

# IPhO 1995

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## Problem 1: Gravitational Red Shift and the Measurement of Stellar Mass

### (a) (3 marks)

A photon of frequency  $f$  possesses an effective inertial mass  $m$  determined by its energy. Assume that it has a gravitational mass equal to this inertial mass. Accordingly, a photon emitted at the surface of a star will lose energy when it escapes from the star's gravitational field. Show that the frequency shift  $\Delta f$  of the photon when it escapes from the surface of the star to infinity is given by

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}$$

for  $\Delta f \ll f$  where:

- $G$  = gravitational constant
- $R$  = radius of the star
- $c$  = velocity of light
- $M$  = mass of the star.

Thus, the red-shift of a known spectral line measured a long way from the star can be used to measure the ratio  $M/R$ . Knowledge of  $R$  will allow the mass of the star to be determined.

### (b) (12 marks)

An unmanned spacecraft is launched in an experiment to measure both the mass  $M$  and radius  $R$  of a star in our galaxy. Photons are emitted from  $\text{He}^+$  ions on the surface of the star. These photons can be monitored through resonant absorption by  $\text{He}^+$  ions contained in a test chamber in the spacecraft. Resonant absorption occurs only if the  $\text{He}^+$  ions are given a velocity towards the star to allow exactly for the red shifts.

As the spacecraft approaches the star radially, the velocity relative to the star ( $v = \beta c$ ) of the  $\text{He}^+$  ions in the test chamber at absorption resonance is measured as a function of the distance  $d$  from the (nearest) surface of the star. The experimental data are displayed in the accompanying table.

Fully utilize the data to determine graphically the mass  $M$  and radius  $R$  of the star. There is no need to estimate the uncertainties in your answer.

### Data for Resonance Condition

| Velocity parameter            | $\beta = v/c \quad (\times 10^{-5})$ | 3.352 | 3.279 | 3.195 | 3.077 | 2.955 |
|-------------------------------|--------------------------------------|-------|-------|-------|-------|-------|
| Distance from surface of star | $d \quad (\times 10^8 \text{m})$     | 38.90 | 19.98 | 13.32 | 8.99  | 6.67  |

#### (c) (5 marks)

In order to determine  $R$  and  $M$  in such an experiment, it is usual to consider the frequency correction due to the recoil of the emitting atom. [Thermal motion causes emission lines to be broadened without displacing emission maxima, and we may therefore assume that all thermal effects have been taken into account.]

#### (i) (4 marks)

Assume that the atom decays at rest, producing a photon and a recoiling atom. Obtain the relativistic expression for the energy  $hf$  of a photon emitted in terms of  $\Delta E$  (the difference in rest energy between the two atomic levels) and the initial rest mass  $m_0$  of the atom.

#### (ii) (1 mark)

Hence make a numerical estimate of the relativistic frequency shift  $\left(\frac{\Delta f}{f}\right)_{\text{recoil}}$  for the case of  $\text{He}^+$  ions.

Your answer should turn out to be much smaller than the gravitational red shift obtained in part (b).

Data:

|                        |           |   |                                                   |
|------------------------|-----------|---|---------------------------------------------------|
| Velocity of light      | $c$       | = | $3.0 \times 10^8 \text{m s}^{-1}$                 |
| Rest energy of He      | $m_0 c^2$ | = | $4 \times 938 (\text{MeV})$                       |
| Bohr energy            | $E_n$     | = | $-\frac{13.6 Z^2}{n^2} (\text{eV})$               |
| Gravitational constant | $G$       | = | $6.7 \times 10^{-11} \text{N m}^2 \text{kg}^{-2}$ |

Official English text of the problem.

### Answers

- (a) With  $\Delta f = f_\infty - f$ ,

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}.$$

- (b) A graph of  $1/\beta$  against  $1/d$  is approximately a straight line. Using all five measurements gives

$$R \approx 1.1 \times 10^8 \text{ m}, \quad M \approx 5.2 \times 10^{30} \text{ kg}.$$

The corresponding surface-to-infinity fractional frequency shift is approximately  $-3.46 \times 10^{-5}$ .

