

IPhO 1996

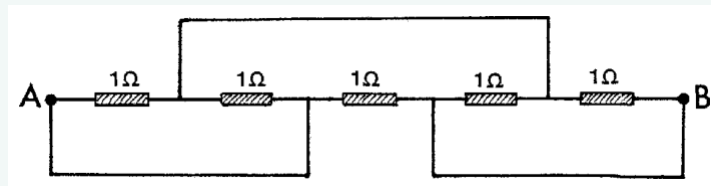
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Problem 1: Five unrelated short problems (resistor network, skier on a hill, heat capacity of a metal, radiation heat shield, magnetic field between two conductors)

(The five parts of this problem are unrelated)

a) Five 1Ω resistances are connected as shown in the figure. The resistance in the conducting wires (fully drawn lines) is negligible.



Determine the resulting resistance R between A and B. (1 point)

b)



A skier starts from rest at point A and slides down the hill, without turning or braking. The friction coefficient is μ . When he stops at point B, his horizontal displacement is s . What is the height difference h between points A and B?

(The velocity of the skier is small so that the additional pressure on the snow due to the curvature can be neglected. Neglect also the friction of air and the dependence of μ on the velocity of the skier.) (1.5 points)

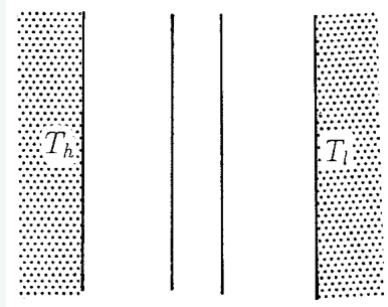
c) A thermally insulated piece of metal is heated under atmospheric pressure by an electric current so that it receives electric energy at a constant power P . This leads to an increase of the absolute temperature T of the metal with time t as follows:

$$T(t) = T_0[1 + a(t - t_0)]^{1/4}.$$

Here a , t_0 and T_0 are constants. Determine the heat capacity $C_p(T)$ of the metal (temperature dependent in the temperature range of the experiment). (2 points)

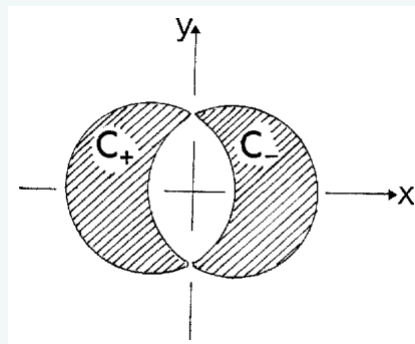
d) A black plane surface at a constant high temperature T_h is parallel to another black plane surface at a constant lower temperature T_l . Between the plates is vacuum.

In order to reduce the heat flow due to radiation, a heat shield consisting of two thin black plates, thermally isolated from each other, is placed between the warm and the cold surfaces and parallel to these. After some time stationary conditions are obtained.



By what factor ξ is the stationary heat flow reduced due to the presence of the heat shield? Neglect end effects due to the finite size of the surfaces. (1.5 points)

e) Two straight and very long nonmagnetic conductors C_+ and C_- , insulated from each other, carry a current I in the positive and the negative z direction, respectively. The cross sections of the conductors (hatched in the figure) are limited by circles of diameter D in the x - y plane, with a distance $D/2$ between the centres. Thereby the resulting cross sections each have an area $\left(\frac{1}{12}\pi + \frac{1}{8}\sqrt{3}\right)D^2$. The current in each conductor is uniformly distributed over the cross section.



Determine the magnetic field $B(x, y)$ in the space between the conductors. (4 points)

Official English text of the problem.

Answers

- **a)** $R = \frac{1}{2} \Omega = 0.5 \Omega$.
- **b)** $h = \mu s$, with A higher than B.
- **c)** $C_p(T) = \frac{4PT^3}{aT_0^4}$, in J K^{-1} when SI units are used.
- **d)** $\dot{Q}_{\text{shield}} = \frac{\dot{Q}_0}{3}$: a threefold reduction, $\xi = \frac{\dot{Q}_0}{\dot{Q}_{\text{shield}}} = 3$.

- e) Throughout the lens-shaped gap,

$$\mathbf{B}(x, y) = \frac{6\mu_0 I}{(2\pi + 3\sqrt{3})D} \hat{\mathbf{y}}.$$

The field is uniform and points in the positive y direction.

