

IPhO 1997

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Problem 1: Scaling

(a) A small mass hangs on the end of a massless ideal spring and oscillates up and down at its natural frequency f . If the spring is cut in half and the mass reattached at the end, what is the new frequency f' ? (1.5 marks)

(b) The radius of a hydrogen atom in its ground state is $a_0 = 0.0529$ nm (the "Bohr radius"). What is the radius a' of a "muonic-hydrogen" atom in which the electron is replaced by an identically charged muon, with mass 207 times that of the electron? Assume the proton mass is much larger than that of the muon and electron. (2 marks)

(c) The mean temperature of the earth is $T = 287$ K. What would the new mean temperature T' be if the mean distance between the earth and the sun was reduced by 1%? (2 marks)

(d) On a given day, the air is dry and has a density $\rho = 1.2500$ kg/m³. The next day the humidity has increased and the air is 2% by mass water vapour. The pressure and temperature are the same as the day before. What is the air density ρ' now? (2 marks)

Mean molecular weight of dry air: 28.8 (g/mol)

Molecular weight of water: 18 (g/mol)

Assume ideal-gas behaviour.

(e) A type of helicopter can hover if the mechanical power output of its engine is P . If another helicopter is made which is an exact $\frac{1}{2}$ -scale replica (in all linear dimensions) of the first, what mechanical power P' is required for it to hover? (2.5 marks)

Official English text of the problem.

Answers

- (a) $f' = \sqrt{2} f$.
- (b) $a' = \frac{a_0}{207} = 2.56 \times 10^{-4}$ nm = 2.56×10^{-13} m.
- (c) $T' = \frac{287}{\sqrt{0.99}}$ K ≈ 288.4 K, an increase of approximately 1.4 K.
- (d) $\rho' = \frac{1.2500}{1.012}$ kg/m³ ≈ 1.2352 kg/m³.

- (e) $P' = 2^{-7/2}P = \frac{P}{8\sqrt{2}} \approx 0.0884P.$

Solution

(a)

Let the original spring constant be k , and the attached mass be m . Its natural frequency is

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}.$$

For a uniform spring, the two halves are identical springs connected in series. If each half has spring constant k_h , then

$$\frac{1}{k} = \frac{1}{k_h} + \frac{1}{k_h},$$

so $k_h = 2k$. Equivalently, under the same tension each half extends half as much as the whole spring.

The mass remains unchanged, giving

$$f' = \frac{1}{2\pi} \sqrt{\frac{2k}{m}} = \sqrt{2} f.$$

Gravity changes the equilibrium extension but does not affect the frequency of an ideal spring.

(b)

Under the stated approximation, the proton is effectively fixed, so the orbiting particle's mass m determines the atomic length scale.

In the Bohr model, Coulomb attraction supplies the centripetal force, and the ground-state angular momentum is $\hbar$:

$$\frac{mv^2}{a} = \frac{e^2}{4\pi\epsilon_0 a^2}, \quad mva = \hbar.$$

Eliminating v gives

$$a = \frac{4\pi\epsilon_0 \hbar^2}{me^2}.$$

Thus, for unchanged charge, the radius is inversely proportional to the particle's mass. Since $m_\mu = 207m_e$,

$$\frac{a'}{a_0} = \frac{m_e}{m_\mu} = \frac{1}{207}.$$

Therefore,

$$a' = \frac{0.0529}{207} \text{ nm} = 2.56 \times 10^{-4} \text{ nm} = 2.56 \times 10^{-13} \text{ m}.$$

(c)

Assume radiative equilibrium, constant solar luminosity, and unchanged terrestrial albedo and effective thermal-radiation properties.

At Earth–Sun distance r , the incident solar flux is proportional to r^{-2} . The absorbed power is therefore

$$P_{\text{abs}} = (1 - \alpha) \frac{L_{\odot}}{4\pi r^2} \pi R_{\oplus}^2,$$

where α is the albedo. By the Stefan–Boltzmann law, the emitted power can be represented as

$$P_{\text{emit}} = 4\pi R_{\oplus}^2 \varepsilon_{\text{eff}} \sigma T^4,$$

with a fixed effective factor ε_{eff} .

Equating absorbed and emitted power gives

$$T^4 \propto r^{-2}, \quad T \propto r^{-1/2}.$$

The new distance is $r' = 0.99r$, so

$$T' = T \left(\frac{r}{r'} \right)^{1/2} = \frac{287}{\sqrt{0.99}} \text{ K} \approx 288.4 \text{ K}.$$

As a first-order check,

$$\frac{\Delta T}{T} \approx -\frac{1}{2} \frac{\Delta r}{r} = 0.005,$$

giving $\Delta T \approx 1.435 \text{ K}$, close to the unrounded result 1.446 K .

