

IPhO 1998

Theory problems and solutions · Reykjavik, Iceland · 3 theory problems with complete solutions

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Problem 1: Rolling of a hexagonal prism

Consider a long, solid, rigid, regular hexagonal prism like a common type of pencil (Figure 1.1). The mass of the prism is M and it is uniformly distributed. The length of each side of the cross-sectional hexagon is a . The moment of inertia I of the hexagonal prism about its central axis is

$$I = \frac{5}{12}Ma^2 \quad (1.1)$$

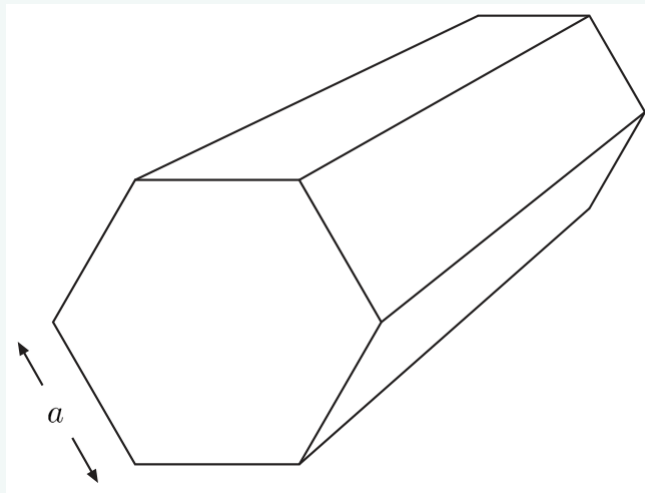


Figure 1.1: A solid prism with the cross section of a regular hexagon.

The moment of inertia I' about an edge of the prism is

$$I' = \frac{17}{12}Ma^2 \quad (1.2)$$

a) (3.5 points) The prism is initially at rest with its axis horizontal on an inclined plane which makes a small angle θ with the horizontal (Figure 1.2). Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The effect of this concavity on the moment of inertia can be ignored. The prism is now displaced from rest and starts an uneven rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is ω_i while ω_f is the angular velocity immediately after the impact.

Show that we may write

$$\omega_f = s\omega_i \quad (1.3)$$

and write the value of the coefficient s on the answer sheet.

¹Authors: Leó Kristjánsson and Thorsteinn Vilhjálmsson

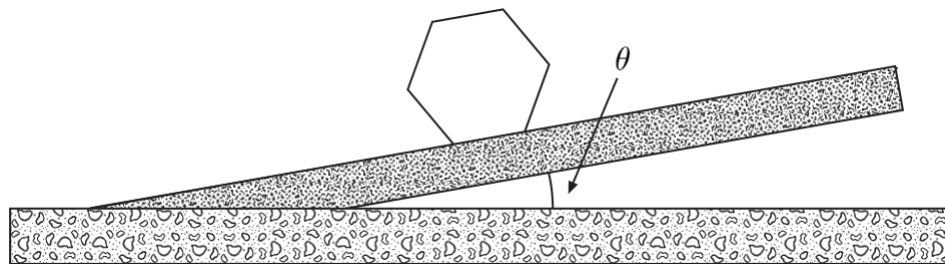


Figure 1.2: A hexagonal prism lying on an inclined plane.

b) (1 point) The kinetic energy of the prism just before and after impact is similarly K_i and K_f .

Show that we may write

$$K_f = rK_i \quad (1.4)$$

and write the value of the coefficient r on the answer sheet.

c) (1.5 points) For the next impact to occur K_i must exceed a minimum value $K_{i,min}$ which may be written in the form

$$K_{i,min} = \delta Mga \quad (1.5)$$

where $g = 9.81 \text{ m/s}^2$ is the acceleration of gravity.

Find the coefficient δ in terms of the slope angle θ and the coefficient r . Write your answer on the answer sheet. (Use the algebraic symbol r , not its value.)

d) (2 points) If the condition of part (c) is satisfied, the kinetic energy K_i will approach a fixed value $K_{i,0}$ as the prism rolls down the incline.

Given that the limit exists, show that $K_{i,0}$ may be written as:

$$K_{i,0} = \kappa Mga \quad (1.6)$$

and write the coefficient κ in terms of θ and r on the answer sheet.

e) (2 points) Calculate, to within 0.1° , the minimum slope angle θ_0 for which the uneven rolling, once started, will continue indefinitely. Write your numerical answer on the answer sheet.

Official English text of the problem.

Answers

- a) $s = \frac{11}{17} \approx 0.6471$.
- b) $r = \frac{121}{289} \approx 0.4187$.
- c) $\delta = \frac{1 - \cos(\pi/6 - \theta)}{r}$.
- d) $\kappa = \frac{\sin \theta}{1 - r}$.
- e) $\theta_0 = 6.5724 \dots^\circ \approx \boxed{6.6^\circ}$.

The coefficients s, r, δ, κ are dimensionless.

Solution

We treat each impact as instantaneous, neglecting gravity's impulse during it. Between impacts the prism rotates about its stationary contact edge; the contact force does no work, so mechanical energy is conserved. We retain the exact trigonometric functions throughout.

a) Change of angular velocity at impact

In a cross section, let A denote the old contact edge, B the next edge to strike the plane, and C the centre of mass. For a regular hexagon,

$$AC = BC = AB = a, \quad \angle ACB = 60^\circ.$$

Immediately before impact, the prism rotates about A , so

$$\mathbf{v}_C = \boldsymbol{\omega}_i \times \overrightarrow{AC}.$$

At impact the old edge releases and the new edge becomes the pivot. The impulsive contact force acts through B , so angular momentum about the fixed line where B strikes is conserved.

Before impact, the angular momentum about B comprises rotation about the centre of mass and the centre of mass's orbital angular momentum:

$$\begin{aligned} \mathbf{L}_{B,i} &= I\boldsymbol{\omega}_i + \overrightarrow{BC} \times M\mathbf{v}_C \\ &= \left(I + M\overrightarrow{BC} \cdot \overrightarrow{AC} \right) \boldsymbol{\omega}_i \\ &= \left(I + \frac{1}{2}Ma^2 \right) \boldsymbol{\omega}_i. \end{aligned}$$

After impact, the prism rotates about B , giving

$$\mathbf{L}_{B,f} = I'\boldsymbol{\omega}_f.$$

Consequently,

$$\left(\frac{5}{12} + \frac{1}{2} \right) Ma^2 \omega_i = \frac{17}{12} Ma^2 \omega_f,$$

and therefore

$$\boxed{\omega_f = \frac{11}{17}\omega_i}, \quad \boxed{s = \frac{11}{17}}.$$

b) Change of kinetic energy at impact

Immediately before and after impact, the motion is pure rotation about a stationary edge. Both edges have the same moment of inertia I' , so

$$K_i = \frac{1}{2}I'\omega_i^2, \quad K_f = \frac{1}{2}I'\omega_f^2.$$

It follows that

$$\frac{K_f}{K_i} = \left(\frac{\omega_f}{\omega_i} \right)^2 = s^2.$$

Thus

$$\boxed{K_f = \frac{121}{289} K_i}, \quad \boxed{r = \frac{121}{289}}.$$

c) Minimum energy needed for the next impact

After impact at B , the centre of mass must rise until it is vertically above B , then descend toward the next impact.