Solution

a)

Let each resistance be $r = 1\ \Omega$, and label the four interior junctions N_1, N_2, N_3, N_4 from left to right. The ideal connecting wires identify the following nodes:

$$A = N_2, \quad N_1 = N_4 = X, \quad N_3 = B.$$

Consequently, the first and second resistors both connect A to X and are in parallel. The fourth and fifth both connect X to B and are also in parallel:

$$R_{AX} = R_{XB} = \frac{r}{2}.$$

These two equivalent resistances form a series branch from A to B, of resistance $r/2 + r/2 = r$. The third resistor connects A directly to B, providing another branch of resistance r . Thus

$$\frac{1}{R} = \frac{1}{r} + \frac{1}{r}, \quad \boxed{R = \frac{r}{2} = 0.5\ \Omega.}$$

b)

Let θ be the local angle of the skier's path to the horizontal and $d\ell$ an element of path length. Neglecting the additional normal force due to curvature, as instructed, gives

$$N = mg \cos \theta.$$

The friction force has magnitude μN and opposes the motion. The energy dissipated over the path is therefore

$$E_{\text{fr}} = \int_A^B \mu mg \cos \theta \, d\ell.$$

For the path shown, the horizontal coordinate increases throughout the motion, and

$$dx = \cos \theta \, d\ell.$$

It follows that

$$E_{\text{fr}} = \mu mg \int_A^B dx = \mu mgs.$$

The skier starts and ends at rest, so the loss of gravitational potential energy equals the energy dissipated by friction. Writing $h = y_A - y_B$,

$$mgh = \mu mgs, \quad \boxed{h = \mu s.}$$

Within the stated approximation, the result depends only on the horizontal displacement.

c)

The metal is thermally insulated, but electrical work supplies energy at rate P . At constant atmospheric pressure p , the first law gives

$$dU = P dt - p dV.$$

Since the enthalpy is $H = U + pV$, and $dp = 0$,

$$dH = P dt.$$

Using the total heat capacity of the piece at constant pressure,

$$C_p(T) = \left(\frac{\partial H}{\partial T} \right)_p,$$

we obtain

$$P = C_p(T) \frac{dT}{dt}.$$

Raise the given temperature law to the fourth power:

$$T^4 = T_0^4 [1 + a(t - t_0)].$$

Differentiating,

$$4T^3 \frac{dT}{dt} = aT_0^4.$$

Hence

$$C_p(T) = \frac{4PT^3}{aT_0^4} = \frac{4P}{aT_0} \left(\frac{T}{T_0} \right)^3.$$

Thus the heat capacity is proportional to T^3 over the experimental temperature range. Since a has units s^{-1} , the expression has units $J K^{-1}$.

d)

For two black parallel surfaces with negligible edge effects, the Stefan–Boltzmann law gives the net radiative heat flux

$$q_0 = \sigma (T_h^4 - T_l^4)$$

before the shield is inserted.

Let T_1 and T_2 be the temperatures of the thin shield plates, ordered from the hot side to the cold side. Each thin plate has a common temperature on its two faces. At steady state, neither plate accumulates energy, so the net heat flux q is the same across all three vacuum gaps:

$$q = \sigma(T_h^4 - T_1^4) = \sigma(T_1^4 - T_2^4) = \sigma(T_2^4 - T_l^4).$$

Adding the three temperature differences gives

$$3q = \sigma(T_h^4 - T_l^4) = q_0.$$

The same ratio applies to total heat flow:

$$\frac{\dot{Q}_{\text{shield}}}{\dot{Q}_0} = \frac{1}{3}.$$

Therefore the heat flow is reduced by a factor

$$\boxed{\xi = 3.}$$

e)

Let the area of either actual conductor's cross section be

$$A_c = \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8} \right) D^2.$$

The magnitude of the uniform current density is

$$J = \frac{I}{A_c}.$$

The circles have radius $D/2$. With the origin shown in the figure, their centres are

$$\mathbf{r}_+ = -\frac{D}{4} \hat{\mathbf{x}}, \quad \mathbf{r}_- = \frac{D}{4} \hat{\mathbf{x}}.$$

Use superposition to represent the actual current distribution by two complete circular cylinders:

- a left cylinder carrying uniform current density $+J\hat{\mathbf{z}}$;
- a right cylinder carrying uniform current density $-J\hat{\mathbf{z}}$.

In their overlap, the two current densities cancel. In each crescent, the remaining current density is exactly that of the actual conductor. Thus this construction reproduces the physical current distribution.