(d)

Let $M_d = 28.8 \text{ g/mol}$ and $M_w = 18 \text{ g/mol}$ be the molar masses of dry air and water. The water **mass fraction** is $w = 0.02$.

A mass m of moist air contains masses $(1 - w)m$ of dry air and $w m$ of water vapour. Using consistent mass units, its total number of moles is

$$n = \frac{(1 - w)m}{M_d} + \frac{w m}{M_w}.$$

Since $m/V = \rho'$, the ideal-gas law gives

$$\frac{p}{RT} = \frac{n}{V} = \rho' \left(\frac{1 - w}{M_d} + \frac{w}{M_w} \right).$$

For the original dry air at the same pressure and temperature,

$$\frac{p}{RT} = \frac{\rho}{M_d}.$$

Equating these expressions and rearranging,

$$\rho' = \frac{\rho}{(1 - w) + w M_d / M_w}.$$

Hence

$$\rho' = \frac{1.2500}{0.98 + 0.02(28.8/18)} = \frac{1.2500}{1.012} \approx 1.2352 \text{ kg/m}^3.$$

The density decreases because water molecules are lighter than the average dry-air molecule, while the total molecular number density remains fixed at fixed pressure and temperature.

(e)

Assume the replica uses the same material densities and hovers in air of the same density. Use ideal momentum-flow scaling, with unchanged aerodynamic efficiency; neglect changes in viscous effects and mechanical losses.

Let the linear scale factor be λ . The helicopter's mass m scales with volume, while the rotor's swept area A scales with area:

$$m' = \lambda^3 m, \quad A' = \lambda^2 A.$$

Let v characterize the downward air velocity produced by the rotor. For geometrically similar flow, the mass of air accelerated per unit time scales as

$$\dot{m}_{\text{air}} \propto \rho_{\text{air}} A v.$$

By conservation of momentum, the upward thrust scales as

$$F \propto \dot{m}_{\text{air}} v \propto \rho_{\text{air}} A v^2.$$

Hovering requires $F = mg$, so

$$v \propto \sqrt{\frac{mg}{\rho_{\text{air}} A}}.$$

Consequently,

$$\frac{v'}{v} = \sqrt{\frac{\lambda^3}{\lambda^2}} = \lambda^{1/2}.$$

The required power is proportional to the rate at which kinetic energy is imparted to the air:

$$P \propto \dot{m}_{\text{air}} v^2 \propto \rho_{\text{air}} A v^3.$$

Thus

$$\frac{P'}{P} = \lambda^2 (\lambda^{1/2})^3 = \lambda^{7/2}.$$

For $\lambda = \frac{1}{2}$,

$$P' = \left(\frac{1}{2}\right)^{7/2} P = \frac{P}{8\sqrt{2}} \approx 0.0884P.$$

The solution was checked against the official solution.