- **(c)(i)** The exact photon energy is

$$hf = \Delta E \left( 1 - \frac{\Delta E}{2m_0 c^2} \right).$$

- **(c)(ii)** For a transition  $n_i \rightarrow n_f$  in  $\text{He}^+$ ,

$$\left( \frac{\Delta f}{f} \right)_{\text{recoil}} \simeq -7.25 \times 10^{-9} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right).$$

Its magnitude is below  $7.3 \times 10^{-9}$  for bound-bound transitions. For example,  $2 \rightarrow 1$  gives  $-5.4 \times 10^{-9}$ , several thousand times smaller than the gravitational shift.

## Solution

### (a)

The photon has energy  $E = hf$ , so its effective inertial and gravitational mass, under the assumptions of the question, is

$$m = \frac{hf}{c^2}.$$

The gravitational potential outside the star is

$$\Phi(r) = -\frac{GM}{r}.$$

As the photon travels from the surface to infinity, its gravitational potential energy increases by  $GMm/R$ . Conservation of energy therefore gives

$$h\Delta f \simeq -\frac{GMm}{R}.$$

Here  $m$  can be treated as constant to first order because the fractional change in photon energy is small. Substituting  $m = hf/c^2$ ,

$$h\Delta f \simeq -\frac{GM}{R} \frac{hf}{c^2},$$

and hence

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}.$$

The negative sign represents a red shift. The approximation requires  $GM/(Rc^2) \ll 1$ .

### (b)

At the spacecraft, the distance from the star's centre is  $r = R + d$ . Applying the same energy argument between  $R$  and  $R + d$ , the frequency  $f_r$  measured by an observer stationary relative to the star is

$$f_r \simeq f_0 \left[ 1 - \frac{GM}{c^2} \left( \frac{1}{R} - \frac{1}{R+d} \right) \right],$$

where  $f_0$  is the emitted transition frequency. Recoil corrections are neglected here and assessed in part (c).

Define the positive fractional gravitational frequency loss

$$\epsilon(d) = \frac{GM}{c^2} \left( \frac{1}{R} - \frac{1}{R+d} \right).$$

The photons travel outwards, while the absorbing ions move towards the star. Thus the ions see a Doppler blueshift:

$$f_{\text{ion}} = \gamma(1 + \beta)f_r = f_r \sqrt{\frac{1 + \beta}{1 - \beta}}.$$

Since  $\beta \sim 3 \times 10^{-5}$ ,

$$f_{\text{ion}} \simeq f_r(1 + \beta).$$

Resonant absorption requires  $f_{\text{ion}} = f_0$ , so, to first order,

$$(1 - \epsilon)(1 + \beta) \simeq 1 \implies \beta \simeq \epsilon.$$

Consequently,

$$\boxed{\beta = \frac{GM}{c^2} \left( \frac{1}{R} - \frac{1}{R+d} \right) = \frac{GM}{Rc^2} \frac{d}{R+d}}$$

to the accuracy required.

Taking reciprocals produces a linear relation:

$$\frac{1}{\beta} = \frac{Rc^2}{GM} + \frac{R^2c^2}{GM} \frac{1}{d}.$$

Therefore, plot

$$Y = \frac{1}{\beta} \quad \text{against} \quad X = \frac{1}{d}.$$

Writing the straight line as  $Y = A + BX$ , its intercept and slope are

$$A = \frac{Rc^2}{GM}, \quad B = \frac{R^2c^2}{GM}.$$

These determine both unknowns:

$$\boxed{R = \frac{B}{A}}, \quad \boxed{M = \frac{c^2 B}{GA^2}}.$$

The five points to plot, with convenient axis scales, are:

| $X/(10^{-9} \text{ m}^{-1})$ | $Y/10^4$ |
|------------------------------|----------|
| 0.25707                      | 2.98329  |
| 0.50050                      | 3.04971  |
| 0.75075                      | 3.12989  |
| 1.11235                      | 3.24992  |
| 1.49925                      | 3.38409  |

The graph is an approximately straight, rising line. A best-fitting line using all five points is

$$\frac{Y}{10^4} \simeq 2.8917 + 0.3249 \frac{X}{10^{-9} \text{ m}^{-1}}.$$

Thus the vertical intercept and slope in unscaled units are

$$A \simeq 2.8917 \times 10^4, \quad B \simeq 3.2489 \times 10^{12} \text{ m}.$$

The extrapolated horizontal intercept is  $-A/B \simeq -8.90 \times 10^{-9} \text{ m}^{-1}$ .