For a very long circular cylinder carrying uniform current density $J\hat{\mathbf{z}}$, cylindrical symmetry and Ampère's law give, at an interior radius ρ ,

$$B(\rho) 2\pi\rho = \mu_0 J \pi \rho^2, \quad B(\rho) = \frac{\mu_0 J \rho}{2}.$$

In vector form, for a cylinder centred at $\mathbf{r}_c$,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 J}{2} \hat{\mathbf{z}} \times (\mathbf{r} - \mathbf{r}_c).$$

Every point in the lens-shaped gap is inside both fictitious cylinders. Their fields there are

$$\begin{aligned} \mathbf{B}_+(x, y) &= \frac{\mu_0 J}{2} \left[-y \hat{\mathbf{x}} + \left(x + \frac{D}{4} \right) \hat{\mathbf{y}} \right], \\ \mathbf{B}_-(x, y) &= \frac{\mu_0 J}{2} \left[y \hat{\mathbf{x}} - \left(x - \frac{D}{4} \right) \hat{\mathbf{y}} \right]. \end{aligned}$$

Adding them cancels every coordinate-dependent term:

$$\mathbf{B}(x, y) = \frac{\mu_0 J D}{4} \hat{\mathbf{y}}.$$

Substituting $J = I/A_c$,

$$\boxed{\mathbf{B}(x, y) = \frac{\mu_0 I D}{4 A_c} \hat{\mathbf{y}} = \frac{6 \mu_0 I}{(2\pi + 3\sqrt{3}) D} \hat{\mathbf{y}}.}$$

This uniform field applies throughout the gap defined by the simultaneous inequalities

$$\left(x + \frac{D}{4}\right)^2 + y^2 < \frac{D^2}{4}, \quad \left(x - \frac{D}{4}\right)^2 + y^2 < \frac{D^2}{4}.$$

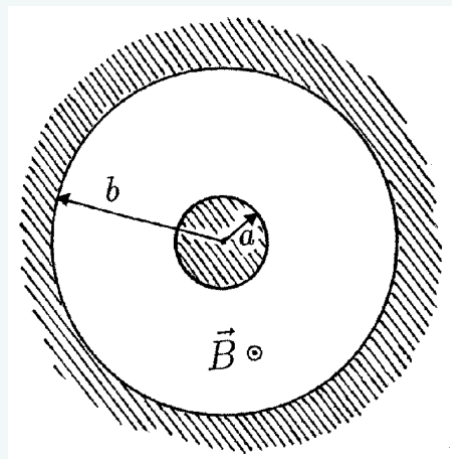
Its direction is upward in the figure, consistent with the right-hand rule for both currents.

The solution was checked against the official solution.

Problem 2: Electrons between coaxial cylinders in a magnetic field (magnetic cut-off)

The space between a pair of coaxial cylindrical conductors is evacuated. The radius of the inner cylinder is a , and the inner radius of the outer cylinder is b , as shown in the figure below. The outer cylinder, called the anode, may be given a positive potential V relative to the inner cylinder. A static homogeneous magnetic field $\vec{B}$ parallel to the cylinder axis, directed out of the plane of the figure, is also present. Induced charges in the conductors are neglected.

We study the dynamics of electrons with rest mass m and charge $-e$. The electrons are released at the surface of the inner cylinder.



a) First the potential V is turned on, but $\vec{B} = 0$. An electron is set free with negligible velocity at the surface of the inner cylinder. Determine its speed v when it hits the anode. Give the answer both when a non-relativistic treatment is sufficient, and when it is not. (1 point)

For the remaining parts of this problem a non-relativistic treatment suffices.

b) Now $V = 0$, but the homogeneous magnetic field $\vec{B}$ is present. An electron starts out with an initial velocity $\vec{v}_0$ in the radial direction. For magnetic fields larger than a critical value B_c , the electron will not reach the anode. Make a sketch of the trajectory of the electron when B is slightly more than B_c . Determine B_c . (2 points)

From now on both the potential V and the homogeneous magnetic field $\vec{B}$ are present.

