

IPhO 1999

Theory problems and solutions · Padua, Italy · 3 theory problems with complete solutions

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Problem 1: Absorption of radiation by a gas

A cylindrical vessel, with its axis vertical, contains a molecular gas at thermodynamic equilibrium. The upper base of the cylinder can be displaced freely and is made out of a glass plate; let's assume that there is no gas leakage and that the friction between glass plate and cylinder walls is just sufficient to damp oscillations but doesn't involve any significant loss of energy with respect to the other energies involved. Initially the gas temperature is equal to that of the surrounding environment. The gas can be considered as perfect within a good approximation. Let's assume that the cylinder walls (including the bases) have a very low thermal conductivity and capacity, and therefore the heat transfer between gas and environment is very slow, and can be neglected in the solution of this problem.

Through the glass plate we send into the cylinder the light emitted by a constant power laser; this radiation is easily transmitted by air and glass but is completely absorbed by the gas inside the vessel. By absorbing this radiation the molecules reach excited states, where they quickly emit infrared radiation returning in steps to the molecular ground state; this infrared radiation, however, is further absorbed by other molecules and is reflected by the vessel walls, including the glass plate. The energy absorbed from the laser is therefore transferred in a very short time into thermal movement (molecular chaos) and thereafter stays in the gas for a sufficiently long time.

We observe that the glass plate moves upwards; after a certain irradiation time we switch the laser off and we measure this displacement.

1. Using the data below and - if necessary - those on the sheet with physical constants, compute the temperature and the pressure of the gas after the irradiation. [2 points]
2. Compute the mechanical work carried out by the gas as a consequence of the radiation absorption. [1 point]
3. Compute the radiant energy absorbed during the irradiation. [2 points]
4. Compute the power emitted by the laser that is absorbed by the gas, and the corresponding number of photons (and thus of elementary absorption processes) per unit time. [1.5 points]
5. Compute the efficiency of the conversion process of optical energy into a change of mechanical potential energy of the glass plate. [1 point]

Thereafter the cylinder axis is slowly rotated by 90° , bringing it into a horizontal direction. The heat exchanges between gas and vessel can still be neglected.

6. State whether the pressure and/or the temperature of the gas change as a consequence of such a rotation, and - if that is the case - what is its/their new value. [2.5 points]

Data

Room pressure: $p_0 = 101.3 \text{ kPa}$

Room temperature: $T_0 = 20.0^\circ\text{C}$

Inner diameter of the cylinder: $2r = 100 \text{ mm}$

Mass of the glass plate: $m = 800 \text{ g}$

Quantity of gas within the vessel: $n = 0.100 \text{ mol}$

Molar specific heat at constant volume of the gas: $c_V = 20.8 \text{ J}/(\text{mol} \cdot \text{K})$

Emission wavelength of the laser: $\lambda = 514 \text{ nm}$

Irradiation time: $\Delta t = 10.0 \text{ s}$

Displacement of the movable plate after irradiation: $\Delta s = 30.0 \text{ mm}$

Physical constants and general data

In addition to the numerical data given within the text of the individual problems, the knowledge of some general data and physical constants may be useful, and you may find them among the following ones. These are nearly the most accurate data currently available, and they have thus a large number of digits; you are expected, however, to write your results with a number of digits that must be appropriate for each problem.

Speed of light in vacuum: $c = 299792458 \text{ m} \cdot \text{s}^{-1}$

Magnetic permeability of vacuum: $\mu_0 = 4\pi \cdot 10^{-7} \text{ H} \cdot \text{m}^{-1}$

Dielectric constant of vacuum: $\varepsilon_0 = 8.8541878 \text{ pF} \cdot \text{m}^{-1}$

Gravitational constant: $G = 6.67259 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$

Gas constant: $R = 8.314510 \text{ J}/(\text{mol} \cdot \text{K})$

Boltzmann's constant: $k = 1.380658 \cdot 10^{-23} \text{ J} \cdot \text{K}^{-1}$

Stefan's constant: $\sigma = 56.703 \text{ nW}/(\text{m}^2 \cdot \text{K}^4)$

Proton charge: $e = 1.60217733 \cdot 10^{-19} \text{ C}$

Electron mass: $m_e = 9.1093897 \cdot 10^{-31} \text{ kg}$

Planck's constant: $h = 6.6260755 \cdot 10^{-34} \text{ J} \cdot \text{s}$

Base of centigrade scale: $T_K = 273.15 \text{ K}$

Sun mass: $M_S = 1.991 \cdot 10^{30} \text{ kg}$

Earth mass: $M_E = 5.979 \cdot 10^{24} \text{ kg}$

Mean radius of Earth: $r_E = 6.373 \text{ Mm}$

Major semiaxis of Earth orbit: $R_E = 1.4957 \cdot 10^{11} \text{ m}$

Sidereal day: $d_S = 86.16406 \text{ ks}$

Year: $y = 31.558150 \text{ Ms}$

Standard value of the gravitational field at the Earth surface: $g = 9.80665 \text{ m} \cdot \text{s}^{-2}$

Standard value of the atmospheric pressure at sea level: $p_0 = 101325 \text{ Pa}$

Refractive index of air for visible light, at standard pressure and 15°C : $n_{\text{air}} = 1.000277$

Solar constant: $S = 1355 \text{ W} \cdot \text{m}^{-2}$

Jupiter mass: $M = 1.901 \cdot 10^{27} \text{ kg}$

Equatorial Jupiter radius: $R_B = 69.8 \text{ Mm}$

Average radius of Jupiter's orbit: $R = 7.783 \cdot 10^{11} \text{ m}$

Jovian day: $d_J = 35.6 \text{ ks}$

Jovian year: $y_J = 374.32 \text{ Ms}$

π : 3.14159265

Answers

1. After irradiation:

$$T_1 = 322.1 \text{ K} = 49.0^\circ\text{C}, \quad p_1 = 102.3 \text{ kPa.}$$

2. Mechanical work done by the gas:

$$W = 24.1 \text{ J.}$$

3. Radiant energy absorbed:

$$E_{\text{abs}} = 84.4 \text{ J.}$$

4. Absorbed laser power and photon absorption rate:

$$P_{\text{abs}} = 8.44 \text{ W}, \quad \dot{N} = 2.18 \times 10^{19} \text{ s}^{-1}.$$

5. Efficiency for increasing the plate's gravitational potential energy:

$$\eta = 2.79 \times 10^{-3} = 0.279\%.$$

6. Both pressure and temperature decrease. After rotation:

$$p_2 = 101.3 \text{ kPa}, \quad T_2 = 321.2 \text{ K} = 48.1^\circ\text{C.}$$

Solution

We use the ideal-gas approximation, the given constant molar heat capacity, and negligible heat leakage and frictional energy loss. Initial and final states have no macroscopic kinetic energy. The room pressure used below is the problem-specific value $p_0 = 101.3 \text{ kPa}$.