It follows that

$$R = \frac{3.2489 \times 10^{12}}{2.8917 \times 10^4} \simeq 1.12 \times 10^8 \text{ m},$$

and

$$M = \frac{(3.0 \times 10^8)^2 (3.2489 \times 10^{12})}{(6.7 \times 10^{-11})(2.8917 \times 10^4)^2} \simeq 5.2 \times 10^{30} \text{ kg}.$$

To sensible precision for a graphical determination,

$$\boxed{R \approx 1.1 \times 10^8 \text{ m}, \quad M \approx 5.2 \times 10^{30} \text{ kg}}.$$

The surface-to-infinity red shift is

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2} = -\frac{1}{A} \simeq -3.46 \times 10^{-5}.$$

This also verifies the weak-field approximation used throughout.

### (c)(i)

Let the initial atom have rest mass  $m_0$ , and let the final atom have rest mass  $m_1$ . The difference between the two internal rest energies is

$$\Delta E = (m_0 - m_1)c^2.$$

Write the emitted photon energy as  $E_\gamma = hf$ . Since the initial atom is at rest, conservation of momentum requires the recoiling atom to have momentum of magnitude

$$p = \frac{E_\gamma}{c}.$$

Using the relativistic energy–momentum relation for the final atom, conservation of energy gives

$$m_0 c^2 = E_\gamma + \sqrt{m_1^2 c^4 + p^2 c^2} = E_\gamma + \sqrt{m_1^2 c^4 + E_\gamma^2}.$$

Rearranging and squaring,

$$(m_0 c^2 - E_\gamma)^2 = m_1^2 c^4 + E_\gamma^2,$$

so

$$E_\gamma = \frac{(m_0^2 - m_1^2)c^2}{2m_0}.$$

Substituting  $m_1 = m_0 - \Delta E/c^2$  yields

$$\boxed{hf = \Delta E - \frac{(\Delta E)^2}{2m_0 c^2} = \Delta E \left(1 - \frac{\Delta E}{2m_0 c^2}\right)}.$$

This expression is exact for the stated two-body decay. The photon energy is smaller than the internal energy difference because some energy becomes kinetic energy of the recoiling atom.

**(c)(ii)**

Without recoil, the transition frequency would be  $f_0 = \Delta E/h$ . Therefore,

$$\frac{f - f_0}{f_0} = -\frac{\Delta E}{2m_0c^2}.$$

Since the correction is very small, the fractional recoil shift is

$$\left(\frac{\Delta f}{f}\right)_{\text{recoil}} \simeq -\frac{\Delta E}{2m_0c^2}.$$

For  $\text{He}^+$ ,  $Z = 2$ , and the given Bohr formula gives

$$E_n = -\frac{54.4}{n^2} \text{ eV}.$$

Thus, for emission from  $n_i$  to  $n_f < n_i$ ,

$$\Delta E = 54.4 \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \text{ eV}.$$

The given rest energy is

$$m_0c^2 = 4 \times 938 \text{ MeV} = 3.752 \times 10^9 \text{ eV}.$$

Hence

$$\left[\left(\frac{\Delta f}{f}\right)_{\text{recoil}} \simeq -7.25 \times 10^{-9} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right)\right].$$

For any bound-bound transition,  $\Delta E < 54.4 \text{ eV}$ , giving the numerical upper estimate

$$\left[\left|\frac{\Delta f}{f}\right|_{\text{recoil}} < 7.3 \times 10^{-9}\right].$$

For example, the  $2 \rightarrow 1$  transition has

$$\Delta E = 54.4 \left( 1 - \frac{1}{4} \right) = 40.8 \text{ eV},$$

and therefore

$$\left[\left(\frac{\Delta f}{f}\right)_{\text{recoil}, 2 \rightarrow 1} \simeq -5.4 \times 10^{-9}\right].$$

Even the upper estimate is several thousand times smaller than the gravitational shifts of approximately  $3 \times 10^{-5}$  in part (b).

