accurate result.

C(e)

Retain both subtractions of 1. Writing

$$u = \frac{T}{10^9 \text{ yr}},$$

we must solve

$$F(u) \equiv \frac{2^{u/0.710} - 1}{2^{u/4.50} - 1} = 83.13675\dots$$

Nearby evaluations give

$$F(4.55) = 82.666\dots, \quad F(4.56) = 83.232\dots$$

Thus the required age lies between 4.55 and 4.56 billion years. A more precise numerical solution is $u = 4.5583157\dots$, giving

$$\boxed{T \simeq 4.56 \times 10^9 \text{ yr}},$$

well within the requested 2%.

D(a)

The uniform volume charge density is

$$\rho_q = \frac{Q}{(4/3)\pi R^3}.$$

Spherical symmetry makes the electric field radial and constant in magnitude on a concentric spherical Gaussian surface.

For $r \leq R$, the enclosed charge is

$$Q_{\text{enc}} = \rho_q \frac{4\pi r^3}{3} = Q \frac{r^3}{R^3}.$$

Gauss's law gives

$$4\pi r^2 E_r = \frac{Q_{\text{enc}}}{\varepsilon_0},$$

and hence

$$E_r = \frac{Qr}{4\pi\varepsilon_0 R^3} \quad (r \leq R).$$

For $r > R$, the enclosed charge is Q , so

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (r > R).$$

The field is zero at the centre and continuous at the surface, where

$$E_r(R) = \frac{Q}{4\pi\varepsilon_0 R^2}.$$

D(b)

In vacuum the electrostatic energy density is

$$u_E = \frac{1}{2}\varepsilon_0 E^2.$$

The total energy includes the field both inside and outside the charged sphere:

$$U = \frac{\varepsilon_0}{2} \left[\int_0^R E^2 4\pi r^2 dr + \int_R^\infty E^2 4\pi r^2 dr \right].$$

Substituting the fields found above,

$$U = \frac{Q^2}{8\pi\varepsilon_0} \left[\frac{1}{R^6} \int_0^R r^4 dr + \int_R^\infty \frac{dr}{r^2} \right].$$

Therefore

$$U = \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{5R} + \frac{1}{R} \right),$$

and

$$U = \frac{3Q^2}{20\pi\epsilon_0 R}.$$

E

Let the ring have radius a , and let the wire's cross-sectional area be S . Denote copper's mass density by ρ_m and its electrical resistivity by ρ_e .

The ring's mass and resistance are

$$M = 2\pi a S \rho_m, \quad \mathcal{R} = \rho_e \frac{2\pi a}{S}.$$

For a thin ring rotating about a diameter, the moment of inertia is

$$I = \frac{1}{2} M a^2 = \pi \rho_m S a^3.$$

The ring's plane remains vertical, so its normal is horizontal. Only the horizontal component of the Earth's field contributes to the magnetic flux:

$$B_h = B \cos 64^\circ.$$

Let θ be the angle between the ring's normal and this horizontal field. The flux is

$$\Phi = B_h \pi a^2 \cos \theta.$$

With $\dot{\theta} = \omega$, Faraday's law gives

$$\mathcal{E} = -\frac{d\Phi}{dt} = B_h \pi a^2 \omega \sin \theta.$$

Neglecting self-inductance as instructed, the current is

$$i = \frac{\mathcal{E}}{\mathcal{R}}.$$

The instantaneous rate of Joule heating is consequently

$$P = i^2 \mathcal{R} = \frac{B_h^2 \pi^2 a^4}{\mathcal{R}} \omega^2 \sin^2 \theta.$$

The stated damping time is much longer than one revolution, so ω changes only slightly during each revolution. We may therefore use

$$\langle \sin^2 \theta \rangle = \frac{1}{2},$$

giving

$$\langle P \rangle = \frac{B_h^2 \pi^2 a^4}{2\mathcal{R}} \omega^2.$$

With friction absent, this heat comes from the rotational kinetic energy:

$$\frac{d}{dt} \left(\frac{1}{2} I \omega^2 \right) = -\langle P \rangle.$$

Thus the averaged angular velocity satisfies

$$\frac{d\omega}{dt} = -\frac{B_h^2 \pi^2 a^4}{2I\mathcal{R}} \omega.$$

But

$$I\mathcal{R} = (\pi\rho_m S a^3) \left(\rho_e \frac{2\pi a}{S} \right) = 2\pi^2 \rho_m \rho_e a^4,$$

so the geometrical quantities cancel:

$$\frac{d\omega}{dt} = -\frac{B_h^2}{4\rho_m \rho_e} \omega.$$

Hence

$$\omega(t) = \omega(0) \exp \left(-\frac{B_h^2 t}{4\rho_m \rho_e} \right).$$

Setting $\omega(t_{1/2}) = \omega(0)/2$ yields

$$t_{1/2} = \frac{4\rho_m \rho_e \ln 2}{B_h^2} = \frac{4\rho_m \rho_e \ln 2}{B^2 \cos^2 64^\circ}.$$

Numerically,

$$B_h = (44.5 \times 10^{-6}) \cos 64^\circ = 1.95075 \dots \times 10^{-5} \text{ T},$$

and

$$t_{1/2} = \frac{4(8.90 \times 10^3)(1.70 \times 10^{-8}) \ln 2}{(44.5 \times 10^{-6})^2 \cos^2 64^\circ}.$$

Therefore

$$t_{1/2} \simeq 1.10 \times 10^6 \text{ s} \simeq 12.8 \text{ days}.$$

The solution was checked against the official solution.

Problem 2: Charge-to-mass ratio of the electron (magnetic focusing in a CRT; beta particles between charged plates)

(a) A cathode ray tube (CRT), consisting of an electron gun and a screen, is placed within a uniform constant magnetic field of magnitude $\mathbf{B}$ such that the magnetic field is parallel to the beam axis of the gun, as shown in figure 2.3.