The cylinder cross-sectional area and the measured volume increase are

$$A = \pi r^2 = \pi(0.0500)^2 = 7.85398 \times 10^{-3} \text{ m}^2,$$

$$\Delta V = A\Delta s = (7.85398 \times 10^{-3})(0.0300) = 2.35619 \times 10^{-4} \text{ m}^3.$$

1. Temperature and pressure after irradiation

In vertical mechanical equilibrium, the upward force of the gas balances atmospheric pressure and the plate's weight:

$$p_1 A = p_0 A + mg.$$

Thus

$$p_1 = p_0 + \frac{mg}{A} = 101300 + \frac{(0.800)(9.80665)}{7.85398 \times 10^{-3}} = 102298.9 \text{ Pa.}$$

The same balance applies before irradiation, so the initial and final gas pressures are equal:

$$p_1 = 102.3 \text{ kPa.}$$

Let V_0 and V_1 be the gas volumes before and after irradiation. Applying the ideal-gas law to both states,

$$p_1 V_0 = nRT_0, \quad p_1 V_1 = nRT_1.$$

Subtracting gives

$$p_1 \Delta V = nR(T_1 - T_0).$$

With $T_0 = 293.15$ K,

$$T_1 - T_0 = \frac{(102298.9)(2.35619 \times 10^{-4})}{(0.100)(8.314510)} = 28.9898 \text{ K}.$$

Therefore

$$\boxed{T_1 = 322.1 \text{ K} = 49.0^\circ\text{C}.}$$

2. Mechanical work done by the gas

The opposing force $p_0 A + mg$ is constant. With negligible frictional loss and zero initial and final plate kinetic energy, the gas does work

$$W = (p_0 A + mg)\Delta s = p_0 \Delta V + mg\Delta s = p_1 \Delta V.$$

Numerically,

$$W = (102298.9)(2.35619 \times 10^{-4}) = 24.1036 \text{ J},$$

so

$$\boxed{W = 24.1 \text{ J}.}$$

This consists of 23.8683 J of work against atmospheric pressure and 0.235360 J of work lifting the plate.

3. Radiant energy absorbed

The absorbed radiation rapidly becomes thermal energy. Energy conservation, with work positive when done by the gas, gives

$$E_{\text{abs}} = \Delta U + W.$$

For the ideal gas,

$$\Delta U = nc_V(T_1 - T_0) = (0.100)(20.8)(28.9898) = 60.2988 \text{ J}.$$

Consequently,

$$E_{\text{abs}} = 60.2988 + 24.1036 = 84.4024 \text{ J},$$

or

$$\boxed{E_{\text{abs}} = 84.4 \text{ J}.}$$

Equivalently, because $W = nR(T_1 - T_0)$,

$$E_{\text{abs}} = n(c_V + R)(T_1 - T_0) = nc_P(T_1 - T_0),$$

where

$$c_P = c_V + R = 29.114510 \text{ J mol}^{-1}\text{K}^{-1}.$$

4. Absorbed power and photon absorption rate

The laser power is constant, hence

$$P_{\text{abs}} = \frac{E_{\text{abs}}}{\Delta t} = \frac{84.4024}{10.0} = \boxed{8.44 \text{ W}}.$$

Each incident laser photon has energy

$$\varepsilon_{\gamma} = \frac{hc}{\lambda} = \frac{(6.6260755 \times 10^{-34})(299792458)}{514 \times 10^{-9}} = 3.86468 \times 10^{-19} \text{ J}.$$

All incident laser photons entering the gas are absorbed, so their absorption rate is

$$\dot{N} = \frac{P_{\text{abs}}}{\varepsilon_{\gamma}} = \frac{8.44024}{3.86468 \times 10^{-19}} = \boxed{2.18 \times 10^{19} \text{ s}^{-1}}.$$

This is also the rate of elementary absorption events for the laser photons.

5. Efficiency of conversion into the plate's potential energy

The plate's gravitational potential energy increases by

$$\Delta E_{\text{pot}} = mg\Delta s = (0.800)(9.80665)(0.0300) = 0.235360 \text{ J}.$$

Therefore the requested efficiency is

$$\eta = \frac{\Delta E_{\text{pot}}}{E_{\text{abs}}} = \frac{0.235360}{84.4024} = 2.78854 \times 10^{-3}.$$

Thus

$$\boxed{\eta = 2.79 \times 10^{-3} = 0.279\%}.$$

Its small value reflects the energy used to heat the gas and to displace the surrounding atmosphere.

6. Pressure and temperature after rotation

Let θ be the angle between the cylinder axis and the upward vertical. During the slow rotation, axial mechanical equilibrium gives

$$p(\theta)A = p_0A + mg \cos \theta,$$

so

$$p(\theta) = p_0 + \frac{mg}{A} \cos \theta.$$

At $\theta = 90^\circ$, the plate's weight has no axial component. Hence

$$\boxed{p_2 = p_0 = 101.3 \text{ kPa.}}$$

The rotation is slow, and heat exchange and frictional dissipation are negligible. The gas therefore undergoes a quasistatic adiabatic expansion. The first law gives

$$nc_V dT = -p dV.$$

Using $p = nRT/V$,

$$c_V \frac{dT}{T} = -R \frac{dV}{V}.$$

Integration yields

$$TV^{R/c_V} = \text{constant},$$

or, defining

$$\gamma = \frac{c_P}{c_V} = \frac{c_V + R}{c_V} = 1.399736,$$

$$TV^{\gamma-1} = \text{constant}.$$

Eliminating volume with the ideal-gas law gives

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} = \left(\frac{p_0}{p_1} \right)^{R/(c_V+R)}.$$