At the instant of impact, C lies a perpendicular distance $\sqrt{3}a/2$ above the plane and a distance $a/2$ uphill from B . Its vertical height relative to B is therefore

$$h_{\text{start}} = \frac{\sqrt{3}a}{2} \cos \theta + \frac{a}{2} \sin \theta = a \cos \left(\frac{\pi}{6} - \theta \right).$$

For the small slopes under consideration, $0 \leq \theta < \pi/6$, the centre of mass subsequently reaches the maximum height

$$h_{\text{top}} = a.$$

The gravitational potential-energy barrier is

$$\Delta U = Mga \left[1 - \cos \left(\frac{\pi}{6} - \theta \right) \right].$$

The available kinetic energy just after impact is rK_i . To pass the highest point and reach the next edge, it must satisfy

$$rK_i > \Delta U.$$

Hence the threshold pre-impact energy is

$$K_{i,\min} = \frac{Mga}{r} \left[1 - \cos \left(\frac{\pi}{6} - \theta \right) \right],$$

so

$$\boxed{\delta = \frac{1 - \cos(\pi/6 - \theta)}{r}}.$$

The inequality is strict: at exactly the threshold energy, the prism approaches the unstable balance position without crossing it.

d) Limiting pre-impact kinetic energy

Successive impact configurations are identical apart from a translation by one side length a down the plane. Thus the centre of mass loses a vertical height $a \sin \theta$ between impacts.

Let E_n be the kinetic energy immediately before impact n . Impact reduces it to rE_n , and gravity supplies $Mga \sin \theta$ before the following impact. Therefore, for successive completed steps,

$$E_{n+1} = rE_n + Mga \sin \theta.$$

Taking the stated limit $E_n \rightarrow K_{i,0}$,

$$K_{i,0} = rK_{i,0} + Mga \sin \theta.$$

Thus

$$\boxed{K_{i,0} = \frac{Mga \sin \theta}{1 - r}}, \quad \boxed{\kappa = \frac{\sin \theta}{1 - r}}.$$

Indeed, the recurrence gives

$$E_n = K_{i,0} + r^n(E_0 - K_{i,0}),$$

which converges because $0 < r < 1$, provided the prism continues to cross each energy barrier.

e) Minimum slope for indefinite rolling

If $K_{i,0} < K_{i,\min}$, any finite initial energy eventually decreases enough that the prism cannot cross the next barrier. If $K_{i,0} \geq K_{i,\min}$ and the initial energy satisfies $E_0 > K_{i,\min}$, the recurrence keeps every finite subsequent E_n above the threshold.

The critical angle is therefore determined by

$$K_{i,0} = K_{i,\min},$$

or

$$\frac{\sin \theta_0}{1 - r} = \frac{1 - \cos(\pi/6 - \theta_0)}{r}.$$

Substituting $r = 121/289$,

$$121 \sin \theta_0 = 168 \left[1 - \cos \left(\frac{\pi}{6} - \theta_0 \right) \right].$$

Expanding the cosine yields

$$205 \sin \theta_0 + 84\sqrt{3} \cos \theta_0 = 168.$$

Using $t = \tan(\theta_0/2)$, this becomes

$$(168 + 84\sqrt{3})t^2 - 410t + (168 - 84\sqrt{3}) = 0.$$

The root in the physical interval $0 < \theta_0 < 30^\circ$ is

$$t = \frac{3(2 - \sqrt{3})}{14}.$$

Consequently,

$$\theta_0 = 2 \arctan \left[\frac{3(2 - \sqrt{3})}{14} \right] = 6.5723657 \dots^\circ,$$

and to the requested precision,

$$\boxed{\theta_0 = 6.6^\circ}.$$

At the critical slope, an initial energy strictly above the threshold permits every subsequent step, although the prism spends progressively longer near each unstable balance position.

The solution was checked against the official solution.

Problem 2: Water under an ice cap

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km. In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may

assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

Density of water:	$\rho_w = 1.000 \cdot 10^3 \text{ kg/m}^3$
Density of ice:	$\rho_i = 0.917 \cdot 10^3 \text{ kg/m}^3$
Specific heat of ice:	$c_i = 2.1 \cdot 10^3 \text{ J/(kg } ^\circ\text{C)}$
Specific latent heat of ice:	$L_i = 3.4 \cdot 10^5 \text{ J/kg}$
Density of rock and magma:	$\rho_r = 2.9 \cdot 10^3 \text{ kg/m}^3$
Specific heat of rock and magma:	$c_r = 700 \text{ J/(kg } ^\circ\text{C)}$
Specific latent heat of rock and magma:	$L_r = 4.2 \cdot 10^5 \text{ J/kg}$
Average outward heat flow through the surface of the earth:	$J_Q = 0.06 \text{ W/m}^2$
Melting point of ice:	$T_0 = 0^\circ\text{C}$, constant

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness d of the ice layer melted every year and write your answer in the designated box on the answer sheet.

b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle α . The upper surface of the cap slopes by an angle β as shown in Figure 2.1. The vertical thickness of the ice at $x = 0$ is h_0 . Hence the lower and upper surfaces of the ice cap can be described by the equations

$$y_1 = x \tan \alpha, \quad y_2 = h_0 + x \tan \beta \quad (2.1)$$

Derive an expression for the pressure p at the bottom of the ice cap as a function of the horizontal coordinate x and write it on the answer sheet.

Formulate mathematically a condition between β and α , so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan \beta = s \tan \alpha$. Find the coefficient s and write the result in a symbolic form on the answer sheet.

The line $y_1 = 0.8x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness h_0 at $x = 0$ is 2 km. Assume that water at the bottom is in equilibrium. On a graph answer sheet draw the line y_1 and add a line y_2 showing the upper surface of the ice. Indicate on the figure which line is which.

⁶Authors: Gudni Axelsson and Thorsteinn Vilhjálmsson

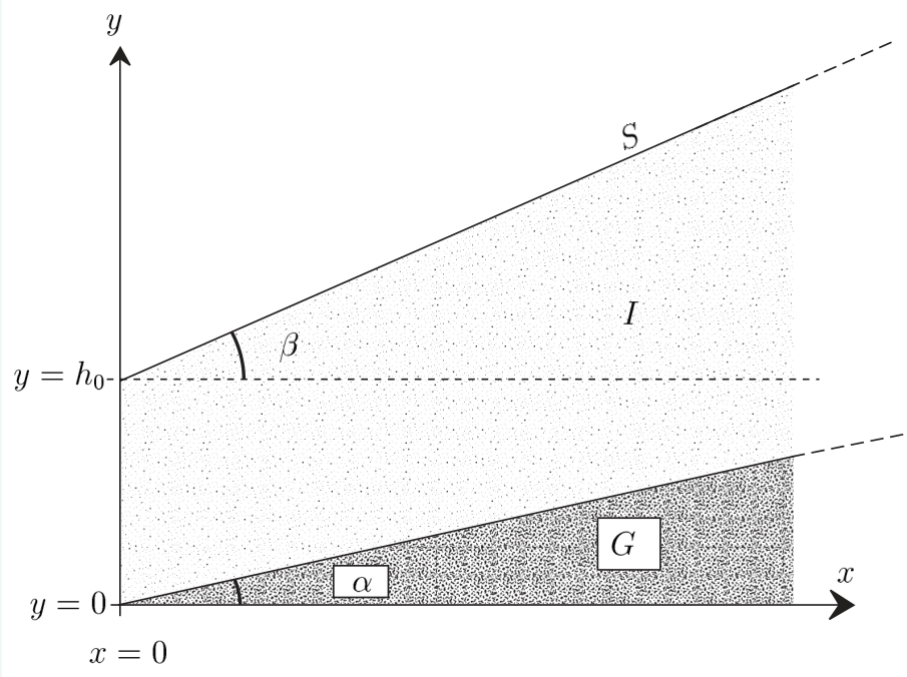


Figure 2.1: Cross section of an ice cap with a plane surface resting on an inclined plane ground. S : surface, G : ground, I : ice cap.