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The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

## Problem 2: Sound Propagation

### Introduction

The speed of propagation of sound in the ocean varies with depth, temperature and salinity. Figure 1(a) below shows the variation of sound speed  $c$  with depth  $z$  for a case where a minimum speed value  $c_0$  occurs midway between the ocean surface and the sea bed. Note that for convenience  $z = 0$  at the depth of this sound speed minimum,  $z = z_S$  at the surface and  $z = -z_b$  at the sea bed. Above  $z = 0$ ,  $c$  is given by

$$c = c_0 + bz .$$

Below  $z = 0$ ,  $c$  is given by

$$c = c_0 - bz .$$

In each case  $b = \left| \frac{dc}{dz} \right|$ , that is,  $b$  is the magnitude of the sound speed gradient with depth;  $b$  is assumed constant.

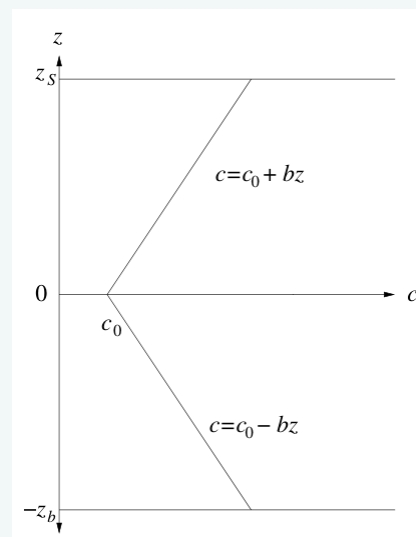


Figure 1 (a)

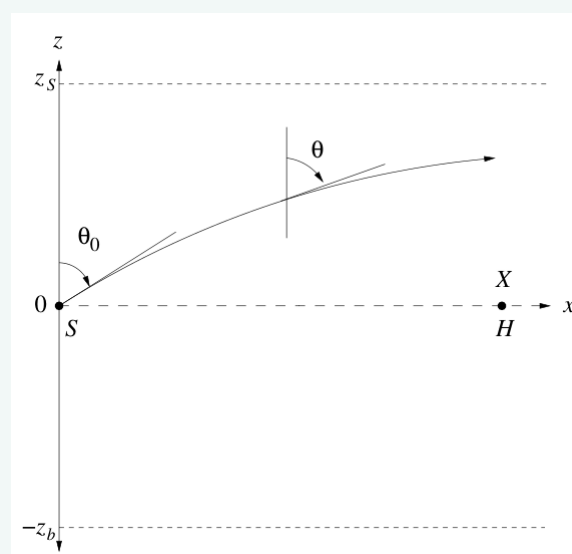


Figure 1 (b)

Figure 1(b) shows a section of the  $z$ - $x$  plane through the ocean, where  $x$  is a horizontal direction. The variation of  $c$  with respect to  $z$  is shown in figure 1(a). At the position  $z = 0$ ,  $x = 0$ , a sound source  $S$  is located. A 'sound ray' is emitted from  $S$  at an angle  $\theta_0$  as shown. Because of the variation of  $c$  with  $z$ , the ray will be refracted.

**(a) (6 marks)**

Show that the trajectory of the ray, leaving the source  $S$  and constrained to the  $z$ - $x$  plane forms an arc of a circle with radius  $R$  where

$$R = \frac{c_0}{b \sin \theta_0} \quad \text{for} \quad 0 \leq \theta_0 < \frac{\pi}{2}.$$

**(b) (3 marks)**

Derive an expression involving  $z_S$ ,  $c_0$  and  $b$  to give the smallest value of the angle  $\theta_0$  for upwardly directed rays which can be transmitted without the sound wave reflecting from the sea surface.

**(c) (4 marks)**

Figure 1(b) shows the position of a sound receiver  $H$  which is located at the position  $z = 0$ ,  $x = X$ . Derive an expression involving  $b$ ,  $X$  and  $c_0$  to give the series of angles  $\theta_0$  required for the sound ray emerging from  $S$  to reach the receiver  $H$ . Assume that  $z_S$  and  $z_b$  are sufficiently large to remove the possibility of reflection from sea surface or sea bed.