Therefore

$$T_2 = 322.1398 \left(\frac{101300}{102298.9} \right)^{8.314510/29.114510} = 321.2384 \text{ K}.$$

Thus

$$T_2 = 321.2 \text{ K} = 48.1^\circ\text{C}.$$

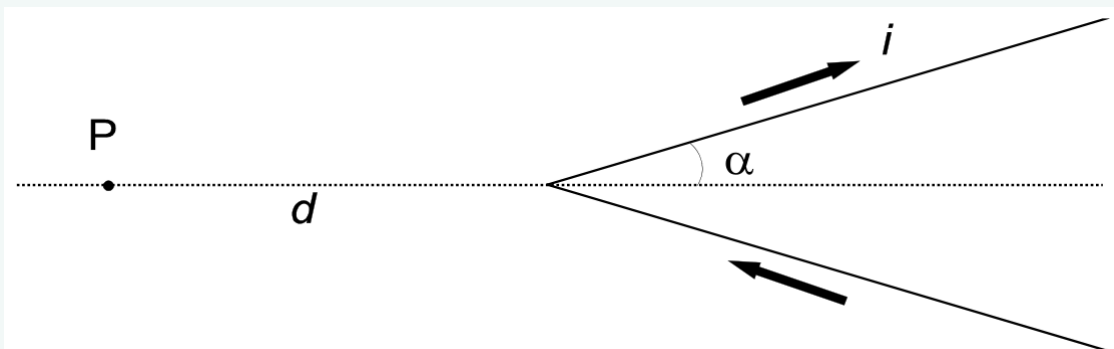
Both pressure and temperature decrease; the temperature falls by approximately 0.90 K because the expanding gas supplies work from its internal energy.

The solution was checked against the official solution.

Problem 2: Magnetic field with a V-shaped wire

Among the first successes of the interpretation by Ampère of magnetic phenomena, we have the computation of the magnetic field $\mathbf{B}$ generated by wires carrying an electric current, as compared to early assumptions originally made by Biot and Savart.

A particularly interesting case is that of a very long thin wire, carrying a constant current i , made out of two rectilinear sections and bent in the form of a "V", with angular half-span¹ α (see figure). According to Ampère's computations, the magnitude B of the magnetic field in a given point P lying on the axis of the "V", outside of it and at a distance d from its vertex, is proportional to $\tan\left(\frac{\alpha}{2}\right)$. Ampère's work was later embodied in Maxwell's electromagnetic theory, and is universally accepted.



Using our contemporary knowledge of electromagnetism,

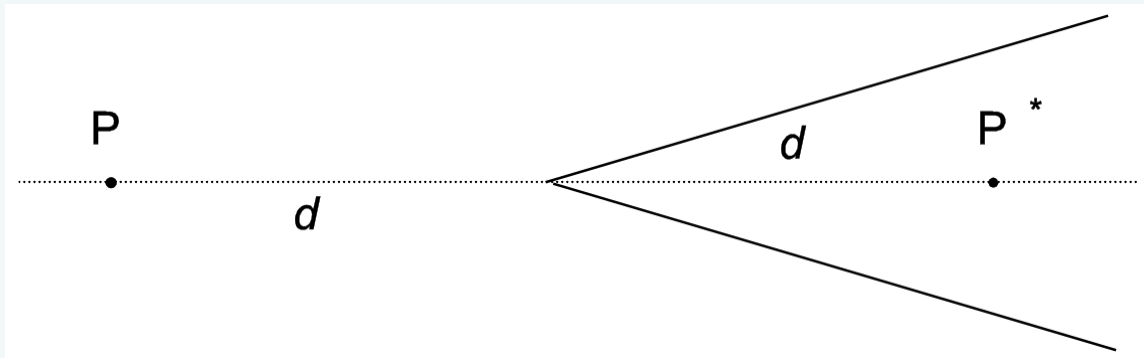
1. Find the direction of the field $\mathbf{B}$ in P. [1 point]
2. Knowing that the field is proportional to $\tan\left(\frac{\alpha}{2}\right)$, find the proportionality factor k in

$$|\mathbf{B}(\mathbf{P})| = k \tan\left(\frac{\alpha}{2}\right).$$

[1.5 points]

3. Compute the field $\mathbf{B}$ in a point $\mathbf{P}^*$ symmetric to $\mathbf{P}$ with respect to the vertex, i.e. along the axis and at the same distance d , but inside the "V" (see figure). [2 points]

¹ Throughout this problem α is expressed in radians



4. In order to measure the magnetic field, we place in $\mathbf{P}$ a small magnetic needle with moment of inertia I and magnetic dipole moment μ ; it oscillates around a fixed point in a plane containing the direction of $\mathbf{B}$. Compute the period of small oscillations of this needle as a function of B . [2.5 points]

In the same conditions Biot and Savart had instead assumed that the magnetic field in $\mathbf{P}$ might have been (we use here the modern notation) $B(\mathbf{P}) = \frac{i\mu_0\alpha}{\pi^2 d}$, where μ_0 is the magnetic permeability of vacuum. In fact they attempted to decide with an experiment between the two interpretations (Ampère's and Biot and Savart's) by measuring the oscillation period of the magnetic needle as a function of the "V" span. For some α values, however, the differences are too small to be easily measurable.

5. If, in order to distinguish experimentally between the two predictions for the magnetic needle oscillation period T in $\mathbf{P}$, we need a difference by at least 10%, namely $T_1 > 1.10 T_2$ (T_1 being the Ampère prediction and T_2 the Biot-Savart prediction) state in which range, approximately, we must choose the "V" half-span α for being able to decide between the two interpretations. [3 points]

Hint

Depending on which path you follow in your solution, the following trigonometric equation might be useful:

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin \alpha}{1 + \cos \alpha}$$

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 π : 3.14159265

Official English text of the problem.

Answers

Let $\hat{\mathbf{z}}$ point out of the page, towards the reader, and let $\mu = |\boldsymbol{\mu}|$.

1. The field at P is **perpendicular to the wire's plane, out of the page**.

2.
$$k = \frac{\mu_0 i}{2\pi d}, \quad \mathbf{B}(\text{P}) = \frac{\mu_0 i}{2\pi d} \tan\left(\frac{\alpha}{2}\right) \hat{\mathbf{z}}.$$

3.
$$\mathbf{B}(\text{P}^*) = -\frac{\mu_0 i}{2\pi d} \cot\left(\frac{\alpha}{2}\right) \hat{\mathbf{z}},$$

directed **into the page**.