Problem 2: Nuclear Masses and Stability

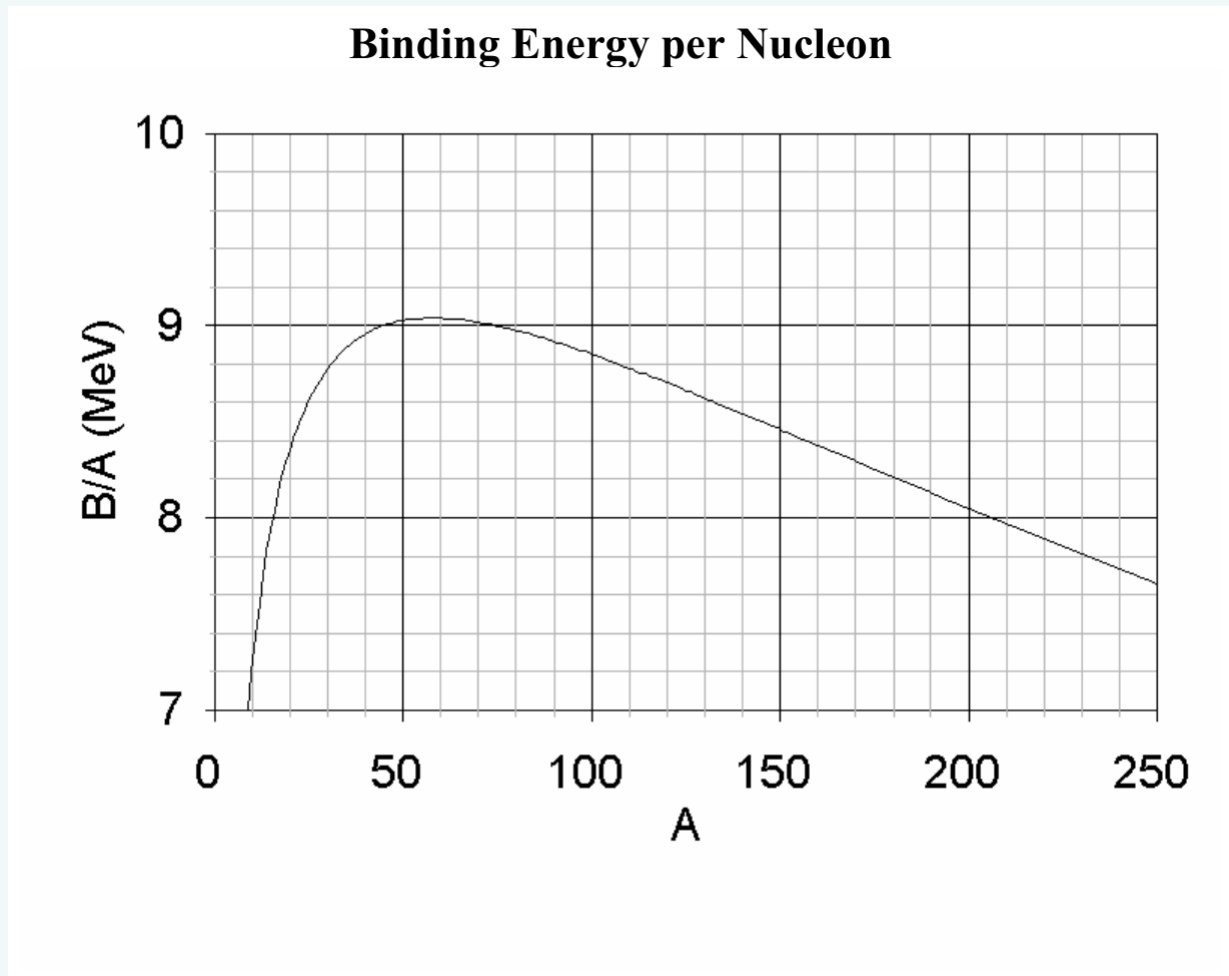
All energies in this question are expressed in MeV - millions of electron volts.

One MeV = 1.6×10^{-13} J, but it is not necessary to know this to solve the problem.

The mass M of an atomic nucleus with Z protons and N neutrons (i.e. the mass number $A = N + Z$) is the sum of masses of the free constituent nucleons (protons and neutrons) minus the binding energy B/c^2 .

$$Mc^2 = Zm_p c^2 + Nm_n c^2 - B$$

The graph shown below plots the maximum value of B/A for a given value of A , vs. A .
The greater the value of B/A , in general, the more stable is the nucleus.



(a) Above a certain mass number A_α , nuclei have binding energies which are always small enough to allow the emission of alpha-particles ($A = 4$). Use a linear approximation to this curve above $A = 100$ to estimate A_α . (3 marks)

For this model, assume the following:

- Both initial and final nuclei are represented on this curve.
- The total binding energy of the alpha-particle is given by $B_4 = 25.0$ MeV (this cannot be read off the graph!).

(b) The binding energy of an atomic nucleus with Z protons and N neutrons ($A = N + Z$) is given by a semi-empirical formula:

$$B = a_v A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - a_a \frac{(N - Z)^2}{A} - \delta$$

The value of δ is given by:

$+a_p A^{-3/4}$ for odd- N /odd- Z nuclei

0 for even- N /odd- Z or odd- N /even- Z nuclei

$-a_p A^{-3/4}$ for even- N /even- Z nuclei

The values of the coefficients are:

$$a_v = 15.8 \text{ MeV}; \quad a_s = 16.8 \text{ MeV}; \quad a_c = 0.72 \text{ MeV}; \quad a_a = 23.5 \text{ MeV}; \quad a_p = 33.5 \text{ MeV}.$$

(i) Derive an expression for the proton number $Z_{\max}$ of the nucleus with the largest binding energy for a given mass number A . Ignore the δ -term for this part only. (2 marks)

(ii) What is the value of Z for the $A = 200$ nucleus with the largest B/A ? Include the effect of the δ -term. (2 marks)

(iii) Consider the three nuclei with $A = 128$ listed in the table on the answer sheet. Determine which ones are energetically stable and which ones have sufficient energy to decay by the processes listed below. Determine $Z_{\max}$ as defined in part (i) and fill out the table on your answer sheet.

In filling out the table, please:

- Mark processes which are energetically allowed thus: $\checkmark$
- Mark processes which are NOT energetically allowed thus: 0
- Consider only transitions between these three nuclei.