**(d) (2 marks)**

Calculate the smallest four values of  $\theta_0$  for refracted rays from  $S$  to reach  $H$  when

- $X = 10000 \text{ m}$
- $c_0 = 1500 \text{ m s}^{-1}$
- $b = 0.02000 \text{ s}^{-1}$

**(e) (5 marks)**

Derive an expression to give the time taken for sound to travel from  $S$  to  $H$  following the ray path associated with the smallest value of angle  $\theta_0$ , as determined in part (c). Calculate the value of this transit time for the conditions given in part (d). The following result may be of assistance:

$$\int \frac{dx}{\sin x} = \ln \tan \left( \frac{x}{2} \right)$$

Calculate the time taken for the direct ray to travel from  $S$  to  $H$  along  $z = 0$ . Which of the two rays will arrive first, the ray for which  $\theta_0 = \pi/2$ , or the ray with the smallest value of  $\theta_0$  as calculated for part (d)?

Official English text of the problem.

## Answers

- **(a)** Each segment lying entirely above or below  $z = 0$  is a circular arc with

$$R = \frac{c_0}{b \sin \theta_0}.$$

The case  $\theta_0 = 0$  is the straight vertical ray, corresponding to  $R \rightarrow \infty$ .

- **(b)** The limiting angle is

$$\theta_{0,\min} = \sin^{-1} \left( \frac{c_0}{c_0 + bz_S} \right).$$

Equality gives a ray tangent to the surface; larger angles turn below it.

- **(c)** For initially upward, refracted rays,

$$\theta_{0,n} = \tan^{-1} \left( \frac{2nc_0}{bX} \right), \quad n = 1, 2, 3, \dots,$$

where  $n$  counts the circular arcs between source and receiver. The direct ray has  $\theta_0 = \pi/2$ .

- **(d)** The four smallest angles are

$$86.19^\circ, \quad 88.09^\circ, \quad 88.73^\circ, \quad 89.05^\circ.$$

- **(e)** For the smallest angle,  $n = 1$ ,

$$t_1 = \frac{2}{b} \ln \left[ \sqrt{1 + \left( \frac{bX}{2c_0} \right)^2} + \frac{bX}{2c_0} \right] \approx 6.662 \text{ s}.$$

The direct-ray time is

$$t_{\text{direct}} = \frac{X}{c_0} \approx 6.667 \text{ s}.$$

The refracted ray arrives first, by approximately  $4.93 \text{ ms}$ .

## Solution

We use geometrical acoustics: the wavelength is sufficiently short for propagation to be described by rays in a stationary, locally isotropic medium. The sound speed depends only on  $z$ .

### (a)

Consider the medium as a succession of thin horizontal layers. Snell's law, with the angle measured from the vertical normal to these layers, gives

$$\frac{\sin \theta}{c(z)} = \frac{\sin \theta_0}{c_0} \equiv p,$$

where  $p$  is constant along the ray.

Let  $s$  be distance along the ray. For a ray travelling toward increasing  $x$ ,

$$\frac{dx}{ds} = \sin \theta, \quad \frac{dz}{ds} = \cos \theta.$$

In the upper region,  $c = c_0 + bz$ . Differentiating  $\sin \theta = pc$  along the ray gives

$$\cos \theta \frac{d\theta}{ds} = pb \frac{dz}{ds} = pb \cos \theta.$$

Thus, with the value at the horizontal turning point obtained by continuity,

$$\frac{d\theta}{ds} = pb = \frac{b \sin \theta_0}{c_0}.$$

The rate of change of the tangent direction with arc length is the curvature. Since it is constant, the trajectory is circular, with

$$R = \frac{1}{pb} = \frac{c_0}{b \sin \theta_0}.$$

The circle can also be found explicitly. Using  $ds = R d\theta$  and the initial position  $x = z = 0$ ,

$$x = R(\cos \theta_0 - \cos \theta), \quad z = R(\sin \theta - \sin \theta_0).$$

Since  $R \sin \theta_0 = c_0/b$ , these equations imply

$$(x - R \cos \theta_0)^2 + \left(z + \frac{c_0}{b}\right)^2 = R^2.$$

The upper arc therefore has its centre at

$$\left(R \cos \theta_0, -\frac{c_0}{b}\right).$$

Below  $z = 0$ , the gradient changes sign, so  $d\theta/ds = -pb$ . The curvature reverses, while its magnitude remains  $1/R$ . A ray crossing  $z = 0$  consequently follows successive circular arcs of equal radius.

For  $\theta_0 = 0$ , Snell's law gives a vertical ray with zero curvature. The stated radius formula is understood in the limiting sense  $R \rightarrow \infty$ .