4.
$$T = 2\pi \sqrt{\frac{I}{\mu B}}.$$

5. For the stated strict inequality $T_1 > 1.10T_2$,

$$0 < \alpha < \alpha_c, \quad \alpha_c \simeq 0.768 \text{ rad} \simeq 44.0^\circ.$$

At $\alpha = \alpha_c$, the difference is exactly 10%.

Solution

We treat the two long, thin arms as semi-infinite straight wires in vacuum. The current enters the vertex along the lower arm and leaves along the upper arm, as shown. Take the vertex as the origin, with the positive x -axis pointing into the V and $\hat{\mathbf{z}}$ pointing out of the page.

1. Direction of the field at P

The modern Biot–Savart law is

$$d\mathbf{B} = \frac{\mu_0 i}{4\pi} \frac{d\boldsymbol{\ell} \times \mathbf{R}}{R^3},$$

where $d\boldsymbol{\ell}$ follows the current and $\mathbf{R}$ points from the current element to the observation point.

Both vectors lie in the plane of the figure, so every contribution is perpendicular to that plane. At P, the right-hand rule gives an outward contribution from each arm: the lower arm carries current towards the vertex, while the upper arm carries current away from it. Thus their contributions add:

$$\mathbf{B}(\text{P}) \text{ points out of the page.}$$

2. Determination of k

The angular dependence is given:

$$B(\text{P}) = k \tan\left(\frac{\alpha}{2}\right).$$

Consider the limiting configuration $\alpha = \pi/2$. The two arms then form one infinite straight wire, and P lies at perpendicular distance d from it.

For this straight wire, cylindrical symmetry and Ampère's circuital law give

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = B(2\pi d) = \mu_0 i.$$

Consequently,

$$B = \frac{\mu_0 i}{2\pi d}.$$

Since $\tan(\pi/4) = 1$, this determines

$$k = \frac{\mu_0 i}{2\pi d}.$$

Using the supplied permeability,

$$k = (2 \times 10^{-7} \text{ T m A}^{-1}) \frac{i}{d}.$$

Therefore,

$$\mathbf{B}(\text{P}) = \frac{\mu_0 i}{2\pi d} \tan\left(\frac{\alpha}{2}\right) \hat{\mathbf{z}}.$$

3. Field at P^*

At P^* , the right-hand rule gives a field into the page from each original arm.

To find its magnitude, continue both arms through the vertex, preserving their current directions, so that each becomes an infinite straight wire. The perpendicular distance from P^* to either completed wire is

$$\rho = d \sin \alpha.$$

Each completed wire produces a field into the page of magnitude

$$\frac{\mu_0 i}{2\pi\rho}.$$

Their total field therefore has magnitude

$$B_{\text{complete}} = \frac{\mu_0 i}{\pi d \sin \alpha}.$$

The two added half-lines form a left-opening V, which is the mirror image of the original wire. Relative to this added V, P^* is an exterior axial point at distance d . Its field consequently has magnitude $B(P)$, and the right-hand rule shows that it also points into the page.

Superposition thus gives

$$B(P^*) + B(P) = \frac{\mu_0 i}{\pi d \sin \alpha}.$$

Substituting the result of part 2,

$$B(P^*) = \frac{\mu_0 i}{2\pi d} \left[\frac{2}{\sin \alpha} - \tan \left(\frac{\alpha}{2} \right) \right].$$

Using

$$\tan \left(\frac{\alpha}{2} \right) = \frac{1 - \cos \alpha}{\sin \alpha},$$

the bracket becomes

$$\frac{2 - (1 - \cos \alpha)}{\sin \alpha} = \frac{1 + \cos \alpha}{\sin \alpha} = \cot \left(\frac{\alpha}{2} \right).$$

Hence

$$\boxed{\mathbf{B}(P^*) = -\frac{\mu_0 i}{2\pi d} \cot \left(\frac{\alpha}{2} \right) \hat{\mathbf{z}}}.$$

4. Period of small oscillations

Treat the small needle as a magnetic dipole in a field uniform over its length, neglecting damping and other torques. Let θ be the angle between its magnetic moment and the field.

The magnetic torque is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}.$$

The equilibrium with $\boldsymbol{\mu}$ parallel to $\mathbf{B}$ is stable. Taking θ as displacement from this equilibrium, the rotational equation of motion is

$$I\ddot{\theta} = -\mu B \sin \theta.$$

For small oscillations, $|\theta| \ll 1$ rad and $\sin \theta \simeq \theta$, so

$$\ddot{\theta} + \frac{\mu B}{I} \theta = 0.$$

This is simple harmonic motion with angular frequency

$$\omega_0 = \sqrt{\frac{\mu B}{I}},$$

and therefore

$$T = 2\pi \sqrt{\frac{I}{\mu B}}.$$

With SI units for I , μ , and B , the period is in seconds.

5. Angular range giving a period difference greater than 10%

The two field predictions at P are

$$B_1 = \frac{\mu_0 i}{2\pi d} \tan\left(\frac{\alpha}{2}\right), \quad B_2 = \frac{\mu_0 i \alpha}{\pi^2 d},$$

where the second expression is the historical assumption quoted in the problem.

Since $T \propto B^{-1/2}$,

$$\frac{T_1}{T_2} = \sqrt{\frac{B_2}{B_1}} = \sqrt{\frac{2\alpha}{\pi \tan(\alpha/2)}}.$$

The experimental requirement becomes

$$\frac{2\alpha}{\pi \tan(\alpha/2)} > 1.10^2 = 1.21.$$

The period ratio decreases as α increases. To verify this, put $x = \alpha/2$; its square is proportional to $x \cot x$, and

$$\frac{d}{dx}(x \cot x) = \frac{\sin x \cos x - x}{\sin^2 x} < 0,$$

because $\sin(2x) < 2x$ for the positive angles considered.

Its limiting values are

$$\lim_{\alpha \rightarrow 0^+} \frac{T_1}{T_2} = \frac{2}{\sqrt{\pi}} \simeq 1.1284, \quad \left. \frac{T_1}{T_2} \right|_{\alpha=\pi/2} = 1.$$

Thus a single threshold separates the distinguishable and indistinguishable ranges.