c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D = 2.0$ km, a conical body of water of height $H = 1.0$ km and radius $r = 1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.

d) (5 points) In its annual expedition, an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth h of 100 m and a radius r of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m.

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at 1200°C at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of 0°C . Subsequently,

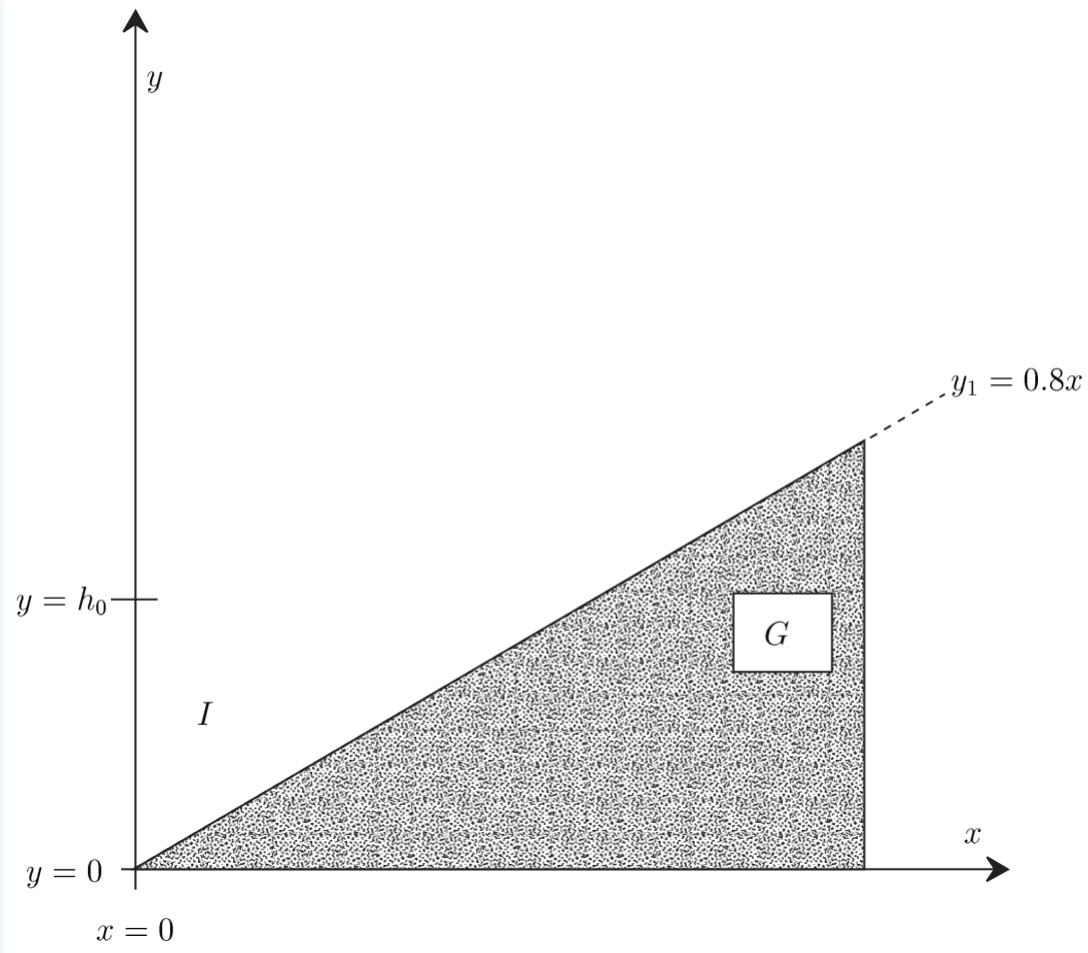


Figure 2.2: Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G : ground, I : ice cap.

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height H of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height h_1 of the intrusion.
3. The total mass m_{tot} of the water produced and the mass m' of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

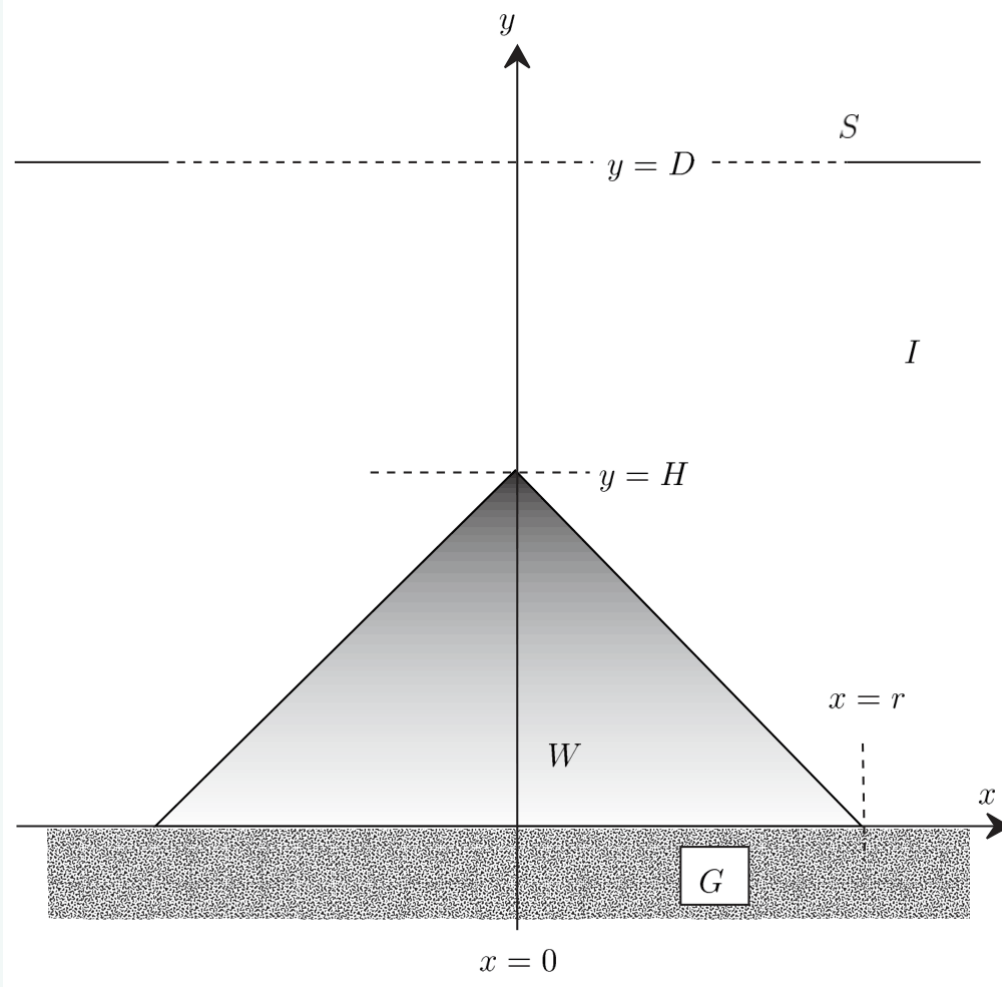


Figure 2.3: A vertical section through the mid-plane of a water cone inside an ice cap. S : surface, W : water, G : ground, I : ice cap.