c) The magnetic field will give the electron a non-zero angular momentum L with respect to the cylinder axis. Write down an equation for the rate of change dL/dt of the angular momentum. Show that this equation implies that

$$L = keBr^2$$

is constant during the motion, where k is a definite pure number. Here r is the distance from the cylinder axis. Determine the value of k . (3 points)

d) Consider an electron, released from the inner cylinder with negligible velocity, that does not reach the anode, but has a maximal distance from the cylinder axis equal to r_m . Determine the speed v at the point where the radial distance is maximal, in terms of r_m . (1 point)

e) We are interested in using the magnetic field to regulate the electron current to the anode. For B larger than a critical magnetic field B_c , an electron, released with negligible velocity, will not reach the anode. Determine B_c . (1 point)

f) If the electrons are set free by heating the inner cylinder an electron will in general have an initial nonzero velocity at the surface of the inner cylinder. The component of the initial velocity parallel to $\vec{B}$ is v_B , the components orthogonal to $\vec{B}$ are v_r (in the radial direction) and v_φ (in the azimuthal direction, i.e. orthogonal to the radial direction).

Determine for this situation the critical magnetic field B_c for reaching the anode. (2 points)

Official English text of the problem.

Answers

Here B is the magnetic-field magnitude, c is the speed of light, and positive azimuthal velocity means counterclockwise motion in the figure. Thus v_φ in part (f) is signed.

- **a)** For $eV \ll mc^2$,

$$v = \sqrt{\frac{2eV}{m}}.$$

Relativistically,

$$v = c \sqrt{1 - \left(1 + \frac{eV}{mc^2}\right)^{-2}}.$$

- **b)**

$$B_c = \frac{2mbv_0}{e(b^2 - a^2)}.$$

The trajectory is a counterclockwise circular arc of radius $R = mv_0/(eB)$. For B just above B_c , its maximum distance from the cylinder axis is just below b , and it returns to the inner cylinder.

- **c)**

$$\frac{dL}{dt} = eBr\dot{r}, \quad L - \frac{eB}{2}r^2 = \text{constant}, \quad k = \frac{1}{2}.$$

• d)

$$v = \frac{eB(r_m^2 - a^2)}{2mr_m}.$$

Equivalently,

$$v = \sqrt{\frac{2eV}{m} \frac{\ln(r_m/a)}{\ln(b/a)}}.$$

• e)

$$B_c = \frac{2mb}{e(b^2 - a^2)} \sqrt{\frac{2eV}{m}} = \frac{b}{b^2 - a^2} \sqrt{\frac{8mV}{e}}.$$

• f) For an electron emitted into the gap,

$$B_c = \frac{2m}{e(b^2 - a^2)} \left[b \sqrt{v_r^2 + v_\phi^2} + \frac{2eV}{m} - av_\phi \right].$$

This is independent of v_B .

Solution

Take the positive z -axis along $\vec{B}$, out of the figure. The positive azimuthal direction is then counterclockwise. Let L denote the signed z -component of the electron's mechanical angular momentum. After part (a), we use non-relativistic mechanics as stipulated.

a)

The electric potential increases by V between the inner cylinder and the anode. Since the electron's charge is $-e$, its potential energy decreases by eV . Conservation of energy therefore gives

$$K_{\text{final}} = eV.$$

When this energy is small compared with the rest energy, $eV \ll mc^2$,

$$\frac{1}{2}mv^2 = eV,$$

so

$$v = \sqrt{\frac{2eV}{m}}.$$

For a relativistic treatment,

$$(\gamma - 1)mc^2 = eV, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Thus

$$\gamma = 1 + \frac{eV}{mc^2},$$

and

$$v = c \sqrt{1 - \left(1 + \frac{eV}{mc^2}\right)^{-2}}.$$

Expanding this expression for $eV/(mc^2) \ll 1$ recovers the non-relativistic result.

b)

The magnetic force is perpendicular to the velocity, so it does no work. The electron's speed remains v_0 , and its motion perpendicular to the uniform magnetic field is circular. The orbit radius R follows from

$$\frac{mv_0^2}{R} = ev_0 B,$$

giving

$$R = \frac{mv_0}{eB}.$$

Choose the release point to be $P = (a, 0)$, with initial velocity along the positive x -axis. Because

$$\vec{F} = -e \vec{v} \times \vec{B},$$

the initial force points along the positive y -axis. The circle therefore has centre

$$C = (a, R)$$

and equation

$$(x - a)^2 + (y - R)^2 = R^2.$$

The distance from the cylinder axis O to the orbit centre is

$$OC = \sqrt{a^2 + R^2}.$$

Consequently, the maximum radial distance attained is

$$r_{\max} = \sqrt{a^2 + R^2} + R.$$

At the critical field, this maximum equals b :