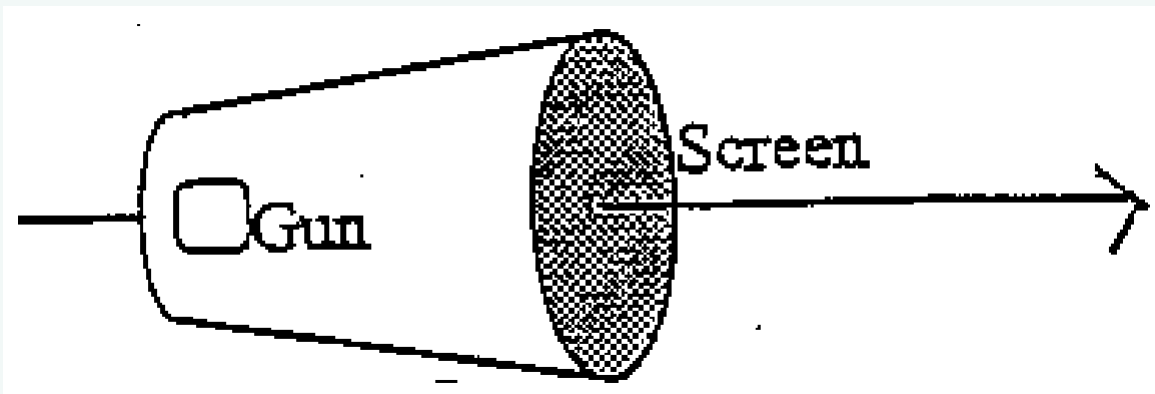


Figure 2.3

The electron beam emerges from the anode of the electron gun on the axis, but with a divergence of up to 5° from the axis, as illustrated in figure 2.4. In general a diffuse spot is produced on the screen, but for certain values of the magnetic field a sharply focused spot is obtained.

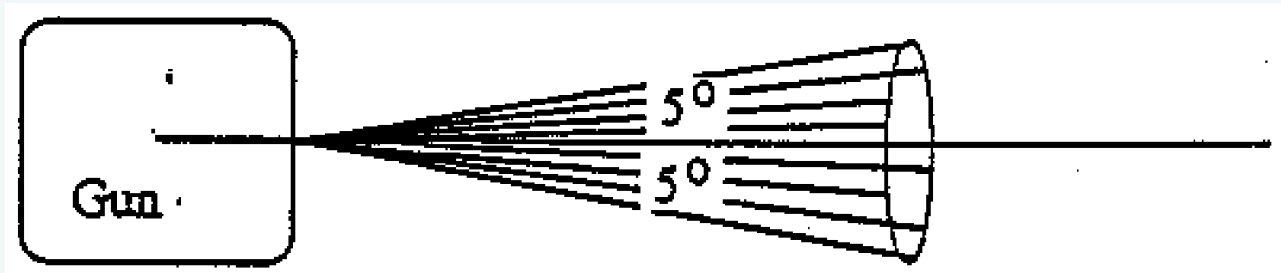


Figure 2.4

By considering the motion of an electron initially moving at an angle β (where $0 \leq \beta \leq 5^\circ$) to the axis as it leaves the electron gun, and considering the components of its motion parallel and perpendicular to the axis, derive an expression for the charge to mass ratio e/m for the electron in terms of the following quantities:

the smallest magnetic field for which a focused spot is obtained,
 the accelerating potential difference across the electron gun V
 (note that $V < 2 \text{ kV}$),
 and D , the distance between the anode and the screen.

Write your expression in the box provided in section 2a of the answer sheet.

(b) Consider another method of evaluating the charge to mass ratio of the electron. The arrangement is shown from a side view and in plan view (from above) in figure 2.5, with the direction of the magnetic field marked $\mathbf{B}$. Within this uniform magnetic field $\mathbf{B}$ are placed two brass circular plates of radius ρ which are separated by a very small distance t . A potential difference V is maintained between them. The plates are mutually parallel and co-axial, however their axis is perpendicular to the magnetic field. A photographic film, covers the inside of the curved surface of a cylinder of radius $\rho + s$, which is held co-axial with the plates. In other words, the film is at a radial distance s from the edges of the plates. The entire arrangement is placed *in vacuo*. Note that t is very much smaller than both s and ρ .