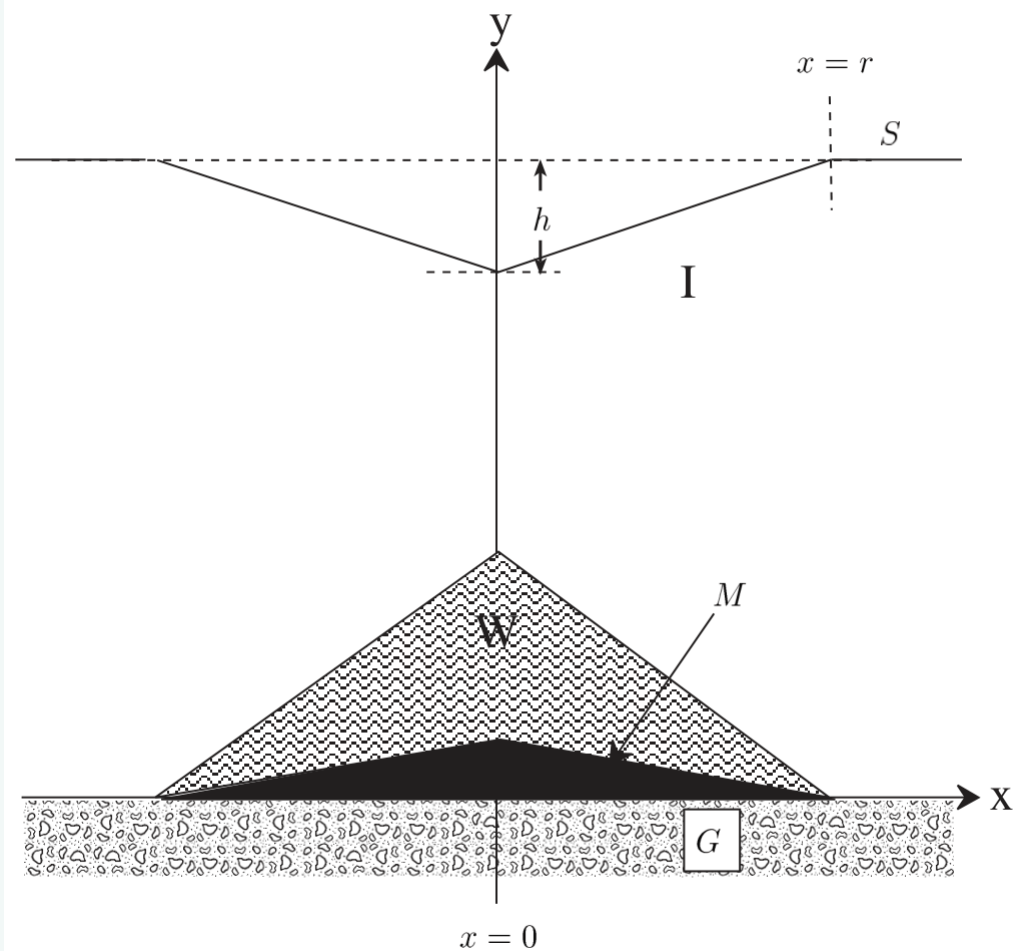


Figure 2.4: A vertical and central cross section of a conical depression in a temperate ice cap. S : surface, G : ground, I : ice cap, M : rock/magma intrusion, W : water. Note that the figure is NOT drawn to scale.

Official English text of the problem.

Answers

- **a)** The thickness melted each year is

$$d \simeq 6.1 \times 10^{-3} \text{ m} = 6.1 \text{ mm.}$$

- **b)** With uniform atmospheric pressure p_a ,

$$p(x) - p_a = \rho_i g [h_0 + x(\tan \beta - \tan \alpha)].$$

No flow requires

$$\tan \beta = s \tan \alpha, \quad s = \frac{\rho_i - \rho_w}{\rho_i} \simeq -0.09051.$$

For the requested graph, with both coordinates in kilometres,

$$y_1 = 0.8x, \quad y_2 = 2.000 - 0.07241x.$$

The lines intersect at approximately (2.293, 1.834) km.

- **c)** Let R be horizontal distance from the cone axis. The surface is

$$y_s(R) = \begin{cases} D - \frac{\rho_w - \rho_i}{\rho_i} H \left(1 - \frac{R}{r}\right), & 0 \leq R \leq r, \\ D, & R \geq r. \end{cases}$$

It is a conical depression of radius 1.0 km and depth approximately 91 m. Its central elevation is 1.9095 km above the bed.

- **d), 1.**

$$H = \frac{\rho_i h}{\rho_w - \rho_i} \simeq 1.10 \times 10^3 \text{ m.}$$

- **d), 2.**

$$h_1 \simeq 103 \text{ m.}$$

The intrusion volume is approximately $2.7 \times 10^7 \text{ m}^3$.

- **d), 3.**

$$m_{tot} \simeq 2.9 \times 10^{11} \text{ kg}, \quad m' \simeq 2.7 \times 10^{10} \text{ kg.}$$

About 9.3% of the meltwater escapes. For the plot, define $f(x) = 1 - |x|/(500 \text{ m})$ for $|x| \leq 500 \text{ m}$. The rock surface and water–ice boundary are

$$y_M = (102.8 \text{ m})f(x), \quad y_W = (1104.8 \text{ m})f(x).$$

Water occupies the region between these two surfaces.

Solution

a) Annual melting

The ice is already at its melting point, so the supplied geothermal heat goes into latent heat. For a horizontal area A , conservation of energy over one year gives

$$J_Q A t = \rho_i A d L_i.$$

Thus

$$d = \frac{J_Q t}{\rho_i L_i}.$$

Taking one year as $365 \times 24 \times 3600 = 3.1536 \times 10^7 \text{ s}$,

$$d = \frac{0.06(3.1536 \times 10^7)}{(917)(3.4 \times 10^5)} = 6.069 \times 10^{-3} \text{ m.}$$

Therefore approximately 6.1 mm of ice melts each year. The specific heat of ice is unnecessary because no initial warming is required.

b) Pressure and the condition for no water flow

The vertical ice thickness is

$$h_i(x) = y_2 - y_1 = h_0 + x(\tan \beta - \tan \alpha).$$

Using the stated model, the basal pressure equals atmospheric pressure plus the weight of the overlying ice per unit horizontal area:

$$p(x) = p_a + \rho_i g [h_0 + x(\tan \beta - \tan \alpha)].$$

If pressure is measured relative to the atmosphere, the p_a term is omitted.