Solving

$$\frac{2\alpha_c}{\pi \tan(\alpha_c/2)} = 1.21$$

numerically gives

$$\alpha_c \simeq 0.7682 \text{ rad} \simeq 44.01^\circ.$$

Therefore the required half-span is approximately

$$0 < \alpha < 0.77 \text{ rad} \quad (\text{approximately below } 44^\circ).$$

The exact threshold is included if "at least 10%" is interpreted inclusively; it is excluded by the explicitly printed strict inequality. The degenerate value $\alpha = 0$ is excluded because both fields vanish and there is no finite magnetic oscillation period.

The solution was checked against the official solution.

Problem 3: A space probe to Jupiter

We consider in this problem a method frequently used to accelerate space probes in the desired direction. The space probe flies by a planet, and can significantly increase its speed and modify considerably its flight direction, by taking away a very small amount of energy from the planet's orbital motion. We analyze here this effect for a space probe passing near Jupiter.

The planet Jupiter orbits around the Sun along an elliptical trajectory, that can be approximated by a circumference of average radius R ; in order to proceed with the analysis of the physical situation we must first:

1. Find the speed V of the planet along its orbit around the Sun. [1.5 points]
2. When the probe is between the Sun and Jupiter (on the segment Sun-Jupiter), find the distance from Jupiter where the Sun's gravitational attraction balances that by Jupiter. [1 point]

A space probe of mass $m = 825 \text{ kg}$ flies by Jupiter. For simplicity assume that the trajectory of the space probe is entirely in the plane of Jupiter's orbit; in this way we neglect the important case in which the space probe is expelled from Jupiter's orbital plane.

We only consider what happens in the region where Jupiter's attraction overwhelms all other gravitational forces.

In the reference frame of the Sun's center of mass the initial speed of the space probe is $v_0 = 1.00 \cdot 10^4 \text{ m/s}$ (along the positive y direction) while Jupiter's speed is along the negative x direction (see figure 1); by "initial speed" we mean the space probe speed when it's in the interplanetary space, still far from Jupiter but already in the region where the Sun's attraction is negligible with respect to Jupiter's. We assume that the encounter occurs in a sufficiently short time to allow neglecting the change of direction of Jupiter along its orbit around the Sun. We also assume that the probe passes behind Jupiter, i.e. the x coordinate is greater for the probe than for Jupiter when the y coordinate is the same.

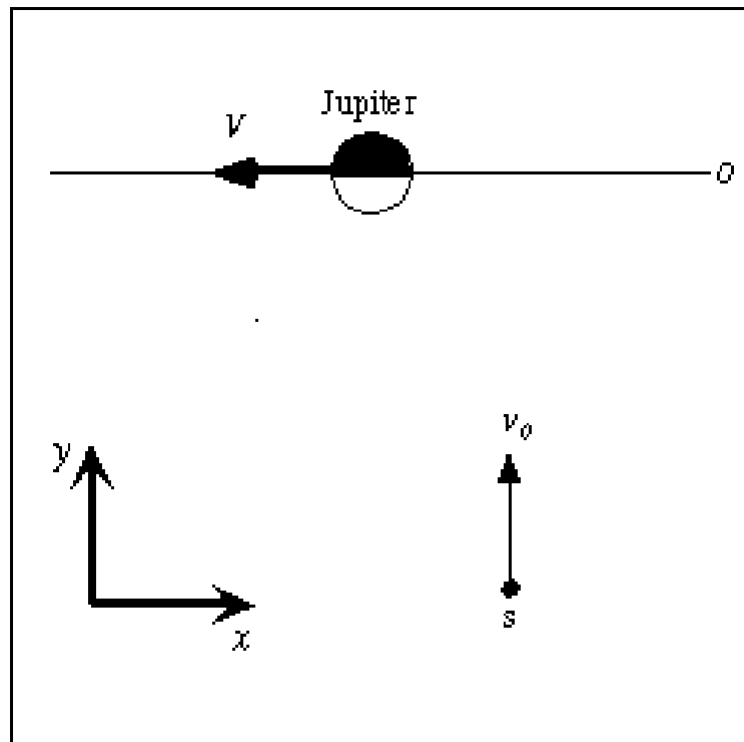


Figure 1: View in the Sun center of mass system. O denotes Jupiter's orbit, s is the space probe.

3. Find the space probe's direction of motion (as the angle φ between its direction and the x axis) and its speed v' in Jupiter's reference frame, when it's still far away from Jupiter. [2 points]
4. Find the value of the space probe's total mechanical energy E in Jupiter's reference frame, putting – as usual – equal to zero the value of its potential energy at a very large distance, in this case when it is far enough to move with almost constant velocity owing to the smallness of all gravitational interactions. [1 point]

The space probe's trajectory in the reference frame of Jupiter is a hyperbola and its equation in polar coordinates in this reference frame is

$$\frac{1}{r} = \frac{GM}{v'^2 b^2} \left(1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos \theta \right). \quad (1)$$

where b is the distance between one of the asymptotes and Jupiter (the so called *impact parameter*), E is the probe's total mechanical energy in Jupiter's reference frame, G is the gravitational constant, M is the mass of Jupiter, r and θ are the polar coordinates (the radial distance and the polar angle).

Figure 2 shows the two branches of a hyperbola as described by equation (1); the asymptotes and the polar co-ordinates are also shown. Note that equation (1) has its origin in the "attractive focus" of the hyperbola. The space probe's trajectory is the attractive trajectory (the emphasized branch).