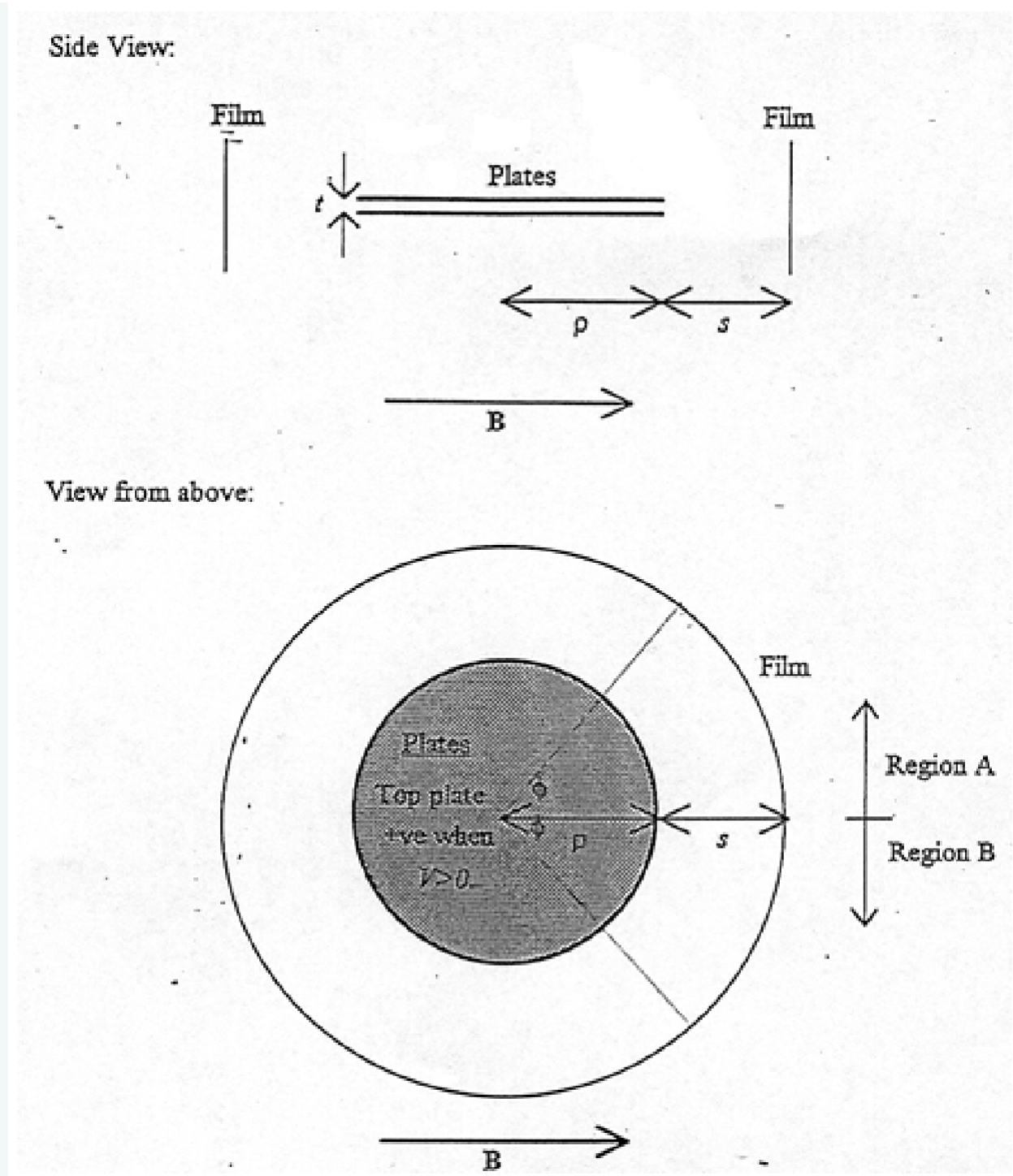


Figure 2.5

A point source of β particles, which emits the β particles uniformly in all directions with a range of velocities, is placed between the centres of the plates, and the *same piece of film* is exposed under three different conditions:

- firstly with $B = 0$, and $V = 0$,
- secondly with $B = B_0$, and $V = V_0$, and
- thirdly with $B = -B_0$, and $V = -V_0$;

where V_0 and B_0 are positive constants. Please note that the upper plate is positively charged when $V > 0$ (negative when $V < 0$), and that the magnetic field is in the direction defined by figure 2.5 when $B > 0$ (in the opposite direction when $B < 0$). For this part you may assume the

gap is negligibly small.

Two regions of the film are labelled A and B on figure 2.5. After exposure and development, a sketch of one of these regions is given in figure 2.6. From which region was this piece taken (on your answer sheet write A or B)? Justify your answer by showing the directions of the forces acting on the electron.

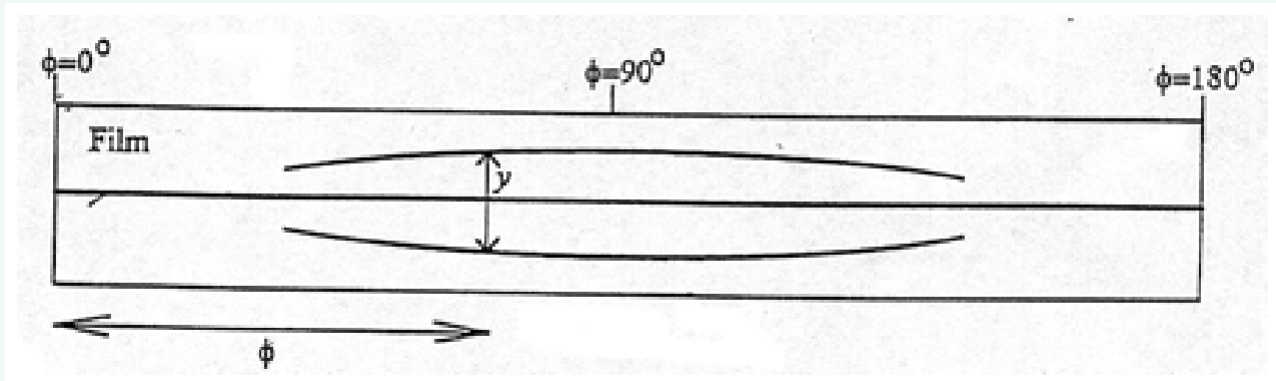


Figure 2.6

(c) After exposure and development, a sketch of the film is given in figure 2.6. Measurements are made of the separation of the two outermost traces with a microscope, and this distance (y) is also indicated for one particular angle on figure 2.6. The results are given in the table below, the angle ϕ being defined in figure 2.5 as the angle between the magnetic field and a line joining the centre of the plates to the point on the film.

Angle to field /degrees	ϕ	90	60	50	40	30	23
Separation /mm	y	17.4	12.7	9.7	6.4	3.3	End of trace

Numerical values of the system parameters are given below:

$$B_0 = 6.91 \text{ mT} \quad V_0 = 580 \text{ V} \quad t = 0.80 \text{ mm} \\ s = 41.0 \text{ mm}$$

In addition, you may take the speed of light in vacuum to be $3.00 \times 10^8 \text{ m s}^{-1}$, and the rest mass of the electron to be $9.11 \times 10^{-31} \text{ kg}$.

Determine the maximum β particle kinetic energy observed.