Decay processes:

- (1) β^- - decay; emission from the nucleus of an electron
- (2) β^+ - decay; emission from the nucleus of a positron
- (3) $\beta^-\beta^-$ - decay; emission from the nucleus of two electrons simultaneously
- (4) Electron capture; capture of an *atomic* electron by the nucleus.

The rest mass energy of an electron (and positron) is $m_e c^2 = 0.51 \text{ MeV}$; that of a proton is $m_p c^2 = 938.27 \text{ MeV}$; that of a neutron is $m_n c^2 = 939.57 \text{ MeV}$.

(3 marks)

(b) (iii)

Nucleus/Process	β^- - decay	β^+ - decay	Electron-capture	$\beta^-\beta^-$ - decay
$^{128}_{53}\text{I}$				
$^{128}_{54}\text{Xe}$				
$^{128}_{55}\text{Cs}$				

Notation : $^A_Z X$

X = Chemical Symbol

Official English text of the problem.

Answers

- (a) $A_\alpha \approx 2.1 \times 10^2$. The straight-line fit used below gives approximately 215.
- (b)(i) $Z_{\max} = \frac{2a_a A}{4a_a + a_c A^{2/3}} = \frac{A}{2 + \frac{a_c}{2a_a} A^{2/3}}$. This is the continuous maximum; with pairing omitted, choose the nearest integer.
- (b)(ii) $Z = 80$, with $N = 120$ and $B/A \approx 8.05 \text{ MeV}$.

- **(b)(iii)** The continuous result is $Z_{\max} \approx 53.58$; the integer binding-energy maximum is $Z = 54$. Within the specified transitions, ^{128}Xe is stable, while ^{128}I and ^{128}Cs can decay.

Nucleus/Process	β^- - decay	β^+ - decay	Electron-capture	$\beta^-\beta^-$ - decay
$^{128}_{53}\text{I}$	$\checkmark$	0	0	$\checkmark$
$^{128}_{54}\text{Xe}$	0	0	0	0
$^{128}_{55}\text{Cs}$	0	$\checkmark$	$\checkmark$	0

Entries whose daughters lie outside these three nuclei are marked 0 under the stated restriction.

Solution

(a)

Alpha decay conserves the total numbers of protons and neutrons. Consequently, the free-nucleon rest energies cancel when calculating the energy release:

$$Q_\alpha = [M(A, Z) - M(A - 4, Z - 2) - M(4, 2)]c^2 = B(A - 4) + B_4 - B(A).$$

The decay is energetically allowed when $Q_\alpha > 0$.

Write the straight-line approximation to the plotted quantity as

$$b(A) \equiv \frac{B(A)}{A} \simeq b_0 - kA.$$

Representative readings from the graph are

$$b(100) \simeq 8.85 \text{ MeV}, \quad b(200) \simeq 8.05 \text{ MeV}.$$

Thus,

$$k \simeq 8.0 \times 10^{-3} \text{ MeV}, \quad b_0 \simeq 9.65 \text{ MeV},$$

where A is dimensionless. The total binding energy is therefore

$$B(A) \simeq b_0 A - kA^2.$$

Using the finite change of four nucleons,

$$\begin{aligned} Q_\alpha &= B_4 + b_0(A - 4) - k(A - 4)^2 - b_0 A + kA^2 \\ &= B_4 - 4b_0 + 8kA - 16k. \end{aligned}$$

At the threshold, $Q_\alpha = 0$, giving

$$A_\alpha = \frac{b_0 - B_4/4}{2k} + 2.$$

Numerically,

$$A_\alpha \simeq \frac{9.65 - 25.0/4}{0.016} + 2 = 214.5.$$

Since Q_α increases with A , alpha emission is energetically possible above this threshold. Allowing for the precision of the graph,

$$A_\alpha \approx 2.1 \times 10^2.$$

(b)(i)

At fixed A , substitute $N = A - Z$, so that $N - Z = A - 2Z$. Omitting pairing,

$$B(Z) = a_v A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - \frac{a_a}{A} (A - 2Z)^2.$$

Treating Z initially as continuous, the maximum satisfies

$$\frac{\partial B}{\partial Z} = -2a_c Z A^{-1/3} + \frac{4a_a}{A} (A - 2Z) = 0.$$

Multiplying by A and collecting the terms in Z ,

$$(2a_c A^{2/3} + 8a_a)Z = 4a_a A.$$

Hence,

$$Z_{\max} = \frac{2a_a A}{4a_a + a_c A^{2/3}} = \frac{A}{2 + \frac{a_c}{2a_a} A^{2/3}}.$$

This stationary point is a maximum because

$$\frac{\partial^2 B}{\partial Z^2} = -2a_c A^{-1/3} - \frac{8a_a}{A} < 0.$$

Without pairing, the physical integer maximum is the integer nearest this value.