## (b)

At the highest point of an upward arc, the ray is horizontal, so  $\theta = \pi/2$ . Snell's law then gives

$$\frac{1}{c_0 + bz_{\max}} = \frac{\sin \theta_0}{c_0},$$

and hence

$$z_{\max} = \frac{c_0}{b} \left( \frac{1}{\sin \theta_0} - 1 \right).$$

The limiting ray that avoids crossing the sea surface satisfies  $z_{\max} = z_S$ . Therefore,

$$\sin \theta_{0,\min} = \frac{c_0}{c_0 + bz_S},$$

so

$$\theta_{0,\min} = \sin^{-1} \left( \frac{c_0}{c_0 + bz_S} \right).$$

Because increasing  $\theta_0$  decreases  $z_{\max}$ , angles above this threshold turn below the surface. Equality is the grazing case; strict separation from the surface requires  $\theta_0 > \theta_{0,\min}$ .

**(c)**

During the first upper arc, the ray returns to  $z = 0$  when

$$\sin \theta = \sin \theta_0, \quad \theta = \pi - \theta_0.$$

Its horizontal displacement at that crossing is

$$D = R [\cos \theta_0 - \cos(\pi - \theta_0)] = 2R \cos \theta_0.$$

Using the radius from part (a),

$$D = \frac{2c_0}{b} \cot \theta_0.$$

The speed is continuous at  $z = 0$ , so the direction of the ray is continuous there. The equal gradient magnitudes above and below  $z = 0$  produce alternating upper and lower arcs, each spanning the same horizontal distance  $D$ .

The receiver is reached after an integer number  $n$  of these arcs:

$$X = nD = \frac{2nc_0}{b} \cot \theta_0, \quad n = 1, 2, 3, \dots$$

Therefore,

$$\theta_{0,n} = \tan^{-1} \left( \frac{2nc_0}{bX} \right).$$

These angles increase with  $n$ , so the smallest angle corresponds to a single upper arc:

$$\theta_{0,1} = \tan^{-1} \left( \frac{2c_0}{bX} \right).$$

Symmetry also gives corresponding paths launched downward. The horizontal direct ray,  $\theta_0 = \pi/2$ , is an additional path.

**(d)**

The numerical factor in the angle formula is

$$\frac{2c_0}{bX} = \frac{2(1500)}{(0.02000)(10000)} = 15.$$

Thus  $\theta_{0,n} = \tan^{-1}(15n)$ , giving

| $n$ | $\tan \theta_{0,n}$ | $\theta_{0,n}$ |
|-----|---------------------|----------------|
| 1   | 15                  | $86.19^\circ$  |
| 2   | 30                  | $88.09^\circ$  |
| 3   | 45                  | $88.73^\circ$  |
| 4   | 60                  | $89.05^\circ$  |

**(e)**

Write the smallest launch angle as  $\theta_1 = \theta_{0,1}$ . This ray travels along one upper arc, with  $\theta$  increasing from  $\theta_1$  to  $\pi - \theta_1$ .

The travel time is

$$t_1 = \int \frac{ds}{c}.$$

Along this arc,

$$ds = R d\theta, \quad c = \frac{c_0}{\sin \theta_1} \sin \theta = bR \sin \theta.$$

Consequently,

$$t_1 = \frac{1}{b} \int_{\theta_1}^{\pi-\theta_1} \frac{d\theta}{\sin \theta} = \frac{1}{b} \left[ \ln \tan \left( \frac{\theta}{2} \right) \right]_{\theta_1}^{\pi-\theta_1}.$$

Using  $\tan[(\pi - \theta_1)/2] = \cot(\theta_1/2)$ ,

$$\boxed{t_1 = \frac{2}{b} \ln \cot \left( \frac{\theta_1}{2} \right)}.$$

To express this in terms of the given quantities, define

$$u = \frac{bX}{2c_0} = \cot \theta_1.$$

The half-angle identity gives

$$\cot \left( \frac{\theta_1}{2} \right) = \frac{1 + \cos \theta_1}{\sin \theta_1} = \csc \theta_1 + \cot \theta_1 = \sqrt{1 + u^2} + u.$$

Hence,

$$\boxed{t_1 = \frac{2}{b} \ln \left[ \sqrt{1 + \left( \frac{bX}{2c_0} \right)^2} + \frac{bX}{2c_0} \right]}.$$