For water at rest, hydrostatic equilibrium requires

$$p + \rho_w g y = \text{constant.}$$

Along the thin water layer, $y = y_1(x)$, so the no-flow condition is

$$\frac{dp}{dx} + \rho_w g \frac{dy_1}{dx} = 0.$$

Substituting the pressure and ground slope,

$$\rho_i g (\tan \beta - \tan \alpha) + \rho_w g \tan \alpha = 0.$$

Consequently,

$$\tan \beta = \frac{\rho_i - \rho_w}{\rho_i} \tan \alpha, \quad s = \frac{\rho_i - \rho_w}{\rho_i}.$$

Numerically,

$$s = \frac{917 - 1000}{917} = -0.0905125.$$

The negative sign means that the ice surface must slope downward in the direction in which the ground rises.

For $y_1 = 0.8x$ and $h_0 = 2$ km,

$$\tan \beta = (-0.0905125)(0.8) = -0.0724100,$$

giving, with x and y in kilometres,

$$y_1 = 0.8x, \quad y_2 = 2.000 - 0.0724100x.$$

Draw the ground as a straight rising line through the origin and the ice surface as a gently descending line through $(0, 2)$. Useful plotting coordinates are:

x (km)	Ground y_1 (km)	Ice surface y_2 (km)
0	0	2.000
1.000	0.800	1.928
2.000	1.600	1.855
2.2925	1.834	1.834

For $x \geq 0$, the ice lies between these lines up to their intersection.

c) Surface above a conical water body

Let $z(R)$ denote the elevation of the ice–water boundary above the horizontal bed:

$$z(R) = H \left(1 - \frac{R}{r} \right), \quad 0 \leq R \leq r.$$

At the rim, the water thickness tends to zero and the ice retains its original thickness D . Hence the water pressure at bed level is

$$p_b = p_a + \rho_i g D.$$

Hydrostatic equilibrium gives the pressure at elevation z :

$$p_w(z) = p_a + \rho_i g D - \rho_w g z.$$

The ice column above that point has thickness $y_s - z$, and therefore exerts pressure

$$p_i = p_a + \rho_i g (y_s - z).$$

Equating the pressures at the ice–water boundary,

$$\rho_i (y_s - z) = \rho_i D - \rho_w z.$$

Thus

$$y_s = D - \frac{\rho_w - \rho_i}{\rho_i} z.$$

Substitution of the conical boundary gives

$$y_s(R) = D - \frac{\rho_w - \rho_i}{\rho_i} H \left(1 - \frac{R}{r} \right), \quad 0 \leq R \leq r.$$

Outside the footprint, $y_s = D$. The depression depth is

$$\delta = \frac{\rho_w - \rho_i}{\rho_i} H = \frac{83}{917} (1000 \text{ m}) = 90.51 \text{ m}.$$

Therefore the surface is an inverted cone of radius 1.0 km, meeting the surrounding horizontal surface continuously. Its lowest point is

$$y_s(0) = 2000 - 90.51 = 1909.49 \text{ m}.$$

For a central vertical section, set $R = |x|$. With coordinates in kilometres, join

$$(-1.000, 2.000), \quad (0, 1.90949), \quad (1.000, 2.000)$$

by straight segments, and continue horizontally at $y = 2.000$ km on both sides. Rotating this profile about the vertical axis gives the surface depression.

d) Volcanic intrusion

Let

$$f(R) = 1 - \frac{R}{r}, \quad 0 \leq R \leq r,$$

where $r = 500$ m. The final surface, water–ice boundary, and rock surface have the forms

$$y_s = D - hf(R), \quad y_W = Hf(R), \quad y_M = h_1 f(R).$$

Here $D = 2000$ m and $h = 100$ m.

1. Height of the top of the water cone

The hydrostatic relation derived in part c applies again:

$$D - y_s = \frac{\rho_w - \rho_i}{\rho_i} y_W.$$

At the centre,

$$h = \frac{\rho_w - \rho_i}{\rho_i} H.$$

Hence

$$H = \frac{\rho_i h}{\rho_w - \rho_i} = \frac{917(100)}{83} = 1104.82 \text{ m.}$$

This height is measured from the original bed. The actual central water depth will be $H - h_1$.

2. Height of the intrusion

First determine how much ice has melted. The final ice thickness in a vertical column is

$$y_s - y_w = D - (h + H)f(R).$$

Since ice moves only vertically, each column loses an ice thickness

$$(h + H)f(R).$$

The initial lifting by the magma does not change the amount of ice in a column. Thus the total melted ice volume is

$$V_{\text{melt}} = \int_0^r (h + H) \left(1 - \frac{R}{r}\right) 2\pi R dR = \frac{\pi r^2}{3} (h + H).$$

Write

$$C = \frac{\pi r^2}{3}.$$

Then

$$m_{\text{tot}} = \rho_i C(h + H), \quad V_M = Ch_1, \quad m_M = \rho_r Ch_1.$$

At final thermal equilibrium, the rock and the ice–water mixture are at 0°C . Each kilogram of magma releases

$$q_M = L_r + c_r(1200^\circ\text{C} - T_0) = 4.2 \times 10^5 + 700(1200) = 1.26 \times 10^6 \text{ J/kg.}$$

This includes both solidification and cooling. Neglecting heat losses and small mechanical-energy changes, conservation of energy gives

$$\rho_r Ch_1 q_M = \rho_i C(h + H)L_i.$$

Therefore

$$h_1 = \frac{\rho_i (h + H)L_i}{\rho_r [L_r + c_r(1200^\circ\text{C} - T_0)]}.$$

The hydrostatic relation also implies

$$\rho_i (h + H) = \rho_w H,$$

so equivalently,

$$h_1 = \frac{\rho_w H L_i}{\rho_r [L_r + c_r(1200^\circ\text{C} - T_0)]}.$$

Numerically,

$$h_1 = \frac{1000(1104.819)(3.4 \times 10^5)}{2900(1.26 \times 10^6)} = 102.80 \text{ m} \simeq 103 \text{ m.}$$

The intrusion volume is consequently

$$V_M = \frac{\pi(500)^2}{3}(102.802) = 2.691 \times 10^7 \text{ m}^3.$$

3. Total water production and water escaping

From the melted ice volume,

$$m_{tot} = \rho_i C(h + H) = \rho_w CH.$$

Thus

$$m_{tot} = 1000 \frac{\pi(500)^2}{3}(1104.819) = 2.8924 \times 10^{11} \text{ kg} \simeq 2.9 \times 10^{11} \text{ kg}.$$