(b)(ii)

For $A = 200$, the continuous maximum without pairing is

$$Z_0 = \frac{200}{2 + \frac{0.72}{47.0} 200^{2/3}} \simeq 79.242.$$

Because A is even, even Z also means even N , while odd Z means odd N . The pairing magnitude is

$$P = a_p A^{-3/4} = 33.5(200)^{-3/4} \simeq 0.6299 \text{ MeV}.$$

Since the formula contains $-\delta$, pairing **adds** P to the binding energy for even Z , and **subtracts** P for odd Z .

The unpaired binding energy can be written

$$B_0(Z) = B_0(Z_0) - \left(a_c A^{-1/3} + \frac{4a_a}{A} \right) (Z - Z_0)^2.$$

Therefore, within either parity class, the best integer is the one nearest Z_0 . The best odd choice is 79, and the best even choice is 80. Direct substitution, including pairing, gives:

Z	N	δ (MeV)	B (MeV)
78	122	-0.630	1609.55
79	121	+0.630	1609.17
80	120	-0.630	1610.12

In particular,

$$B(80) - B(79) \simeq 0.954 \text{ MeV} > 0.$$

Thus,

$$\boxed{Z = 80, \quad N = 120,}$$

with

$$\frac{B}{A} = \frac{1610.12}{200} \simeq 8.05 \text{ MeV}.$$

(b)(iii)

First, the result from part (i) gives

$$Z_{\max} = \frac{128}{2 + \frac{0.72}{47.0} 128^{2/3}} \simeq 53.577.$$

Thus the nearest integer is 54, which is also favoured by pairing.

For these nuclei,

$$P = 33.5(128)^{-3/4} \simeq 0.880313 \text{ MeV}.$$

Iodine and caesium have odd N and odd Z , so $\delta = +P$. Xenon has even N and even Z , so $\delta = -P$.

Decay energetics require nuclear masses as well as binding energies. Writing $E_Z = M_Z c^2$,

$$E_Z = 128m_n c^2 + Z(m_p - m_n)c^2 - B_Z = 128m_n c^2 - (1.30 \text{ MeV})Z - B_Z.$$

Substitution gives the following values. Extra digits are retained to resolve the small double-beta energy.

Nucleus	B_Z (MeV)	$E_Z - E_{54}$ (MeV)
$^{128}_{53}\text{I}$	1104.656	3.195830
$^{128}_{54}\text{Xe}$	1106.552	0
$^{128}_{55}\text{Cs}$	1103.172	2.079906

Neglecting atomic electron binding energies and neutrino rest masses, energy conservation gives

$$\begin{aligned} Q_{\beta^-}(Z) &= E_Z - E_{Z+1} - m_e c^2, \\ Q_{\beta^+}(Z) &= E_Z - E_{Z-1} - m_e c^2, \\ Q_{\text{EC}}(Z) &= E_Z + m_e c^2 - E_{Z-1}, \\ Q_{\beta^-\beta^-}(Z) &= E_Z - E_{Z+2} - 2m_e c^2. \end{aligned}$$

The electron rest energy is subtracted for emission and added for capture. A process is energetically allowed when $Q > 0$.

All possible transitions within the specified set are:

Initial nucleus	Process	Daughter	Q (MeV)	Allowed?
^{128}I	β^-	^{128}Xe	+2.686	✓
^{128}Xe	β^-	^{128}Cs	-2.590	0
^{128}Xe	β^+	^{128}I	-3.706	0
^{128}Cs	β^+	^{128}Xe	+1.570	✓
^{128}Xe	Electron capture	^{128}I	-2.686	0
^{128}Cs	Electron capture	^{128}Xe	+2.590	✓
^{128}I	$\beta^-\beta^-$	^{128}Cs	+0.096	✓

For the small double-beta energy specifically,

$$\begin{aligned} Q_{\beta^-\beta^-} &= E_{53} - E_{55} - 2m_e c^2 \\ &= 3.195830 - 2.079906 - 1.020 \\ &\simeq 0.0959 \text{ MeV} > 0. \end{aligned}$$

Thus simultaneous double-beta decay of iodine to caesium is energetically allowed, even though the second step of the sequential route through xenon is forbidden.