Write the maximum kinetic energy as a numerical result in eV in the box on the answer sheet, section 2f.

(d) Using the information given in part (f) [i.e. part (c) above], obtain a value for the charge to rest mass ratio of the electron. This should be done by plotting an appropriate graph on the paper provided.

Indicate *algebraically* the quantities being plotted on the horizontal and vertical axes both on the graph itself and on the answer sheet in the boxes provided in section 2g.

Write your value for the charge to mass ratio of the electron in the box provided on the answer sheet, section 2g.

Please note that the answer you obtain may not agree with the accepted value because of a systematic error in the observations.

Answers

- **(a)** For the smallest positive focusing field $B_{\min}$,

$$\frac{e}{m} \simeq \frac{8\pi^2 V}{B_{\min}^2 D^2}.$$

- **(b)** Region B. For $B = B_0$, $V = V_0$, the electric force is upward and the magnetic force downward. Reversing both fields reverses both forces.

- **(c)**

$$K_{\max} \simeq 6.4 \times 10^5 \text{ eV}$$

corresponding to $v_{\max} \simeq 2.69 \times 10^8 \text{ m s}^{-1}$.

- **(d)** Plot $Y = y$ vertically against the dimensionless horizontal quantity

$$X = \sin \phi \sqrt{\sin^2 \phi - \left(\frac{V_0}{B_0 t c} \right)^2}.$$

The straight line has slope approximately $1.855 \times 10^{-2} \text{ m}$, giving

$$\frac{e}{m_0} \simeq 1.68 \times 10^{11} \text{ C kg}^{-1}.$$

Solution

Throughout, $e > 0$ denotes the magnitude of the electron charge; the electron's charge is $-e$. Its rest mass is m_0 .

(a)

Since the accelerating voltage is below 2 kV, the electron's kinetic energy is much smaller than its rest energy. We use nonrelativistic mechanics, with $m \simeq m_0$:

$$eV = \frac{1}{2}mv^2.$$

The velocity components on leaving the gun are

$$v_{\parallel} = v \cos \beta, \quad v_{\perp} = v \sin \beta.$$

The magnetic field exerts no force parallel to itself. The parallel motion is therefore uniform, while the perpendicular motion is circular, with

$$r = \frac{mv_{\perp}}{eB}, \quad \omega = \frac{eB}{m}, \quad T = \frac{2\pi m}{eB}.$$

The resulting trajectory is a helix. An electron starting on the axis returns to the axis after each complete revolution. The smallest positive focusing field corresponds to one revolution before reaching the screen, so

$$D = v \cos \beta T = \frac{2\pi m v \cos \beta}{e B_{\min}}.$$

Squaring and substituting $v^2 = 2eV/m$ gives

$$\frac{e}{m} = \frac{8\pi^2 V \cos^2 \beta}{B_{\min}^2 D^2}.$$

For the narrow beam, $\beta \leq 5^\circ$, we take $\cos \beta \simeq 1$. Hence

$$\boxed{\frac{e}{m} \simeq \frac{8\pi^2 V}{B_{\min}^2 D^2}}.$$

The focusing is approximate for a beam containing different β : their axial velocities differ slightly. At 5° , $1 - \cos \beta$ is only about 0.0038.

(b)

The Lorentz force is

$$\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

For $V = V_0 > 0$, the upper plate is positive, so the electric field points downward and the electric force on an electron points upward.

For $B = B_0 > 0$, applying the cross-product rule to the plan view gives:

Electron travelling toward	Electric force	Magnetic force
Region A	Upward	Upward
Region B	Upward	Downward

With a negligibly small gap, electrons can traverse the plates only when their vertical forces balance. This is possible in **region B**.

Let θ be the angle between the electron's velocity and the magnetic field while it travels between the plates. Neglecting fringing,

$$E_0 = \frac{V_0}{t}.$$

Force balance requires

$$eE_0 = eB_0 v \sin \theta,$$

or

$$\boxed{v \sin \theta = \frac{V_0}{B_0 t}}.$$

Reversing both B and V reverses both forces, so they still balance for electrons travelling toward region B.

After an electron leaves the plates, the electric force disappears while the magnetic force remains. Thus:

- $B = B_0$, $V = V_0$ produces the lower curved trace.
- $B = -B_0$, $V = -V_0$ produces the upper curved trace.
- $B = V = 0$ produces the central straight trace.

These are the three traces shown in figure 2.6, establishing

Region B.

(c)

We use the thin-gap approximation and, to leading order in the subsequent deflection, identify the electron's departure angle θ with the observed angle ϕ . Define

$$u = \frac{V_0}{B_0 t}.$$

Numerically,

$$u = \frac{580}{(6.91 \times 10^{-3})(0.80 \times 10^{-3})} = 1.0492 \times 10^8 \text{ m s}^{-1}.$$

The velocity-selection condition becomes

$$v(\phi) = \frac{u}{\sin \phi}.$$

As ϕ decreases from 90° , the selected speed increases. The smallest observed angle, 23° , therefore corresponds to the maximum speed:

$$v_{\max} = \frac{1.0492 \times 10^8}{\sin 23^\circ} = 2.6852 \times 10^8 \text{ m s}^{-1} = 0.8951c.$$

The "End of trace" entry specifies this angular cutoff; it supplies no numerical measurement of y .