The remaining water occupies the space between the water–ice cone and the rock cone. Its volume is

$$V_{\text{remaining}} = C(H - h_1),$$

and its mass is

$$m_{\text{remaining}} = \rho_w C(H - h_1) = 2.6233 \times 10^{11} \text{ kg}.$$

Mass conservation therefore gives

$$\begin{aligned} m' &= m_{tot} - m_{\text{remaining}} \\ &= \rho_w CH - \rho_w C(H - h_1) \\ &= \rho_w Ch_1. \end{aligned}$$

Hence

$$m' = 1000 \frac{\pi(500)^2}{3}(102.802) = 2.6914 \times 10^{10} \text{ kg} \simeq 2.7 \times 10^{10} \text{ kg}.$$

The escaped water volume equals the intrusion volume:

$$\frac{m'}{\rho_w} = V_M.$$

This also explains the two stages. Initially the magma lifts the ice without reducing its thickness, so water on the elevated rock surface has excess hydraulic pressure and drains away. At the end of this drainage stage, the ice–water boundary is approximately the rock surface $h_1 f(R)$. Hydrostatic equilibrium then requires an ice thickness

$$D - \frac{\rho_w}{\rho_i} h_1 f(R).$$

The ice melted by that stage has mass $\rho_w Ch_1$, exactly the escaped mass found above. Further melting produces the retained water body.

Description of the plot

For a central vertical section, use $R = |x|$. Within $|x| \leq 500 \text{ m}$,

$$y_M(x) = 102.802 \left(1 - \frac{|x|}{500} \right) \text{ m},$$

$$y_W(x) = 1104.819 \left(1 - \frac{|x|}{500} \right) \text{ m},$$

$$y_s(x) = 2000 - 100 \left(1 - \frac{|x|}{500} \right) \text{ m},$$

where x is numerically expressed in metres.

Use the following coordinates and join adjacent points by straight lines:

x (m)	Rock surface y_M (m)	Water–ice boundary y_W (m)	Ice surface y_s (m)
−500	0	0	2000
−250	51.4	552.4	1950
0	102.8	1104.8	1900
250	51.4	552.4	1950
500	0	0	2000

Draw the original bed along $y = 0$, shade the intrusion between the bed and y_M , and shade the water between y_M and y_W . The ice lies between y_W and y_s . Outside the footprint, the bed remains at $y = 0$ and the ice surface at $y = 2000$ m.

Using equal length scales on both axes, the intrusion is only about 103 m high, while the water boundary reaches about 1105 m. The central water depth is

$$H - h_1 \simeq 1002 \text{ m},$$

and the remaining ice directly above it is about 795 m thick.

The solution was checked against the official solution.

Problem 3: Faster than light?

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as do altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = 1/3600 of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R = 12.5$ kpc. A kiloparsec (kpc) equals $3.09 \cdot 10^{19}$ m. The speed of light is $c = 3.00 \cdot 10^8$ m/s. Error calculations are not required in the solution.

a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_1(t)$ and $\theta_2(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and t is the time of observation. The angular speeds, as seen from the Earth, are ω_1 and ω_2 . The corresponding apparent transverse linear speeds of the two sources are denoted by $v'_{1,\perp}$ and $v'_{2,\perp}$.

Using Figure 3.1, make a graph to find the numerical values of ω_1 and ω_2 in milli-arc-seconds per day (mas/d). Also determine the numerical values of $v'_{1,\perp}$ and $v'_{2,\perp}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).

b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle ϕ ($0 \leq \phi \leq \pi$) to the direction towards a distant observer O (Figure 3.2). The speed may be written as $v = \beta c$, where c is the speed of light. The distance to the source, as measured by the observer, is R . The angular speed of the source, as seen from the observer, is ω , and the apparent linear speed perpendicular to the line of sight is $v'_\perp$.

Find ω and $v'_\perp$ in terms of β , R and ϕ and write your answer on the answer sheet.

c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v = \beta c$. Then the results of part (b) make it possible to calculate β and ϕ from the angular speeds ω_1 and ω_2 and the distance R . Here ϕ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for β and ϕ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.

d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v'_\perp$ to be larger than the speed of light c .

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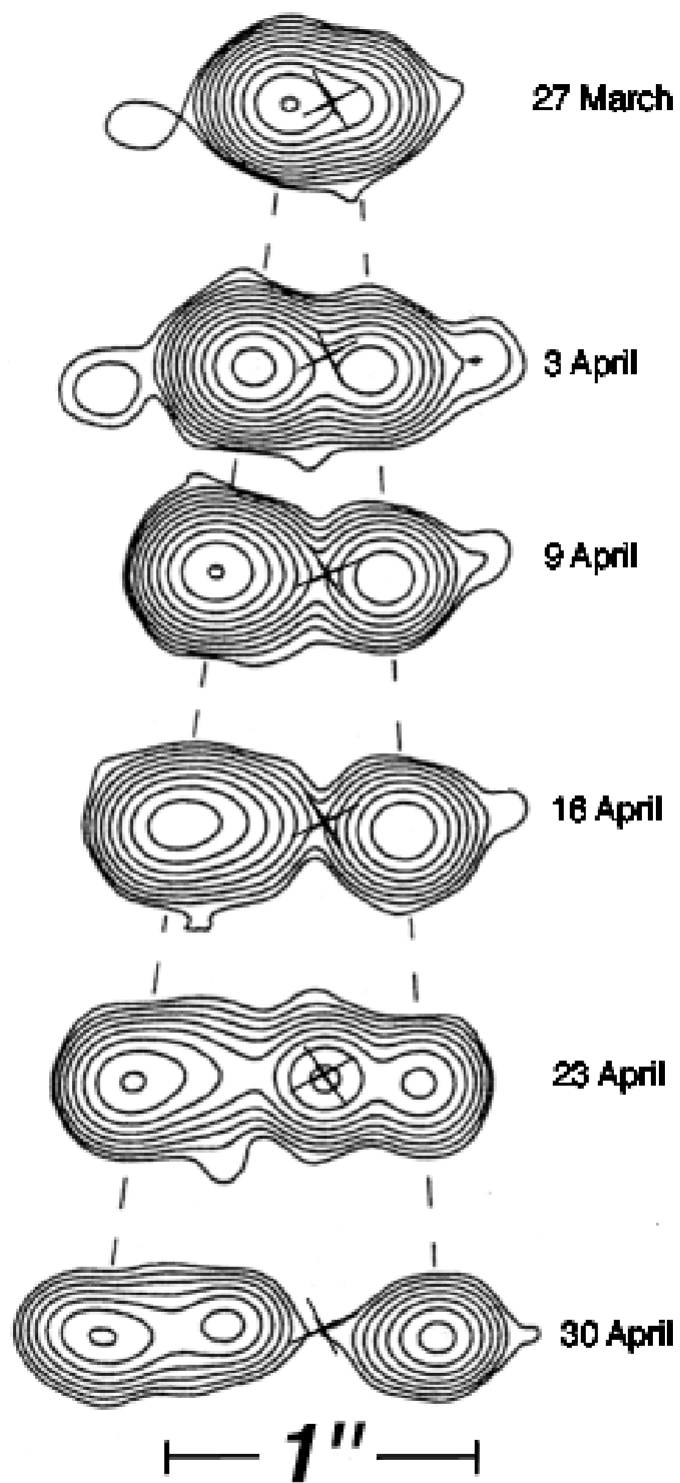


Figure 3.1: Radio emission from a source in our galaxy.