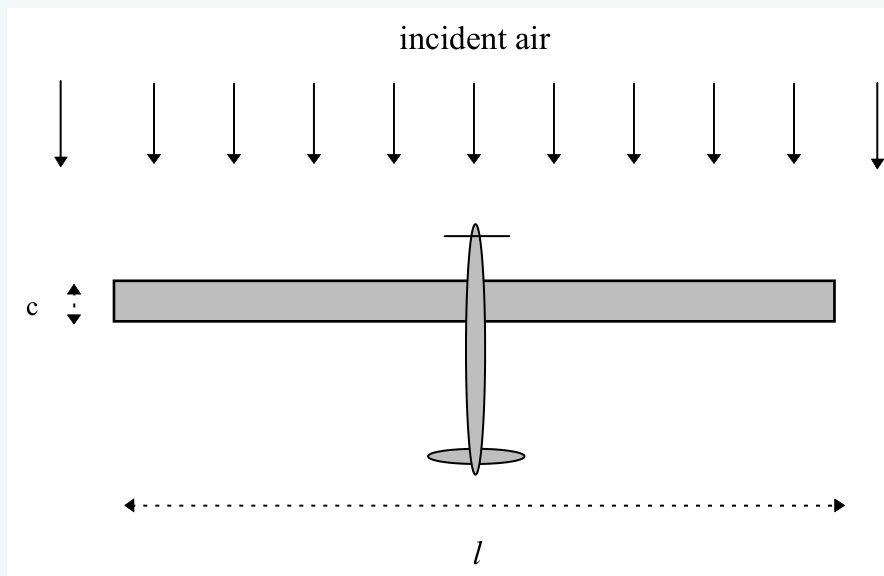
Xenon has no energetically allowed transition within the specified set. Iodine can undergo beta-minus or double-beta-minus decay; caesium can undergo positron emission or electron capture.

The solution was checked against the official solution.

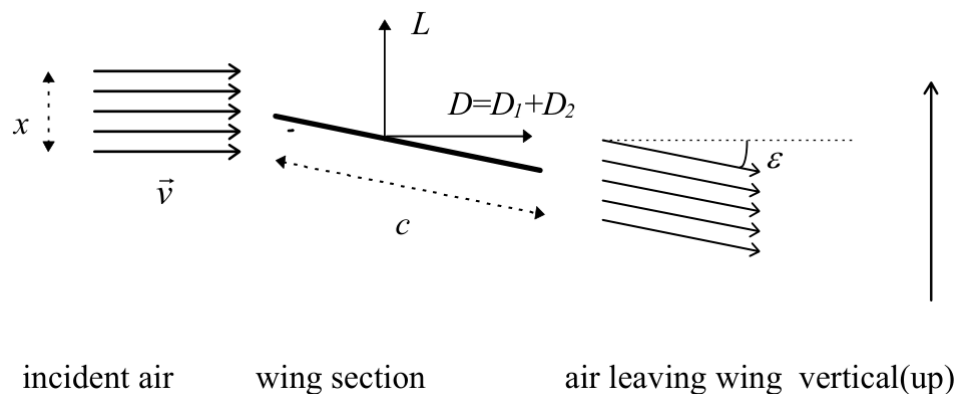
Problem 3: Solar-Powered Aircraft

We wish to design an aircraft which will stay aloft using solar power alone. The most efficient type of layout is one with a wing whose top surface is completely covered in solar cells. The cells supply electrical power with which the motor drives the propeller.

Consider a wing of rectangular plan-form with span l , chord (width) c ; the wing area is $S = cl$, and the wing aspect ratio $A = l/c$. We can get an approximate idea of the wing's performance by considering a slice of air of height x and length l being deflected downward at a small angle ε with only a very small change in speed. Control surfaces can be used to select an optimal value of ε for flight. This simple model corresponds closely to reality if $x = \pi l/4$, and we can assume this to be the case. The total mass of the aircraft is M and it flies horizontally with velocity $\vec{v}$ relative to the surrounding air. In the following calculations consider only the air flow around the wing.



Top view of aircraft (in its own frame of reference):



Side view of wing (in a frame of reference moving with the aircraft):

Ignore the modification of the airflow due to the propeller.

(a) Consider the change in momentum of the air moving past the wing, with *no* change in speed while it does so. Derive expressions for the vertical lift force L and the horizontal drag force D_1 on the wing in terms of wing dimensions, v , ϵ , and the air density ρ . Assume the direction of air flow is always parallel to the plane of the side-view diagram.

(3 marks)

(b) There is an additional horizontal drag force D_2 caused by the friction of air flowing over the surface of the wing. The air slows slightly, with a change of speed Δv ($\ll 1\%$ of v) given by:

$$\frac{\Delta v}{v} = \frac{f}{A}$$

The value of f is independent of ϵ .

Find an expression (in terms of M , f , A , S , ρ and g - the acceleration due to gravity) for the flight speed v_0 corresponding to a minimum power being needed to maintain this aircraft in flight at constant altitude and velocity. Neglect terms of order $(\epsilon^2 f)$ or higher.