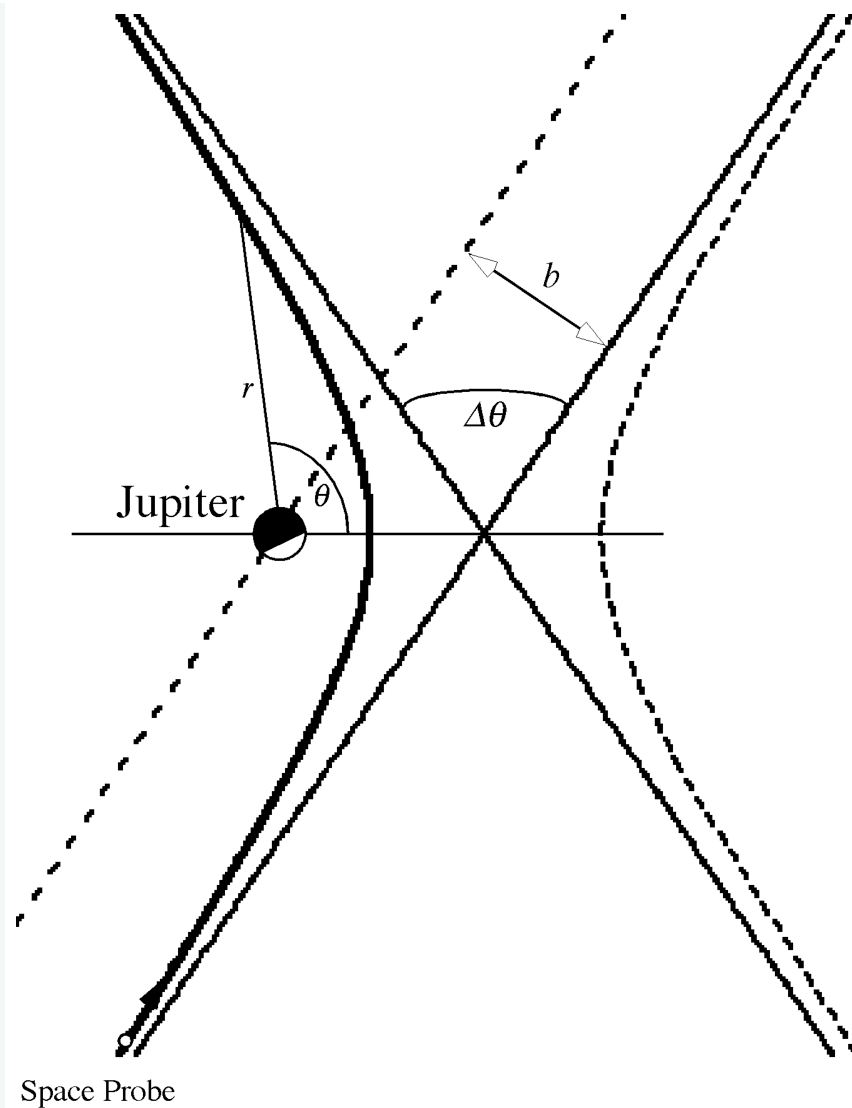


Figure 2

5. Using equation (1) describing the space probe's trajectory, find the total angular deviation $\Delta\theta$ in Jupiter's reference frame (as shown in figure 2) and express it as a function of initial speed v' and impact parameter b . [2 points]
6. Assume that the probe cannot pass Jupiter at a distance less than three Jupiter radii from the center of the planet; find the minimum possible impact parameter and the maximum possible angular deviation. [1 point]
7. Find an equation for the final speed v'' of the probe in the Sun's reference frame as a function only of Jupiter's speed V , the probe's initial speed v_0 and the deviation angle $\Delta\theta$. [1 point]
8. Use the previous result to find the numerical value of the final speed v'' in the Sun's reference frame when the angular deviation has its maximum possible value. [0.5 points]

Hint

Depending on which path you follow in your solution, the following trigonometric formulas might be useful:

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$

Physical constants and general data

In addition to the numerical data given within the text of the individual problems, the knowledge of some general data and physical constants may be useful, and you may find them among the following ones. These are nearly the most accurate data currently available, and they have thus a large number of digits; you are expected, however, to write your results with a number of digits that must be appropriate for each problem.

Speed of light in vacuum:	$c = 299792458 \text{ m} \cdot \text{s}^{-1}$
Magnetic permeability of vacuum:	$\mu_0 = 4\pi \cdot 10^{-7} \text{ H} \cdot \text{m}^{-1}$
Dielectric constant of vacuum:	$\varepsilon_0 = 8.8541878 \text{ pF} \cdot \text{m}^{-1}$
Gravitational constant:	$G = 6.67259 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$
Gas constant:	$R = 8.314510 \text{ J}/(\text{mol} \cdot \text{K})$
Boltzmann's constant:	$k = 1.380658 \cdot 10^{-23} \text{ J} \cdot \text{K}^{-1}$
Stefan's constant:	$\sigma = 56.703 \text{ nW}/(\text{m}^2 \cdot \text{K}^4)$
Proton charge:	$e = 1.60217733 \cdot 10^{-19} \text{ C}$
Electron mass:	$m_e = 9.1093897 \cdot 10^{-31} \text{ kg}$
Planck's constant:	$h = 6.6260755 \cdot 10^{-34} \text{ J} \cdot \text{s}$
Base of centigrade scale:	$T_K = 273.15 \text{ K}$
Sun mass:	$M_S = 1.991 \cdot 10^{30} \text{ kg}$
Earth mass:	$M_E = 5.979 \cdot 10^{24} \text{ kg}$
Mean radius of Earth:	$r_E = 6.373 \text{ Mm}$
Major semiaxis of Earth orbit:	$R_E = 1.4957 \cdot 10^{11} \text{ m}$
Sidereal day:	$d_S = 86.16406 \text{ ks}$
Year:	$y = 31.558150 \text{ Ms}$
Standard value of the gravitational field at the Earth surface:	$g = 9.80665 \text{ m} \cdot \text{s}^{-2}$
Standard value of the atmospheric pressure at sea level:	$p_0 = 101325 \text{ Pa}$
Refractive index of air for visible light, at standard pressure and 15 °C:	$n_{\text{air}} = 1.000277$
Solar constant:	$S = 1355 \text{ W} \cdot \text{m}^{-2}$
Jupiter mass:	$M = 1.901 \cdot 10^{27} \text{ kg}$
Equatorial Jupiter radius:	$R_B = 69.8 \text{ Mm}$
Average radius of Jupiter's orbit:	$R = 7.783 \cdot 10^{11} \text{ m}$
Jovian day:	$d_J = 35.6 \text{ ks}$
Jovian year:	$y_J = 374.32 \text{ Ms}$
π :	3.14159265

Official English text of the problem.