$$\sqrt{a^2 + R_c^2} + R_c = b.$$

Squaring after rearrangement gives

$$a^2 + R_c^2 = (b - R_c)^2, \quad R_c = \frac{b^2 - a^2}{2b}.$$

Hence

$$B_c = \frac{mv_0}{eR_c} = \frac{2mbv_0}{e(b^2 - a^2)}.$$

For the requested sketch, draw the two concentric cylinder boundaries and start the electron at their right-hand radial line, on the inner boundary. Its initial tangent points directly outward, and the path bends upward, following the circle centred at $C = (a, R)$ counterclockwise. Its farthest point from O lies on the ray OC . When B is slightly greater than B_c , this point lies just inside the anode. The electron then moves inward and strikes the inner cylinder again. The return point has polar angle

$$\varphi_{\text{return}} = 2 \arctan(R/a).$$

c)

The electric force is radial and therefore exerts no torque about the cylinder axis.

Writing the velocity as

$$\vec{v} = \dot{r} \hat{r} + r\dot{\varphi} \hat{\varphi} + v_z \hat{z},$$

the magnetic force is

$$-e \vec{v} \times \vec{B} = -eB r \dot{\varphi} \hat{r} + eB \dot{r} \hat{\varphi}.$$

Its azimuthal component is $F_\varphi = eB \dot{r}$. Therefore,

$$\frac{dL}{dt} = rF_\varphi = eB r \dot{r}.$$

Since B is constant,

$$eB r \dot{r} = \frac{eB}{2} \frac{d(r^2)}{dt}.$$

It follows that

$$\frac{d}{dt} \left(L - \frac{eB}{2} r^2 \right) = 0,$$

so

$$\boxed{L - \frac{eB}{2} r^2 = \text{constant}, \quad k = \frac{1}{2}.$$

In particular, between release at $r = a$ and any later position,

$$L(r) = L(a) + \frac{eB}{2} (r^2 - a^2).$$

d)

The electron starts with negligible velocity, so $L(a) = 0$. Thus

$$L(r) = \frac{eB}{2} (r^2 - a^2).$$

Neither field produces an axial force, so the initially vanishing axial velocity remains zero. At the maximum radial distance r_m , the radial velocity also vanishes. The velocity there is therefore entirely azimuthal, giving

$$L(r_m) = m r_m v.$$

Consequently,

$$m r_m v = \frac{eB}{2} (r_m^2 - a^2),$$

and

$$\boxed{v = \frac{eB(r_m^2 - a^2)}{2m r_m}.$$

An equivalent expression follows from energy conservation. In the charge-free cylindrical gap, Gauss's law gives $rE_r = \text{constant}$. With the inner cylinder at zero potential and the anode at V , the potential is

$$\Phi(r) = V \frac{\ln(r/a)}{\ln(b/a)}.$$

Since the magnetic field does no work,

$$\frac{1}{2}mv^2 = e\Phi(r_m),$$

which gives

$$v = \sqrt{\frac{2eV}{m} \frac{\ln(r_m/a)}{\ln(b/a)}}.$$

These two expressions must agree at the actual turning radius r_m .

e)

At the critical field, the electron's maximum radial distance is exactly b . It reaches the anode tangentially, with zero radial velocity.

Energy conservation fixes its speed there:

$$v_b = \sqrt{\frac{2eV}{m}}.$$

Using part (d) with $r_m = b$,

$$v_b = \frac{eB_c(b^2 - a^2)}{2mb}.$$

Therefore,

$$B_c = \frac{2mb}{e(b^2 - a^2)} \sqrt{\frac{2eV}{m}} = \frac{b}{b^2 - a^2} \sqrt{\frac{8mV}{e}}.$$

For $B > B_c$, the azimuthal kinetic energy required at $r = b$ by angular-momentum conservation exceeds the total kinetic energy available there. The electron cannot reach the anode.

f)

Take the initial radial component to point into the gap, so $v_r \geq 0$. Denote the later radial and azimuthal velocity components by u_r and u_φ .