(3 marks)

You may find the following small angle approximation useful:

$$1 - \cos \epsilon \approx \frac{\sin^2 \epsilon}{2}$$

(c) On the answer sheet, sketch a graph of power P versus flight speed v . Show the separate contributions to the power needed from the two sources of drag. Find an expression (in terms of M , f , A , S , ρ and g) for the minimum power, $P_{\min}$.

(2 marks)

(d) If the solar cells can supply sufficient energy so that the electric motors and propellers generate mechanical power of $I = 10$ watts per square metre of wing area, calculate the maximum wing loading Mg/S (N/m^2) for this power and flight speed v_0 (m/s). Assume $\rho = 1.25 \text{ kg/m}^3$, $f = 0.004$, $A = 10$.

(2 marks)

Official English text of the problem.

Answers

- (a)

$$L = \frac{\pi}{4} \rho l^2 v^2 \sin \varepsilon, \quad D_1 = \frac{\pi}{4} \rho l^2 v^2 (1 - \cos \varepsilon).$$

- (b)

$$v_0 = \left[\frac{8(Mg)^2}{3\pi^2 \rho^2 f A S^2} \right]^{1/4}.$$

- (c)

$$P(v) = \underbrace{\frac{2(Mg)^2}{\pi \rho A S v}}_{P_1} + \underbrace{\frac{\pi}{4} \rho f S v^3}_{P_2},$$

$$P_{\min} = \left(\frac{512f}{27\pi^2 A^3} \right)^{1/4} \frac{(Mg)^{3/2}}{\sqrt{\rho S}}.$$

The induced-power contribution P_1 decreases as $1/v$, while the friction contribution P_2 increases as v^3 . Their sum has one minimum, at v_0 , where $P_1 = 3P_2$.

- (d)

$$\left(\frac{Mg}{S} \right)_{\max} \approx 36 \text{ N/m}^2, \quad v_0 \approx 8.6 \text{ m/s}.$$

Solution

- (a)

Work in the aircraft's reference frame. In a time dt , the volume of air passing through the affected cross-section xl is $xlv dt$. Thus the mass flow rate is

$$\dot{m} = \rho xlv.$$

The air initially has horizontal speed v . After deflection, its horizontal velocity is $v \cos \varepsilon$ and its downward velocity is $v \sin \varepsilon$.

By the momentum principle and Newton's third law, the upward force on the wing equals the rate at which the air gains downward momentum:

$$L = \dot{m} v \sin \varepsilon = \rho xlv^2 \sin \varepsilon.$$

Similarly, the air loses horizontal momentum, producing a drag force on the wing:

$$D_1 = \dot{m} v (1 - \cos \varepsilon) = \rho xlv^2 (1 - \cos \varepsilon).$$

Using $x = \pi l/4$,

$$L = \frac{\pi}{4} \rho l^2 v^2 \sin \varepsilon, \quad D_1 = \frac{\pi}{4} \rho l^2 v^2 (1 - \cos \varepsilon).$$

For small ε , these become

$$L \simeq \frac{\pi}{4} \rho l^2 v^2 \varepsilon, \quad D_1 \simeq \frac{\pi}{8} \rho l^2 v^2 \varepsilon^2.$$

(b)

Since $AS = l^2$, define

$$K = \rho x l = \frac{\pi}{4} \rho AS, \quad \dot{m} = Kv.$$

The additional horizontal momentum loss caused by the reduction in speed gives

$$D_2 = \dot{m} \Delta v \cos \varepsilon.$$

Using $\Delta v/v = f/A$ and dropping the correction of order $\varepsilon^2 f$,

$$D_2 \simeq Kv^2 \frac{f}{A} = \boxed{\frac{\pi}{4} \rho f S v^2}.$$

The slowing also reduces the lift by a relative amount f/A . However, its effect on the induced drag, which is proportional to $\sin^2 \varepsilon$, is of order $\varepsilon^2 f$. Therefore, to the required accuracy, we may impose vertical equilibrium using the lift expression from (a):

$$Mg = Kv^2 \sin \varepsilon, \quad \sin \varepsilon = \frac{Mg}{Kv^2}.$$

The supplied small-angle approximation then gives

$$D_1 \simeq \frac{Kv^2}{2} \sin^2 \varepsilon = \frac{(Mg)^2}{2Kv^2} = \frac{2(Mg)^2}{\pi \rho AS v^2}.$$

In steady horizontal flight, propeller thrust balances the total drag. The mechanical power required is therefore

$$P = (D_1 + D_2)v = \frac{2(Mg)^2}{\pi \rho AS v} + \frac{\pi}{4} \rho f S v^3.$$