Answers

1. Jupiter's orbital speed is

$$V = \sqrt{\frac{GM_S}{R}} \simeq 1.31 \times 10^4 \text{ m s}^{-1}.$$

2. The distance from Jupiter toward the Sun is

$$d = \frac{R}{1 + \sqrt{M_S/M}} \simeq 2.33 \times 10^{10} \text{ m}.$$

3. The initial velocity in Jupiter's frame is $\mathbf{v}'_i = (V, v_0)$. Thus

$$\varphi = \tan^{-1}\left(\frac{v_0}{V}\right) = 37.4^\circ, \quad v' = \sqrt{V^2 + v_0^2} = 1.65 \times 10^4 \text{ m s}^{-1}.$$

The angle is measured counterclockwise from the positive x axis.

4. The total mechanical energy in Jupiter's frame is

$$E = \frac{1}{2}m(V^2 + v_0^2) = 1.12 \times 10^{11} \text{ J}.$$

5. The angular deviation is

$$\Delta\theta = 2 \tan^{-1}\left(\frac{GM}{bv'^2}\right).$$

6. With $r_0 = 3R_B$,

$$b_{\min} = \sqrt{r_0^2 + \frac{2GM r_0}{v'^2}} = 4.90 \times 10^8 \text{ m},$$

$$\Delta\theta_{\max} = 2 \sin^{-1}\left(\frac{1}{1 + r_0 v'^2/(GM)}\right) = 87.4^\circ \simeq 1.53 \text{ rad}.$$

7. The final speed in the Sun's frame is

$$v'' = \sqrt{v_0^2 + 2V^2(1 - \cos \Delta\theta) + 2Vv_0 \sin \Delta\theta}.$$

8. At the maximum allowed deviation,

$$v'' = 2.62 \times 10^4 \text{ m s}^{-1} = 26.2 \text{ km s}^{-1}.$$

Solution

We use Newtonian gravitation and the approximations specified in the problem. Jupiter's orbit is circular, and its velocity is effectively constant during the encounter. The probe's mass is negligible compared with Jupiter's mass. During the flyby we neglect the Sun's force, so Jupiter's frame is treated as inertial and the probe moves in the potential

$$U(r) = -\frac{GMm}{r}.$$

The initial and final velocities refer to the asymptotic velocities of this two-body approximation. Numerical calculations retain unrounded intermediate values.

1. Jupiter's orbital speed

The Sun's gravitational attraction supplies Jupiter's centripetal force:

$$\frac{GM_S M}{R^2} = \frac{MV^2}{R}.$$

Therefore

$$V = \sqrt{\frac{GM_S}{R}}.$$

Using the supplied constants,

$$V = \sqrt{\frac{(6.67259 \times 10^{-11})(1.991 \times 10^{30})}{7.783 \times 10^{11}}} = 1.30650 \times 10^4 \text{ m s}^{-1}.$$

Thus

$$\boxed{V = 1.31 \times 10^4 \text{ m s}^{-1}.}$$

The alternative circular-orbit calculation $V = 2\pi R/y_J$ gives $1.30642 \times 10^4 \text{ m s}^{-1}$, agreeing at the reported precision.

2. Point where the gravitational attractions balance

Let d be the distance from Jupiter toward the Sun. The distance from the Sun is then $R - d$. The two gravitational forces point in opposite directions, so their balance requires

$$\frac{GMm}{d^2} = \frac{GM_S m}{(R - d)^2}.$$

Taking positive square roots,

$$\frac{R - d}{d} = \sqrt{\frac{M_S}{M}}.$$

Solving,

$$d = \frac{R}{1 + \sqrt{M_S/M}}.$$

Numerically,

$$d = \frac{7.783 \times 10^{11}}{1 + \sqrt{(1.991 \times 10^{30})/(1.901 \times 10^{27})}} = 2.33285 \times 10^{10} \text{ m}.$$

Hence

$$\boxed{d = 2.33 \times 10^{10} \text{ m}.}$$

3. Initial motion in Jupiter's frame

In the Sun's frame, the initial velocities are

$$\mathbf{v}_i = v_0 \hat{\mathbf{y}}, \quad \mathbf{V}_J = -V \hat{\mathbf{x}}.$$

The Galilean velocity transformation gives

$$\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}_J = V\hat{\mathbf{x}} + v_0\hat{\mathbf{y}}.$$

Both components are positive, so the velocity lies in the first quadrant:

$$\tan \varphi = \frac{v_0}{V}, \quad v' = \sqrt{V^2 + v_0^2}.$$

Numerically,

$$\varphi = \tan^{-1}\left(\frac{1.00 \times 10^4}{1.30650 \times 10^4}\right) = 37.4306^\circ,$$

$$v' = \sqrt{(1.30650 \times 10^4)^2 + (1.00 \times 10^4)^2} = 1.64528 \times 10^4 \text{ m s}^{-1}.$$

Thus

$$\boxed{\varphi = 37.4^\circ, \quad v' = 1.65 \times 10^4 \text{ m s}^{-1}.}$$

4. Total mechanical energy in Jupiter's frame

The potential energy vanishes at infinite separation. Consequently, the conserved total energy equals the initial asymptotic kinetic energy:

$$E = \frac{1}{2}mv'^2 = \frac{1}{2}m(V^2 + v_0^2).$$

Substitution gives

$$E = \frac{1}{2}(825)(1.64528 \times 10^4)^2 = 1.11661 \times 10^{11} \text{ J}.$$

Therefore

$$\boxed{E = 1.12 \times 10^{11} \text{ J}.}$$

Its positive value is consistent with an unbound hyperbolic orbit.