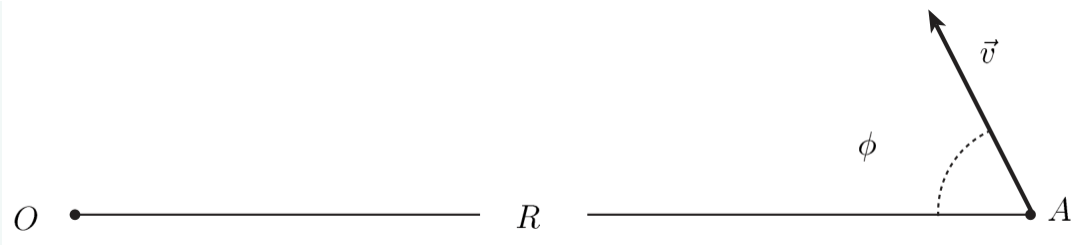


Figure 3.2: The observer is at O and the original position of the light source is at A . The velocity vector is $\vec{v}$.

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ) -plane. Show by shading in which part of this region the condition $v'_{\perp} > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_{\perp})_{\max}$ of the apparent perpendicular speed $v'_{\perp}$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \rightarrow 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method of determining R . One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift, $\lambda = \lambda_0(1 - \beta \cos \phi)(1 - \beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v , show that the unknown $\beta = v/c$ can be expressed in terms of λ_0 , λ_1 , and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha \lambda_0^2}{(\lambda_1 + \lambda_2)^2}}. \quad (3.1)$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

Official English text of the problem.

Answers

- a) Representative graphical values are

$$\omega_1 \simeq 17.5 \text{ mas/d}, \quad \omega_2 \simeq 9.0 \text{ mas/d}.$$

Consequently,

$$v'_{1,\perp} \simeq 3.8 \times 10^8 \text{ m/s} \simeq 1.3c, \quad v'_{2,\perp} \simeq 2.0 \times 10^8 \text{ m/s} \simeq 0.65c.$$

• b)

$$\omega = \frac{\beta c \sin \phi}{R(1 - \beta \cos \phi)}, \quad v'_\perp = \frac{\beta c \sin \phi}{1 - \beta \cos \phi}.$$

• c) With angular speeds expressed in radians per unit time, define

$$X = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}, \quad Y = \frac{2R\omega_1\omega_2}{c(\omega_1 + \omega_2)}.$$

Then

$$\beta = \sqrt{X^2 + Y^2} \simeq 0.92, \quad \phi = \arccos\left(\frac{X}{\beta}\right) \simeq 70^\circ.$$

The left-hand object is approaching.

• d) The superluminal region is

$$0 < \phi < \frac{\pi}{2}, \quad \frac{1}{\sin \phi + \cos \phi} < \beta < 1.$$

Thus $f(\phi) = 1/(\sin \phi + \cos \phi)$ on the relevant angular interval. Such motion requires $\beta > 1/\sqrt{2}$.

• e)

$$(v'_\perp)_{\max} = \frac{\beta c}{\sqrt{1 - \beta^2}}, \quad \phi_{\max} = \arccos \beta.$$

• f)

$$\alpha = 4, \quad \beta = \sqrt{1 - \frac{4\lambda_0^2}{(\lambda_1 + \lambda_2)^2}}.$$

Solution

a) Angular motions and apparent transverse speeds

Take $t = 0$ on 27 March. Measure each component's angular distance from the cross in its own image, using the full marked scale length as $1'' = 1000$ mas. Follow the two outward-moving components indicated by the dashed tracks. We use positive angular distances on both sides of the center.

Approximate readings from the supplied figure are:

Date	t (days)	θ_1 (arcseconds)	θ_2 (arcseconds)
27 March	0	0.12	0.07
3 April	7	0.22	0.14
9 April	13	0.35	0.20
16 April	20	0.46	0.26
23 April	27	0.61	0.31
30 April	34	0.70	0.38

Plot t horizontally and angular distance vertically. Each set of points lies approximately on a straight line; the left-hand component has the greater slope. Representative lines are

$$\frac{\theta_1}{\text{arcsec}} \simeq 0.11 + 0.0175 \frac{t}{\text{d}}, \quad \frac{\theta_2}{\text{arcsec}} \simeq 0.08 + 0.0090 \frac{t}{\text{d}}.$$

Thus their vertical intercepts are approximately $0.11''$ and $0.08''$, and their slopes give

$$\omega_1 \simeq 17.5 \text{ mas/d}, \quad \omega_2 \simeq 9.0 \text{ mas/d}.$$

The angles are extremely small, so the transverse displacement corresponding to an angular displacement θ is $R\theta$. Consequently,

$$v'_{i,\perp} = R\omega_i,$$

where ω_i must be expressed in radians per second. The required conversions are

$$R = 12.5(3.09 \times 10^{19}) = 3.8625 \times 10^{20} \text{ m},$$

$$1 \text{ mas} = \frac{\pi}{180 \times 3600 \times 1000} \text{ rad} = 4.8481 \times 10^{-9} \text{ rad},$$

and therefore

$$1 \text{ mas/d} = 5.6113 \times 10^{-14} \text{ rad/s}.$$

At the stated distance, 1 mas/d corresponds to

$$R(5.6113 \times 10^{-14}) = 2.1674 \times 10^7 \text{ m/s}.$$

Hence

$$v'_{1,\perp} \simeq 17.5(2.1674 \times 10^7) \simeq 3.8 \times 10^8 \text{ m/s} \simeq 1.3c,$$

$$v'_{2,\perp} \simeq 9.0(2.1674 \times 10^7) \simeq 2.0 \times 10^8 \text{ m/s} \simeq 0.65c.$$

The first value exceeds c . These speeds use the time interval between the **arrivals** of signals at Earth; that interval differs from the interval between their emissions.

b) Effect of light-travel time

Work in the inertial frame in which the observer and central source are stationary. Let two signals be emitted a time Δt_e apart in this frame.

During this interval, the emitter moves a transverse distance

$$\Delta x_{\perp} = v \sin \phi \Delta t_e.$$

Its displacement toward the observer is

$$\Delta x_{\parallel} = v \cos \phi \Delta t_e.$$

For a distant observer, the second signal's path is shorter by approximately $\Delta x_{\parallel}$. The arrival-time interval is therefore

$$\Delta t_{\text{obs}} = \Delta t_e - \frac{\Delta x_{\parallel}}{c} = (1 - \beta \cos \phi) \Delta t_e.$$

The same expression covers a receding source: then $\cos \phi < 0$, so the second signal travels farther.

The angular displacement is, in the small-angle approximation,

$$\Delta \theta \simeq \frac{\Delta x_{\perp}}{R}.$$

Thus

$$\omega = \frac{\Delta\theta}{\Delta t_{\text{obs}}} = \frac{\beta c \sin \phi}{R(1 - \beta \cos \phi)}$$

and

$$v'_{\perp} = R\omega = \frac{\beta c \sin \phi}{1 - \beta \cos \phi}.$$

Here the source's displacement is negligible compared with R , allowing R and ϕ to be treated as constant over the observations.