Relativistic kinetic energy is required:

$$\gamma_{\max} = \frac{1}{\sqrt{1 - v_{\max}^2/c^2}} = 2.2426,$$

$$K_{\max} = (\gamma_{\max} - 1)m_0c^2.$$

Therefore

$$K_{\max} = (2.2426 - 1)(9.11 \times 10^{-31})(3.00 \times 10^8)^2 = 1.0188 \times 10^{-13} \text{ J}.$$

Using the standard unit conversion $1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ J}$,

$$K_{\max} = 6.36 \times 10^5 \text{ eV}.$$

To sensible precision for the endpoint angle,

$K_{\max} \simeq 6.4 \times 10^5 \text{ eV} = 0.64 \text{ MeV}.$

(d)

Outside the plates, we neglect the electric fringe field and retain the leading magnetic deflection. The electron's horizontal motion is then approximately uniform, and its flight time from the plate edge to the film is

$$\tau \simeq \frac{s}{v}.$$

The magnetic field does no work, so the electron's speed and Lorentz factor remain constant. To leading order, the vertical magnetic force has magnitude

$$F_z = eB_0v \sin \phi,$$

giving vertical acceleration

$$a_z = \frac{eB_0 v \sin \phi}{\gamma m_0}.$$

The displacement of either curved trace from the central trace is consequently

$$\delta \simeq \frac{1}{2} a_z \tau^2 = \frac{eB_0 s^2 \sin \phi}{2\gamma m_0 v}.$$

The two field settings produce opposite displacements. Their separation is therefore

$$y = 2\delta \simeq \frac{eB_0 s^2 \sin \phi}{\gamma m_0 v}.$$

Substituting

$$v = \frac{V_0}{B_0 t \sin \phi}, \quad \frac{1}{\gamma} = \sqrt{1 - \left(\frac{V_0}{B_0 t c \sin \phi} \right)^2},$$

we obtain

$$y \simeq \frac{e}{m_0} \frac{B_0^2 t s^2}{V_0} \sin^2 \phi \sqrt{1 - \left(\frac{V_0}{B_0 t c \sin \phi} \right)^2}.$$

Since $0 < \phi < 180^\circ$, this can be written

$$y \simeq \frac{e}{m_0} \frac{B_0^2 t s^2}{V_0} \sin \phi \sqrt{\sin^2 \phi - \left(\frac{V_0}{B_0 t c} \right)^2}.$$

Thus choose the graph axes as

$$X = \sin \phi \sqrt{\sin^2 \phi - \left(\frac{V_0}{B_0 t c} \right)^2}, \quad Y = y.$$

The horizontal quantity is dimensionless. Here

$$\frac{V_0}{B_0 t c} = 0.349735.$$

The points to plot are:

ϕ	Horizontal coordinate X	Vertical coordinate Y / mm
90°	0.93685	17.4
60°	0.68612	12.7
50°	0.52210	9.7
40°	0.34667	6.4
30°	0.17867	3.3

The graph is an almost straight line through the origin. Suitable axis ranges are $0 \leq X \leq 1$ and $0 \leq Y \leq 20$ mm. A fit through the origin gives

$$Y \simeq (18.55 \text{ mm})X.$$

Writing the slope in metres,

$$S = 1.855 \times 10^{-2} \text{ m},$$

the theoretical relation gives

$$S = \frac{e}{m_0} \frac{B_0^2 t s^2}{V_0}.$$

Hence

$$\frac{e}{m_0} = \frac{SV_0}{B_0^2 t s^2} \simeq \frac{(1.855 \times 10^{-2})(580)}{(6.91 \times 10^{-3})^2 (0.80 \times 10^{-3})(41.0 \times 10^{-3})^2}.$$

Therefore the value inferred from the observations is

$$\boxed{\frac{e}{m_0} \simeq 1.68 \times 10^{11} \text{ C kg}^{-1}}.$$

This derivation retains the leading deflection. At the largest measured separation, $\delta/s = 8.7/41.0 \simeq 0.21$; finite-curvature corrections can therefore contribute effects of a few percent.

The solution was checked against the official solution.

Problem 3: Gravitational-wave detector (Part A) and gravitational deflection of light by the Sun (Part B)

Gravitational waves and the effects of gravity on light.

Part A

This part is concerned with the difficulties of detecting gravitational waves generated by astronomical events. It should be realised that the explosion of a distant supernova may produce fluctuations in the gravitational field strength at the surface of the Earth of about $10^{-19} \text{ N kg}^{-1}$.

A model for a gravitational wave detector (see figure 3.1) consists of two metal rods each 1m long, held at right angles to each other. One end of each rod is polished optically flat and the other end is held rigidly. The position of one rod is adjusted so there is a minimum signal received from the photocell (see figure 3.1).