There is no axial force, so

$$u_z = v_B$$

throughout the motion. The initial angular momentum is

$$L(a) = mav_\varphi.$$

The conservation law from part (c) gives

$$mru_\varphi = mav_\varphi + \frac{eB}{2}(r^2 - a^2).$$

At the anode,

$$u_\varphi(b) = \frac{a}{b}v_\varphi + \frac{eB(b^2 - a^2)}{2mb}.$$

Energy conservation between the two cylinders gives

$$\frac{1}{2}m [u_r(b)^2 + u_\varphi(b)^2 + v_B^2] = \frac{1}{2}m [v_r^2 + v_\varphi^2 + v_B^2] + eV.$$

The axial kinetic energies cancel. Defining

$$W = \sqrt{v_r^2 + v_\varphi^2 + \frac{2eV}{m}},$$

we obtain

$$u_r(b)^2 + u_\varphi(b)^2 = W^2.$$

Thus the condition for reaching the anode is

$$\left| av_\varphi + \frac{eB}{2m}(b^2 - a^2) \right| \leq bW.$$

At $B = 0$, $|av_\varphi| < bW$. As the positive field is increased, the upper boundary is reached when

$$av_\varphi + \frac{eB_c}{2m}(b^2 - a^2) = bW.$$

Solving,

$$B_c = \frac{2m}{e(b^2 - a^2)} \left[b\sqrt{v_r^2 + v_\varphi^2 + \frac{2eV}{m}} - av_\varphi \right].$$

The absence of v_B reflects the fact that axial motion neither changes the distance from the cylinder axis nor exchanges energy with the transverse motion.

To verify that the anode condition does not overlook an earlier radial turning point, set

$$s = \ln(r/a), \quad \alpha = \frac{eBa}{2m}, \quad \beta = v_\varphi - \alpha.$$

The angular-momentum and energy equations give

$$u_r^2(s) = v_r^2 + v_\varphi^2 + \frac{2eV}{m \ln(b/a)} s - (\alpha e^s + \beta e^{-s})^2.$$

Its second derivative is

$$\frac{d^2 u_r^2}{ds^2} = -4\alpha^2 e^{2s} - 4\beta^2 e^{-2s} \leq 0.$$

Thus u_r^2 is concave as a function of s . If it is nonnegative at both cylinders, it is nonnegative throughout the gap. The condition used above therefore determines the actual cut-off.

Setting $v_r = v_\varphi = 0$ recovers part (e). Setting $V = 0$, $v_\varphi = 0$, and $v_r = v_0$ recovers part (b).

The solution was checked against the official solution.

Problem 3: Mid-ocean tides in the Earth-Moon system

In this problem we consider some gross features of the magnitude of mid-ocean tides on earth. We simplify the problem by making the following assumptions:

- (i) The earth and the moon are considered to be an isolated system,
- (ii) the distance between the moon and the earth is assumed to be constant,
- (iii) the earth is assumed to be completely covered by an ocean,
- (iv) the dynamic effects of the rotation of the earth around its axis are neglected, and
- (v) the gravitational attraction of the earth can be determined as if all mass were concentrated at the centre of the earth.

The following data are given:

Mass of the earth: $M = 5.98 \cdot 10^{24} \text{ kg}$

Mass of the moon: $M_m = 7.3 \cdot 10^{22} \text{ kg}$

Radius of the earth: $R = 6.37 \cdot 10^6 \text{ m}$

Distance between centre of the earth and centre of the moon:

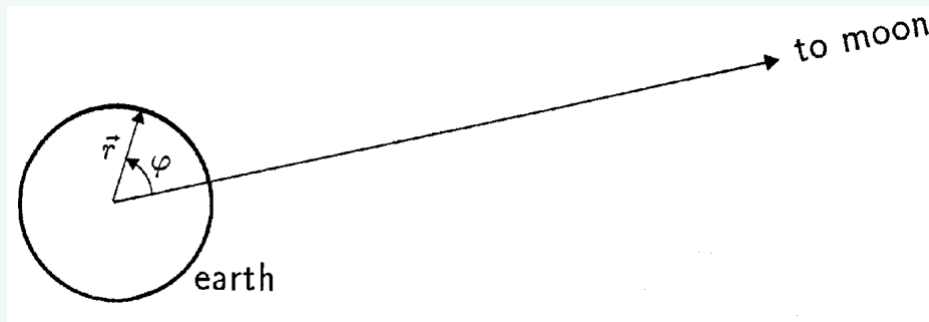
$L = 3.84 \cdot 10^8 \text{ m}$

The gravitational constant: $G = 6.67 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.

a) The moon and the earth rotate with angular velocity ω about their common centre of mass, C . How far is C from the centre of the earth? (Denote this distance by l .)

Determine the numerical value of ω . (2 points)

We now use a frame of reference that is co-rotating with the moon and the center of the earth around C . In this frame of reference the shape of the liquid surface of the earth is static.