5. Angular deviation

Using $E = \frac{1}{2}mv'^2$, write equation (1) as

$$r = \frac{p}{1 + e \cos \theta},$$

where

$$p = \frac{v'^2 b^2}{GM}, \quad e = \sqrt{1 + \frac{b^2 v'^4}{G^2 M^2}} > 1.$$

At either asymptote, $r \rightarrow \infty$, so

$$1 + e \cos \theta = 0.$$

Let the asymptotic polar angles be $\pm\theta_\infty$, where

$$\cos \theta_\infty = -\frac{1}{e}, \quad \frac{\pi}{2} < \theta_\infty < \pi.$$

On the incoming asymptote, the velocity points opposite to the radial direction. Its angle relative to the periapsis axis is therefore $\pi - \theta_\infty$. On the outgoing asymptote, the velocity points along the radial direction and has angle θ_∞ . Their difference is

$$\Delta\theta = \theta_\infty - (\pi - \theta_\infty) = 2\theta_\infty - \pi.$$

It follows that

$$\sin \frac{\Delta\theta}{2} = \sin \left(\theta_\infty - \frac{\pi}{2} \right) = -\cos \theta_\infty = \frac{1}{e}.$$

Since $0 < \Delta\theta/2 < \pi/2$,

$$\tan \frac{\Delta\theta}{2} = \frac{1}{\sqrt{e^2 - 1}} = \frac{GM}{bv'^2}.$$

Thus

$$\boxed{\Delta\theta = 2 \tan^{-1} \left(\frac{GM}{bv'^2} \right)}.$$

Equivalently,

$$\Delta\theta = 2 \sin^{-1} \left[\frac{1}{\sqrt{1 + b^2 v'^4 / (G^2 M^2)}} \right].$$

6. Minimum impact parameter and maximum deviation

Let r_p be the closest distance to Jupiter's center, and v_p the speed there. Conservation of energy gives

$$\frac{1}{2}mv_p^2 - \frac{GMm}{r_p} = \frac{1}{2}mv'^2,$$

so

$$v_p^2 = v'^2 + \frac{2GM}{r_p}.$$

The force is central, so angular momentum about Jupiter is conserved. At closest approach the velocity is perpendicular to the radius:

$$mbv' = mr_p v_p.$$

Squaring and substituting the energy result,

$$b^2 v'^2 = r_p^2 \left(v'^2 + \frac{2GM}{r_p} \right),$$

which yields

$$b^2 = r_p^2 + \frac{2GM r_p}{v'^2}.$$

This expression increases with r_p . Hence the smallest permitted impact parameter occurs at

$$r_p = r_0 = 3R_B = 3(69.8 \times 10^6) \text{ m}.$$

Therefore

$$b_{\min} = \sqrt{r_0^2 + \frac{2GM r_0}{v'^2}}.$$

Using

$$GM = 1.268459359 \times 10^{17} \text{ m}^3 \text{ s}^{-2},$$

we obtain

$$b_{\min} = 4.89996 \times 10^8 \text{ m},$$

or

$$\boxed{b_{\min} = 4.90 \times 10^8 \text{ m.}}$$

The deflection decreases as b increases, so its maximum occurs at $b_{\min}$. Thus

$$\Delta\theta_{\max} = 2 \tan^{-1} \left(\frac{GM}{b_{\min} v'^2} \right).$$

To express this directly through the closest distance, substitute the relation for $b_{\min}^2$ into the eccentricity:

$$e_{\min}^2 = 1 + \frac{b_{\min}^2 v'^4}{G^2 M^2} = \left(1 + \frac{r_0 v'^2}{GM} \right)^2.$$

Hence

$$e_{\min} = 1 + \frac{r_0 v'^2}{GM} = 1.44687$$

and

$$\Delta\theta_{\max} = 2 \sin^{-1} \left(\frac{1}{e_{\min}} \right) = 1.52615 \text{ rad} = 87.4421^\circ.$$

Therefore

$$\boxed{\Delta\theta_{\max} = 87.4^\circ.}$$

7. Final speed in the Sun's frame

Energy conservation in Jupiter's frame ensures that the outgoing asymptotic speed is still v' .

Because the probe passes behind Jupiter, the gravitational deflection rotates its velocity counterclockwise in the given x, y coordinates. The outgoing relative velocity consequently makes angle $\varphi + \Delta\theta$ with the positive x axis:

$$\mathbf{v}'_f = v' \cos(\varphi + \Delta\theta) \hat{\mathbf{x}} + v' \sin(\varphi + \Delta\theta) \hat{\mathbf{y}}.$$

Using

$$v' \cos \varphi = V, \quad v' \sin \varphi = v_0,$$

the addition formulas give

$$\begin{aligned} v'_{f,x} &= V \cos \Delta\theta - v_0 \sin \Delta\theta, \\ v'_{f,y} &= V \sin \Delta\theta + v_0 \cos \Delta\theta. \end{aligned}$$

Transforming back to the Sun's frame requires adding Jupiter's velocity $-V\hat{\mathbf{x}}$:

$$\begin{aligned} v''_x &= V(\cos \Delta\theta - 1) - v_0 \sin \Delta\theta, \\ v''_y &= V \sin \Delta\theta + v_0 \cos \Delta\theta. \end{aligned}$$

Therefore

$$\begin{aligned}
v''^2 &= (v_x'')^2 + (v_y'')^2 \\
&= V^2 [(\cos \Delta\theta - 1)^2 + \sin^2 \Delta\theta] \\
&\quad + v_0^2 [\sin^2 \Delta\theta + \cos^2 \Delta\theta] + 2Vv_0 \sin \Delta\theta \\
&= 2V^2(1 - \cos \Delta\theta) + v_0^2 + 2Vv_0 \sin \Delta\theta.
\end{aligned}$$

Thus the requested expression is

$$v'' = \sqrt{v_0^2 + 2V^2(1 - \cos \Delta\theta) + 2Vv_0 \sin \Delta\theta}.$$

As a consistency check, setting $\Delta\theta = 0$ gives $v'' = v_0$.

8. Final speed at maximum angular deviation

Substituting the unrounded values

$$V = 1.30650 \times 10^4 \text{ m s}^{-1}, \quad v_0 = 1.00 \times 10^4 \text{ m s}^{-1}, \quad \Delta\theta_{\max} = 1.52615 \text{ rad},$$

gives

$$\begin{aligned}
v'' &= \left[(1.00 \times 10^4)^2 + 2(1.30650 \times 10^4)^2(1 - \cos 1.52615) \right. \\
&\quad \left. + 2(1.30650 \times 10^4)(1.00 \times 10^4) \sin 1.52615 \right]^{1/2} \\
&= 2.62144 \times 10^4 \text{ m s}^{-1}.
\end{aligned}$$

Hence

$$v'' = 2.62 \times 10^4 \text{ m s}^{-1} = 26.2 \text{ km s}^{-1}.$$

Jupiter's gravity changes the direction of the probe's asymptotic velocity in Jupiter's frame. This change produces the increase in speed and kinetic energy in the Sun's frame, supplied by a negligible reduction in Jupiter's orbital energy.

The solution was checked against the official solution.