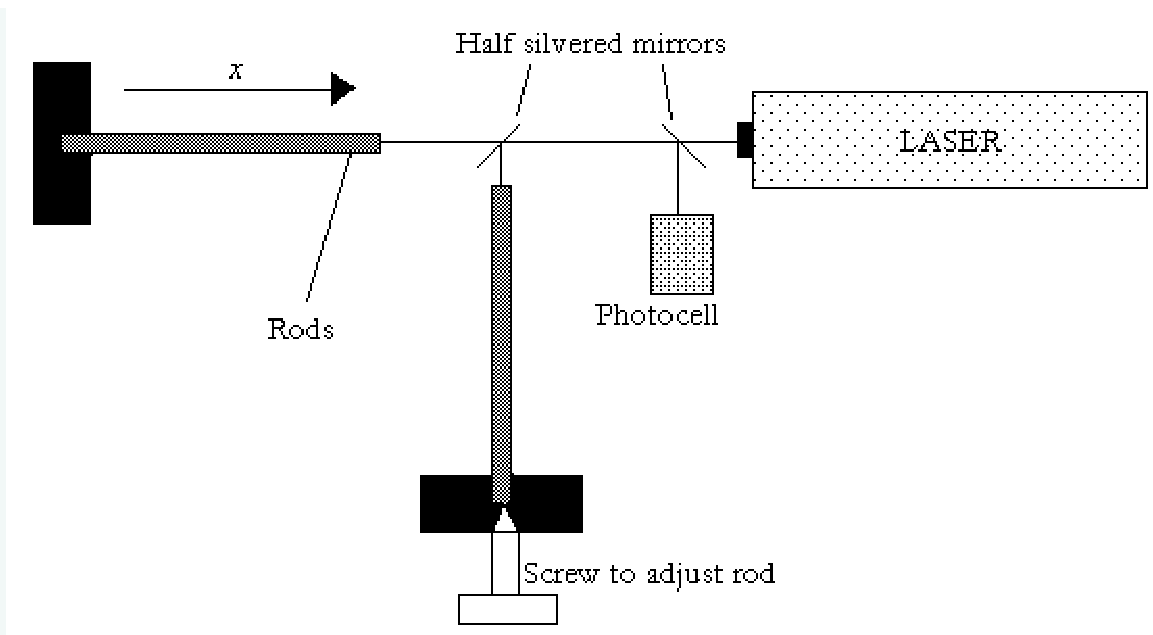


Figure 3.1

The rods are given a short sharp impulse by a piezoelectric device. As a result the free ends of the rods oscillate with a longitudinal displacement Δx_t , where

$$\Delta x_t = ae^{-\mu t} \cos(\omega t + \phi),$$

and a , μ , ω and ϕ are constants.

- If the amplitude of the motion is reduced by 20% during a 50s interval determine a value for μ .
- Determine also a value for ω given that the rods are made of aluminium with a density (ρ) of 2700 kg m^{-3} and a Young modulus (E) of $7.1 \times 10^{10} \text{ Pa}$.
- It is impossible to make the rods exactly the same length so the photocell signal has a beat frequency of 0.005 Hz. What is the difference in length of the rods?
- For the rod of length l , derive an algebraic expression for the change in length, Δl , due to a change, Δg , in the gravitational field strength, g , in terms of l and other constants of the rod material.
- The light produced by the laser is monochromatic with a wavelength of 656nm. If the minimum fringe shift that can be detected is 10^{-4} of the wavelength of the laser, what is the minimum value of l necessary if such a system were to be capable of detecting variations in g of $10^{-19} \text{ N kg}^{-1}$?

A non-directional form of gravitational wave detector consists of a sphere of copper-alloy of mass 1168 kg, suspended in a vacuum from a vibration-reducing assembly. Transducers, containing tuned circuits, are attached to the sphere to detect changes in its dimensions. The transducers will, however, pick up all spurious vibrations due to, for example, temperature effects and noise due to electric pick up.

- To reduce vibrations due to temperature effects the sphere is maintained at a temperature of 100 mK. By what factor will the amplitude of the atomic vibrations been reduced in cooling the assembly from 300 K?

(g) The sphere is initially cooled to 4.2 K using liquid nitrogen and liquid helium. The temperature, T , is further reduced to 100 mK by a refrigeration process, which removes energy from the system at an average rate of 1 mW. Given that the specific thermal capacity, s , of the copper-alloy varies directly as T^3 at these low temperatures, estimate the time taken for the system to cool from 4.2 K to 100 mK, given that $s = 0.072 \text{ J kg}^{-1} \text{ K}^{-1}$ at 4.2 K.

Part B

This part is concerned with the effect of a gravitational field on the propagation of light in space.

(a) A photon emitted from the surface of the Sun (mass M , radius R) is red-shifted. By assuming a rest-mass equivalent for the photon energy, apply Newtonian gravitational theory to show that the effective (or measured) frequency of the photon at infinity is reduced (red-shifted) by the factor $(1 - GM/Rc^2)$.

(b) A reduction of the photon's frequency is equivalent to an increase in its time period, or, using the photon as a standard clock, a dilation of time. In addition, it may be shown that a time dilation is always accompanied by a contraction in the unit of length by the same factor.

We will now try to study the effect that this has on the propagation of light near the Sun. Let us first define an effective refractive index n_r at a point r from the centre of the Sun. Let

$$n_r = \frac{c}{c'_r},$$

where c is the speed of light as measured by a co-ordinate system far away from the Sun's gravitational influence ($r \rightarrow \infty$), and c'_r is the speed of light as measured by a co-ordinate system at a distance r from the centre of the Sun.

Show that n_r may be approximated to:

$$n_r = 1 + \frac{\alpha GM}{rc^2},$$

for small GM/rc^2 , where α is a constant that you determine.

(c) Using this expression for n_r , calculate in radians the deflection of a light ray from its straight path as it passes the edge of the Sun.

Data:

Gravitational constant, $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$.

Mass of Sun, $M = 1.99 \times 10^{30} \text{ kg}$.

Radius of Sun, $R = 6.95 \times 10^8 \text{ m}$.

Velocity of light, $c = 3.00 \times 10^8 \text{ m s}^{-1}$.

You may also need the following integral:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{2}{a^2}.$$

Official English text of the problem.

Answers

- **A(a):** $\mu = 4.46 \times 10^{-3} \text{ s}^{-1}$.
- **A(b):** For the fundamental longitudinal mode, $\omega \simeq 8.06 \times 10^3 \text{ rad s}^{-1}$.
- **A(c):** $|l_1 - l_2| \simeq 3.90 \times 10^{-6} \text{ m} = 3.90 \text{ }\mu\text{m}$.
- **A(d):** $\Delta l = \frac{\rho l^2 \Delta g}{2E}$, with Δg taken along the rod, positive towards its free end.
- **A(e):** $l_{\min} \simeq 1.31 \times 10^8 \text{ m}$, interpreting the threshold as an optical path difference and taking one arm's axial response relative to the other.
- **A(f):** The amplitude decreases by a factor 54.8 , to 0.0183 of its initial value, in the classical harmonic model.
- **A(g):** $t \simeq 8.83 \times 10^4 \text{ s} \simeq 24.5 \text{ h}$.
- **B(a):** $\nu_{\infty} \simeq \nu_R \left(1 - \frac{GM}{Rc^2} \right)$.
- **B(b):** $\alpha = 2, \quad n_r \simeq 1 + \frac{2GM}{rc^2}$.
- **B(c):** $\theta = \frac{4GM}{Rc^2} = 8.49 \times 10^{-6} \text{ rad}$, towards the Sun.