In the plane P through C and orthogonal to the axis of rotation the position of a point mass on the liquid surface of the earth can be described by polar coordinates r, φ as shown in the figure. Here r is the distance from the centre of the earth.

We will study the shape

$$r(\varphi) = R + h(\varphi)$$

of the liquid surface of the earth in the plane P .

b) Consider a mass point (mass m) on the liquid surface of the earth (in the plane P). In our frame of reference it is acted upon by a centrifugal force and by gravitational forces from the moon and the earth. Write down an expression for the potential energy corresponding to these three forces.

Note: Any force $F(r)$, radially directed with respect to some origin, is the negative derivative of a spherically symmetric potential energy $V(r)$:

$$F(r) = -V'(r). \text{ (3 points)}$$

c) Find, in terms of the given quantities M , M_m , etc, the approximate form $h(\varphi)$ of the tidal bulge. What is the difference in meters between high tide and low tide in this model?

You may use the approximate expression

$$\frac{1}{\sqrt{1 + a^2 - 2a \cos \theta}} \approx 1 + a \cos \theta + \frac{1}{2}a^2(3 \cos^2 \theta - 1),$$

valid for a much less than unity.

In this analysis make simplifying approximations whenever they are reasonable. (5 points)

Official English text of the problem.

Answers

- a) The centre of mass lies toward the Moon, at

$$l = \frac{M_m L}{M + M_m} \approx 4.6 \times 10^6 \text{ m}$$

from the Earth's centre. The angular velocity is

$$\omega = \sqrt{\frac{G(M + M_m)}{L^3}} \approx 2.67 \times 10^{-6} \text{ rad s}^{-1}.$$

- b) Up to an additive constant,

$$V(r, \varphi) = -\frac{GMm}{r} - \frac{GM_m m}{\sqrt{L^2 + r^2 - 2Lr \cos \varphi}} - \frac{1}{2}m\omega^2 (r^2 + l^2 - 2lr \cos \varphi).$$

- c) The leading tidal shape is

$$h(\varphi) = h_{\min} + \frac{3M_m R^4}{2ML^3} \cos^2 \varphi,$$

where the constant $h_{\min}$ fixes the reference sea level. Equivalently, relative to the mean height $\bar{h}_P$ around the section,

$$h(\varphi) - \bar{h}_P = \frac{3M_m R^4}{4ML^3} \cos 2\varphi \approx (0.27 \text{ m}) \cos 2\varphi.$$

High tides occur at $\varphi = 0, \pi$, and low tides at $\varphi = \pi/2, 3\pi/2$. Their height difference is

$$\Delta h = \frac{3M_m R^4}{2ML^3} \approx 0.53 \text{ m}.$$

Solution

a)

The centre of mass lies on the line joining the Earth and Moon. Taking distances from it gives

$$Ml = M_m(L - l),$$

and therefore

$$l = \frac{M_m}{M + M_m} L = \frac{7.3 \times 10^{22}}{5.98 \times 10^{24} + 7.3 \times 10^{22}} (3.84 \times 10^8) = 4.631 \dots \times 10^6 \text{ m.}$$

Thus C lies inside the Earth, since $l < R$.

The Earth moves in a circle of radius l about C . Its gravitational attraction to the Moon supplies the centripetal force:

$$M\omega^2 l = \frac{GM M_m}{L^2}.$$

Substituting the expression for l ,

$$\omega^2 = \frac{G(M + M_m)}{L^3}.$$

Numerically,

$$\omega = \sqrt{\frac{(6.67 \times 10^{-11})(5.98 \times 10^{24} + 7.3 \times 10^{22})}{(3.84 \times 10^8)^3}} = 2.67 \times 10^{-6} \text{ rad s}^{-1}.$$

We will also use the exact relation

$$\omega^2 l = \frac{GM_m}{L^2}.$$

b)

Place the origin at the Earth's centre and take $\varphi = 0$ toward the Moon. The Moon and C then lie at distances L and l along that direction.