Write this as

$$P = \frac{\alpha}{v} + \beta v^3, \quad \alpha = \frac{2(Mg)^2}{\pi \rho AS}, \quad \beta = \frac{\pi}{4} \rho f S.$$

Differentiating with respect to flight speed,

$$\frac{dP}{dv} = -\frac{\alpha}{v^2} + 3\beta v^2.$$

Hence the optimum speed satisfies

$$v_0^4 = \frac{\alpha}{3\beta} = \frac{8(Mg)^2}{3\pi^2 \rho^2 f AS^2},$$

so

$$v_0 = \boxed{\left[\frac{8(Mg)^2}{3\pi^2 \rho^2 f AS^2} \right]^{1/4}}.$$

This is the unique minimum for $v > 0$, since

$$\frac{d^2P}{dv^2} = \frac{2\alpha}{v^3} + 6\beta v > 0.$$

(c)

At the minimum, the condition $\alpha = 3\beta v_0^4$ implies

$$P_1(v_0) = \frac{\alpha}{v_0} = 3\beta v_0^3 = 3P_2(v_0).$$

Thus

$$P_{\min} = 4\beta v_0^3 = \pi \rho f S v_0^3.$$

Substituting the expression for v_0 ,

$$P_{\min} = \left(\frac{512f}{27\pi^2 A^3} \right)^{1/4} \frac{(Mg)^{3/2}}{\sqrt{\rho S}}.$$

The requested graph has the following features:

- $P_1 = \alpha/v$ is a decreasing hyperbola. Formally, it diverges as $v \rightarrow 0^+$ and approaches zero as $v \rightarrow \infty$.
- $P_2 = \beta v^3$ starts at the origin with a horizontal tangent and increases monotonically.
- Their sum lies above both curves, decreases to its unique minimum $(v_0, P_{\min})$, and then increases. It diverges at both ends of the positive-speed range and has no axis intercept.
- The two contributions cross at $v = 3^{1/4}v_0 \approx 1.316v_0$, to the right of the total-power minimum.

A precise dimensionless description is obtained by setting $u = v/v_0$:

$$\frac{P_1}{P_{\min}} = \frac{3}{4u}, \quad \frac{P_2}{P_{\min}} = \frac{u^3}{4}, \quad \frac{P}{P_{\min}} = \frac{3}{4u} + \frac{u^3}{4}.$$

Useful points for the sketch are:

v/v_0	$P_1/P_{\min}$	$P_2/P_{\min}$	$P/P_{\min}$
0.500	1.500	0.031	1.531
1.000	0.750	0.250	1.000
1.316	0.570	0.570	1.140
2.000	0.375	2.000	2.375

The limit $v \rightarrow 0^+$ is a formal extrapolation of these equations: sufficiently low speeds would require a deflection angle outside the small-angle approximation.

(d)

Let the wing loading be

$$w = \frac{Mg}{S}.$$

The available mechanical power is IS . The maximum loading occurs when this just equals the minimum required power:

$$IS = P_{\min} = \pi \rho f S v_0^3.$$

Consequently,

$$v_0 = \left(\frac{I}{\pi \rho f} \right)^{1/3}.$$

Numerically,

$$v_0 = \left(\frac{10}{\pi(1.25)(0.004)} \right)^{1/3} = 8.6025 \dots \text{ m/s},$$

giving

$$\boxed{v_0 \approx 8.6 \text{ m/s}}.$$

Rearranging the optimum-speed relation from (b),

$$v_0^4 = \frac{8w^2}{3\pi^2\rho^2 f A},$$

we obtain

$$w_{\max} = \pi\rho v_0^2 \sqrt{\frac{3fA}{8}} = \pi\rho \sqrt{\frac{3fA}{8}} \left(\frac{I}{\pi\rho f} \right)^{2/3}.$$

Substitution gives

$$w_{\max} = \pi(1.25)(8.6025 \dots)^2 \sqrt{\frac{3(0.004)(10)}{8}} = 35.5925 \dots \text{ N/m}^2.$$

Therefore,

$$\boxed{\left(\frac{Mg}{S} \right)_{\max} \approx 36 \text{ N/m}^2}.$$

This is the maximum because $P_{\min}/S$ increases as $w^{3/2}$; a larger loading would require more than the available 10 W/m^2 at every speed.

As a consistency check, at this operating point,

$$\sin \varepsilon_0 = \sqrt{\frac{6f}{A}}, \quad \varepsilon_0 \simeq 0.049 \text{ rad} \simeq 2.8^\circ,$$

and

$$\frac{\Delta v}{v} = \frac{f}{A} = 0.0004 = 0.04\%.$$

Both approximations are consistent with the given data.

The solution was checked against the official solution.