Solution

A(a)

The amplitude envelope is

$$A(t) = ae^{-\mu t}.$$

A reduction by 20% means that the amplitude after 50 s is 0.80 times its previous value:

$$\frac{A(t+50)}{A(t)} = e^{-50\mu} = 0.80.$$

Therefore,

$$\mu = -\frac{\ln(0.80)}{50} = 4.46 \times 10^{-3} \text{ s}^{-1}.$$

A(b)

Treat the rod as a uniform elastic continuum, and take the observed oscillation to be its fundamental longitudinal mode.

Let $u(x, t)$ be the longitudinal displacement and S the cross-sectional area. The longitudinal force is $ES \partial u / \partial x$. Newton's second law for a short element gives

$$\rho S dx \frac{\partial^2 u}{\partial t^2} = ES dx \frac{\partial^2 u}{\partial x^2}.$$

Thus longitudinal waves travel at

$$v = \sqrt{\frac{E}{\rho}}.$$

The fixed end has zero displacement, while the free end has zero longitudinal stress:

$$u(0, t) = 0, \quad \left. \frac{\partial u}{\partial x} \right|_{x=l} = 0.$$

For a mode with spatial dependence $\sin(kx)$, the free-end condition requires

$$\cos(kl) = 0.$$

For the fundamental mode, $kl = \pi/2$, so the rod is one quarter of a wavelength long. Hence

$$\omega_0 = vk = \frac{\pi}{2l} \sqrt{\frac{E}{\rho}}.$$

Numerically,

$$v = \sqrt{\frac{7.1 \times 10^{10}}{2700}} = 5.128 \times 10^3 \text{ m s}^{-1},$$

and, for $l = 1 \text{ m}$,

$$\omega_0 = \frac{\pi}{2} (5.128 \times 10^3) = 8.055 \times 10^3 \text{ rad s}^{-1}.$$

For linear damping, the observed angular frequency is $\omega = \sqrt{\omega_0^2 - \mu^2}$. Since $\mu \ll \omega_0$, this correction is negligible:

$$\boxed{\omega \simeq 8.06 \times 10^3 \text{ rad s}^{-1}}.$$

The corresponding frequency is

$$f = \frac{\omega}{2\pi} = \frac{v}{4l} \simeq 1.282 \times 10^3 \text{ Hz}.$$

A(c)

For rods made of the same material and oscillating in the same fundamental mode,

$$f(l) = \frac{v}{4l}.$$

Consequently, their beat frequency is

$$f_b = |f_1 - f_2| = \frac{v}{4} \left| \frac{1}{l_1} - \frac{1}{l_2} \right|.$$

For a small length difference $\delta l = |l_1 - l_2|$, with both lengths approximately l ,

$$f_b \simeq \frac{v}{4l^2} \delta l = \frac{f}{l} \delta l.$$

Therefore,

$$\delta l \simeq \frac{l f_b}{f} = \frac{(1)(0.005)}{1281.998} \text{ m},$$

giving

$$\boxed{\delta l \simeq 3.90 \times 10^{-6} \text{ m}}.$$

The ratio $\delta l/l \simeq 3.9 \times 10^{-6}$ confirms the validity of the small-difference approximation.

A(d)

Use linear elasticity and a quasi-static gravitational perturbation. Let x run from the fixed end, $x = 0$, to the free end, $x = l$, and let Δg denote the signed component of the field change along this direction.

The material beyond a cross-section at x has mass

$$m(x) = \rho S(l - x).$$

The additional axial force transmitted through that section is therefore

$$\Delta F(x) = \rho S(l - x)\Delta g.$$

By Young's modulus, the change in local strain is

$$\Delta \epsilon(x) = \frac{\Delta F(x)}{ES} = \frac{\rho(l - x)\Delta g}{E}.$$

Integrating the extension of all elements,

$$\Delta l = \int_0^l \Delta \epsilon(x) dx = \frac{\rho \Delta g}{E} \int_0^l (l - x) dx.$$

Thus

$$\boxed{\Delta l = \frac{\rho l^2 \Delta g}{2E}}.$$

A field change directed towards the fixed end gives compression, represented by a negative Δg .

A(e)

A displacement Δl of a reflecting end changes the round-trip optical path by $2\Delta l$. Taking the field change along one rod, with the other acting as the longitudinal reference,

$$|\Delta D| = 2|\Delta l|.$$

A change of optical path by one wavelength moves the interference condition through one fringe. The stated sensitivity therefore corresponds to

$$|\Delta D|_{\min} = 10^{-4} \lambda = 10^{-4} (656 \times 10^{-9}) = 6.56 \times 10^{-11} \text{ m}.$$

The minimum detectable rod-end displacement is consequently

$$|\Delta l|_{\min} = 3.28 \times 10^{-11} \text{ m}.$$

Using A(d),

$$2|\Delta l| = \frac{\rho l^2 |\Delta g|}{E}.$$

At the detection threshold,

$$l_{\min} = \sqrt{\frac{E 10^{-4} \lambda}{\rho |\Delta g|}}.$$

Substitution gives

$$l_{\min} = \sqrt{\frac{(7.1 \times 10^{10})(10^{-4})(656 \times 10^{-9})}{(2700)(10^{-19})}},$$

so

$$l_{\min} \simeq 1.31 \times 10^8 \text{ m}.$$

More generally, if both rods respond, Δg in this expression is replaced by the difference between their signed axial field changes.