For an approaching emitter, successive signals arrive closer together than they were emitted. Dividing the transverse displacement by this shortened interval can give $v'_{\perp} > c$, although the actual emitter speed remains $v = \beta c < c$.

c) Recovering the true speed and direction

The two directions make angles ϕ and $\pi - \phi$ with the direction toward the observer. Their transverse velocity magnitudes are equal, while their radial components have opposite signs. Therefore,

$$\frac{R\omega_1}{c} = \frac{\beta \sin \phi}{1 - \beta \cos \phi}, \quad \frac{R\omega_2}{c} = \frac{\beta \sin \phi}{1 + \beta \cos \phi}.$$

Set

$$X = \beta \cos \phi, \quad Y = \beta \sin \phi.$$

Then

$$\omega_1(1 - X) = \omega_2(1 + X) = \frac{cY}{R}.$$

Equating the first two expressions gives

$$X = \beta \cos \phi = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2}.$$

Substitution into either angular-speed equation gives

$$Y = \beta \sin \phi = \frac{2R\omega_1\omega_2}{c(\omega_1 + \omega_2)}.$$

Consequently,

$$\beta = \frac{\sqrt{(\omega_1 - \omega_2)^2 + (2R\omega_1\omega_2/c)^2}}{\omega_1 + \omega_2}$$

and an expression valid throughout $0 \leq \phi \leq \pi$ is

$$\phi = \arccos \left[\frac{\omega_1 - \omega_2}{\beta(\omega_1 + \omega_2)} \right].$$

For these observations, $\omega_1 > \omega_2$, so ϕ is acute. We can also use

$$\tan \phi = \frac{2R\omega_1\omega_2}{c(\omega_1 - \omega_2)}.$$

Using the representative rates from (a),

$$X \simeq \frac{17.5 - 9.0}{17.5 + 9.0} \simeq 0.321,$$

$$Y \simeq 0.859.$$

Thus

$$\beta \simeq \sqrt{0.321^2 + 0.859^2} \simeq 0.917, \quad \phi \simeq 69.5^\circ.$$

To the precision justified by the graph,

$$\boxed{\beta \simeq 0.92, \quad \phi \simeq 70^\circ.}$$

The true speed is approximately 2.8×10^8 m/s. Since $\beta \cos \phi > 0$, the left-hand component is approaching and the right-hand component is receding.

d) Condition and region for apparent superluminal motion

For a material emitter, $0 \leq \beta < 1$. Hence $1 - \beta \cos \phi > 0$, and the condition $v'_\perp > c$ becomes

$$\frac{\beta \sin \phi}{1 - \beta \cos \phi} > 1,$$

or

$$\boxed{\beta(\sin \phi + \cos \phi) > 1.}$$

For $\phi \geq \pi/2$, the denominator of $v'_\perp/c$ is at least one, while its numerator is at most $\beta < 1$.

Superluminal apparent motion is therefore impossible for a receding or exactly transverse source. It is also impossible at $\phi = 0$, where the transverse motion vanishes.

Within the relevant interval $0 < \phi < \pi/2$,

$$\boxed{\beta > f(\phi), \quad f(\phi) = \frac{1}{\sin \phi + \cos \phi}.$$

The complete physical condition is thus

$$\boxed{0 < \phi < \frac{\pi}{2}, \quad \frac{1}{\sin \phi + \cos \phi} < \beta < 1.}$$

For the requested diagram, the physically relevant region has

$$0 \leq \beta < 1, \quad 0 \leq \phi \leq \pi.$$

With ϕ horizontal and β vertical, draw the threshold curve over $0 \leq \phi \leq \pi/2$. Its useful points are:

ϕ	$f(\phi)$
0	1
$\pi/6$	0.732
$\pi/4$	$1/\sqrt{2} \simeq 0.707$
$\pi/3$	0.732
$\pi/2$	1

The curve is symmetric about $\phi = \pi/4$. It decreases from 1 to its minimum $1/\sqrt{2}$, then rises to 1. Shade the region strictly above this curve and strictly below $\beta = 1$. The threshold curve itself has $v'_\perp = c$, so it is excluded from the shading. No region at $\phi \geq \pi/2$ is shaded.

Since

$$\sin \phi + \cos \phi = \sqrt{2} \cos\left(\phi - \frac{\pi}{4}\right) \leq \sqrt{2},$$

such a region exists only when

$$\beta > \frac{1}{\sqrt{2}}.$$

Equivalently, for any fixed $\beta > 1/\sqrt{2}$, its angular extent is

$$\left|\phi - \frac{\pi}{4}\right| < \arccos\left(\frac{1}{\sqrt{2}\beta}\right).$$

e) Maximum apparent transverse speed

At fixed $0 < \beta < 1$, differentiate the result of (b):

$$\frac{d}{d\phi}\left(\frac{v'_\perp}{c}\right) = \frac{d}{d\phi}\left(\frac{\beta \sin \phi}{1 - \beta \cos \phi}\right) = \frac{\beta(\cos \phi - \beta)}{(1 - \beta \cos \phi)^2}.$$

The derivative changes from positive to negative where

$$\cos \phi = \beta.$$

Because the apparent transverse speed vanishes at both $\phi = 0$ and $\phi = \pi$, this stationary point is the global maximum. At that angle,

$$\sin \phi = \sqrt{1 - \beta^2}, \quad 1 - \beta \cos \phi = 1 - \beta^2.$$

Therefore,

$$(v'_\perp)_{\max} = \frac{\beta c}{\sqrt{1 - \beta^2}}, \quad \phi_{\max} = \arccos \beta.$$

As $\beta \rightarrow 1^-$, the denominator tends to zero and the maximum apparent speed increases without limit. For $\beta = 0$, the apparent speed is zero at every angle.

f) Determination of β from the Doppler shifts

Let

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

For the first object, the given Doppler formula is

$$\lambda_1 = \lambda_0 \gamma (1 - \beta \cos \phi).$$

For the oppositely moving object, $\cos(\pi - \phi) = -\cos \phi$, giving

$$\lambda_2 = \lambda_0 \gamma (1 + \beta \cos \phi).$$

Adding these expressions eliminates the direction:

$$\lambda_1 + \lambda_2 = 2\lambda_0\gamma = \frac{2\lambda_0}{\sqrt{1-\beta^2}}.$$

Thus

$$1 - \beta^2 = \frac{4\lambda_0^2}{(\lambda_1 + \lambda_2)^2},$$

and, taking the nonnegative speed,

$$\boxed{\beta = \sqrt{1 - \frac{4\lambda_0^2}{(\lambda_1 + \lambda_2)^2}}, \quad \alpha = 4.}$$

This also explains how the distance could be obtained. The angular motions determine

$$X = \beta \cos \phi = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2},$$

while the wavelengths determine β . Therefore $\beta \sin \phi = \sqrt{\beta^2 - X^2}$, and the relation derived in (c) gives

$$R = \frac{c(\omega_1 + \omega_2)}{2\omega_1\omega_2} \sqrt{\beta^2 - X^2}.$$

Thus the combined angular and wavelength measurements determine R without adopting its initial estimate.

The solution was checked against the official solution.