By the cosine rule, the distance from the mass point to the Moon is

$$d = \sqrt{L^2 + r^2 - 2Lr \cos \varphi}.$$

Because the point lies in the plane perpendicular to the rotation axis through C , its distance from that axis is

$$s = \sqrt{r^2 + l^2 - 2lr \cos \varphi}.$$

The gravitational potential energies are

$$V_E = -\frac{GMm}{r}, \quad V_m = -\frac{GM_m m}{d}.$$

The centrifugal force points away from the rotation axis and has magnitude $m\omega^2 s$. Hence

$$-\frac{dV_{\text{cf}}}{ds} = m\omega^2 s \implies V_{\text{cf}} = -\frac{1}{2}m\omega^2 s^2.$$

Adding the three contributions gives

$$V(r, \varphi) = -\frac{GMm}{r} - \frac{GM_m m}{\sqrt{L^2 + r^2 - 2Lr \cos \varphi}} - \frac{1}{2}m\omega^2 (r^2 + l^2 - 2lr \cos \varphi) + \text{constant}.$$

c)

For a fluid at rest, hydrostatic equilibrium requires

$$\nabla p = -\rho \nabla \Phi, \quad \Phi = \frac{V}{m}.$$

Pressure is constant along the free surface, so that surface must be an equipotential of Φ .

Since

$$\frac{R}{L} = 0.0166 \ll 1,$$

we expand the lunar potential through second order in r/L :

$$-\frac{GM_m}{d} \approx -\frac{GM_m}{L} - \frac{GM_m r}{L^2} \cos \varphi - \frac{GM_m r^2}{2L^3} (3 \cos^2 \varphi - 1).$$

The centrifugal potential per unit mass is

$$-\frac{1}{2} \omega^2 (r^2 + l^2 - 2lr \cos \varphi) = -\frac{1}{2} \omega^2 r^2 - \frac{1}{2} \omega^2 l^2 + \omega^2 lr \cos \varphi.$$

Using the relation found in part **a**),

$$-\frac{GM_m r}{L^2} \cos \varphi + \omega^2 lr \cos \varphi = 0.$$

Thus the uniform part of the Moon's gravitational field is cancelled by the corresponding centrifugal term. After absorbing terms independent of position into a constant,

$$\Phi(r, \varphi) \approx -\frac{GM}{r} - \frac{1}{2} \omega^2 r^2 - \frac{GM_m r^2}{2L^3} (3 \cos^2 \varphi - 1) + \text{constant}.$$

Now put $r = R + h(\varphi)$, with $|h| \ll R$. The Earth's potential becomes

$$-\frac{GM}{R+h} \approx -\frac{GM}{R} + \frac{GM}{R^2} h = -\frac{GM}{R} + gh,$$

where

$$g = \frac{GM}{R^2} = 9.83 \text{ m s}^{-2}.$$

In the smaller centrifugal and lunar terms, we may set $r = R$. For example, the centrifugal correction to the coefficient of h , relative to g , is only

$$\frac{\omega^2 R}{g} = 4.62 \times 10^{-6}.$$

The centrifugal potential evaluated at R is independent of φ within P , so it contributes only to the constant surface level. The equipotential condition therefore reduces to

$$gh(\varphi) - \frac{GM_m R^2}{2L^3} (3 \cos^2 \varphi - 1) = \text{constant}.$$

Consequently,

$$h(\varphi) = H + \frac{M_m R^4}{2ML^3} (3 \cos^2 \varphi - 1),$$

where H is independent of φ . This constant is fixed by the reference sea level or a volume normalization; the angular tidal variation does not depend on it.

Writing the result relative to its minimum gives

$$h(\varphi) = h_{\min} + \frac{3M_m R^4}{2ML^3} \cos^2 \varphi.$$

Alternatively, using $\cos^2 \varphi = (1 + \cos 2\varphi)/2$,

$$h(\varphi) - \bar{h}_P = \frac{3M_m R^4}{4ML^3} \cos 2\varphi.$$

The surface therefore has two equal maxima along the Earth–Moon line: one facing the Moon and one on the opposite side. Its two minima lie perpendicular to that line in P . Equality of the two bulges holds to the retained order in R/L .

The difference between maximum and minimum is

$$\begin{aligned} \Delta h &= \frac{3M_m R^4}{2ML^3} \\ &= \frac{3}{2} \frac{7.3 \times 10^{22}}{5.98 \times 10^{24}} \frac{(6.37 \times 10^6)^4}{(3.84 \times 10^8)^3} \text{ m} \\ &= 0.532447 \dots \text{ m}. \end{aligned}$$

Hence

$$\Delta h \approx 0.53 \text{ m}.$$

The amplitude relative to the mean in P is half this value, approximately 0.27 m. This small deformation confirms the validity of the linear expansion in h/R .

The solution was checked against the official solution.