A(f)

In the classical harmonic approximation, an atomic displacement x has elastic potential energy $\frac{1}{2} \kappa x^2$. Equipartition gives

$$\frac{1}{2} \kappa \langle x^2 \rangle = \frac{1}{2} k_B T.$$

For unchanged effective stiffness κ , the characteristic vibration amplitude therefore scales as

$$A \propto \sqrt{T}.$$

Since $100 \text{ mK} = 0.100 \text{ K}$,

$$\frac{A_{\text{cold}}}{A_{\text{warm}}} = \sqrt{\frac{0.100}{300}} = \frac{1}{\sqrt{3000}} = 0.0183.$$

Thus the amplitude is reduced by a factor

$$\frac{A_{\text{warm}}}{A_{\text{cold}}} = \sqrt{3000} \simeq 54.8.$$

This estimate uses classical thermal vibrations and neglects quantum zero-point motion.

A(g)

Let

$$T_i = 4.2 \text{ K}, \quad T_f = 0.100 \text{ K}, \quad s_i = 0.072 \text{ J kg}^{-1} \text{ K}^{-1}.$$

The specified heat-capacity law is

$$s(T) = s_i \left(\frac{T}{T_i} \right)^3.$$

The heat that must be removed from the sphere is

$$Q = m \int_{T_f}^{T_i} s(T) dT = \frac{ms_i}{T_i^3} \int_{T_f}^{T_i} T^3 dT.$$

Therefore,

$$Q = \frac{ms_i}{4T_i^3} (T_i^4 - T_f^4) = \frac{ms_i T_i}{4} \left[1 - \left(\frac{T_f}{T_i} \right)^4 \right].$$

Numerically,

$$Q = \frac{(1168)(0.072)(4.2)}{4} \left[1 - \left(\frac{0.100}{4.2} \right)^4 \right] \simeq 88.3 \text{ J}.$$

At average cooling power $P = 1.00 \times 10^{-3} \text{ W}$,

$$t = \frac{Q}{P} = \frac{88.3}{10^{-3}} \text{ s}.$$

Hence

$$t \simeq 8.83 \times 10^4 \text{ s} \simeq 24.5 \text{ h}.$$

B(a)

Let the emitted frequency be ν_R . Following the prescribed Newtonian model, assign the photon an energy-equivalent gravitational mass

$$m_\gamma = \frac{h\nu_R}{c^2}.$$

The work required to move this mass from the solar surface to infinity is

$$W = \int_R^\infty \frac{GMm_\gamma}{r^2} dr = \frac{GMm_\gamma}{R}.$$

Conservation of energy gives

$$h\nu_\infty = h\nu_R - \frac{GMm_\gamma}{R}.$$

Substituting $m_\gamma = h\nu_R/c^2$,

$$h\nu_\infty = h\nu_R \left(1 - \frac{GM}{Rc^2} \right),$$

and therefore

$$\nu_\infty \simeq \nu_R \left(1 - \frac{GM}{Rc^2} \right).$$

This is a first-order weak-field result. Using the emitted energy to calculate m_γ introduces only higher-order corrections. For the Sun,

$$\frac{GM}{Rc^2} = 2.12 \times 10^{-6} \ll 1,$$

so the approximation is well justified.

B(b)

Define

$$\epsilon = \frac{GM}{rc^2}, \quad q = 1 - \epsilon.$$

The frequency reduction by q corresponds to an increase in the clock period by $1/q$. According to the length-contraction rule supplied in the question, the associated length scale decreases by q .

Thus, in the problem's comparison with distant coordinates, the effective speed scales as

$$\frac{c'_r}{c} \simeq \frac{q}{1/q} = q^2.$$

It follows that

$$n_r = \frac{c}{c'_r} \simeq \frac{1}{(1 - \epsilon)^2}.$$

Expanding to first order,

$$(1 - \epsilon)^{-2} = 1 + 2\epsilon + O(\epsilon^2).$$

Therefore,

$$\boxed{n_r \simeq 1 + \frac{2GM}{rc^2}}, \quad \boxed{\alpha = 2}.$$

Here c'_r is the effective coordinate speed used in the question's refractive-index model. A local measurement using proper lengths and times still gives the vacuum speed c .

B(c)

Take the incoming light ray to travel approximately along the x -direction, with impact parameter b . Along its unperturbed path,

$$y = b, \quad r = \sqrt{x^2 + b^2}.$$

The refractive index is

$$n(x, y) = 1 + \frac{\alpha GM}{c^2 \sqrt{x^2 + y^2}}.$$

Snell's law for a continuously varying refractive index gives the ray equation

$$\frac{d}{ds}(n\hat{\mathbf{t}}) = \nabla n,$$

where $\hat{\mathbf{t}}$ is the ray's unit tangent. For a small deflection, $ds \simeq dx$ and $n \simeq 1$, so its transverse component becomes

$$\frac{d\theta}{dx} \simeq \frac{\partial n}{\partial y}.$$

Differentiation gives

$$\frac{\partial n}{\partial y} = -\frac{\alpha GM y}{c^2 (x^2 + y^2)^{3/2}}.$$

To first order, this may be evaluated along the straight path $y = b$:

$$\Delta\theta = -\frac{\alpha GM b}{c^2} \int_{-\infty}^{+\infty} \frac{dx}{(x^2 + b^2)^{3/2}}.$$

Using the supplied integral,

$$\Delta\theta = -\frac{\alpha GM b}{c^2} \frac{2}{b^2} = -\frac{2\alpha GM}{bc^2}.$$

The negative sign indicates bending towards the Sun. For a ray grazing its edge, $b \simeq R$, and $\alpha = 2$, giving the deflection magnitude

$$\theta = \frac{4GM}{Rc^2}.$$

Numerically,

$$\theta = \frac{4(6.67 \times 10^{-11})(1.99 \times 10^{30})}{(6.95 \times 10^8)(3.00 \times 10^8)^2}.$$

Thus

$$\theta = 8.49 \times 10^{-6} \text{ rad} \quad \text{towards the Sun}.$$

This is approximately 1.75 arcseconds.

The solution was checked against the official solution.