For the numerical data,

$$u = \frac{(0.02000)(10000)}{2(1500)} = \frac{1}{15},$$

so

$$t_1 = \frac{2}{0.02000} \ln \left[ \sqrt{1 + \frac{1}{225}} + \frac{1}{15} \right] \text{ s} = \boxed{6.662 \text{ s}}.$$

The direct ray remains at  $z = 0$ , where its speed is  $c_0$ . Its time is therefore

$$t_{\text{direct}} = \frac{X}{c_0} = \frac{10000}{1500} \text{ s} = \boxed{6.667 \text{ s}}.$$

Using unrounded values, the difference is

$$t_{\text{direct}} - t_1 = 0.00492842 \dots \text{ s} \approx \boxed{4.93 \text{ ms}}.$$

Thus **the refracted ray with the smallest launch angle arrives first**. The increased sound speed away from  $z = 0$  more than compensates for its extra path length.

This comparison also follows generally from

$$t_1 = \frac{2}{b} \operatorname{arsinh} u, \quad t_{\text{direct}} = \frac{2u}{b},$$

because, for  $u > 0$ ,

$$\operatorname{arsinh} u = \int_0^u \frac{dv}{\sqrt{1+v^2}} < u.$$

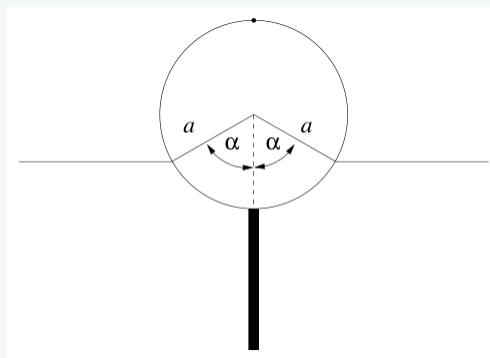
The solution was checked against the official solution.

### Problem 3: Cylindrical Buoy

(a) (3 marks)

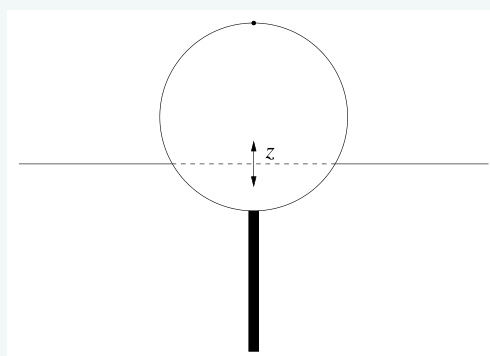
A buoy consists of a solid cylinder, radius  $a$ , length  $l$ , made of lightweight material of uniform density  $d$  with a uniform rigid rod protruding directly outwards from the bottom halfway along the length. The mass of the rod is equal to that of the cylinder, its length is the same as the diameter of the cylinder and the density of the rod is greater than that of seawater. This buoy is floating in sea-water of density  $\rho$ .

In equilibrium derive an expression relating the floating angle  $\alpha$ , as drawn, to  $d/\rho$ . Neglect the volume of the rod.



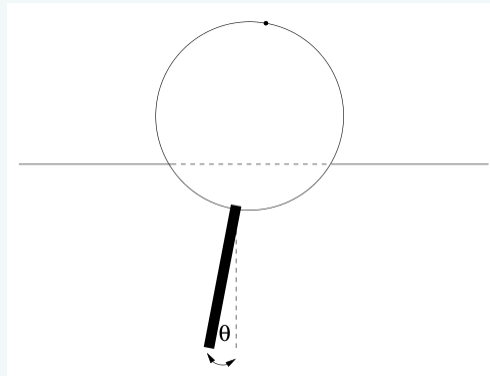
(b) (4 marks)

If the buoy, due to some perturbation, is depressed vertically by a small amount  $z$ , it will experience a nett force, which will cause it to begin oscillating vertically about the equilibrium floating position. Determine the frequency of this vertical mode of vibration in terms of  $\alpha$ ,  $g$  and  $a$ , where  $g$  is the acceleration due to gravity. Assume the influence of water motion on the dynamics of the buoy is such as to increase the effective mass of the buoy by a factor of one third. You may assume that  $\alpha$  is not small.



(c) (8 marks)

In the approximation that the cylinder swings about its horizontal central axis, determine the frequency of swing again in terms of  $g$  and  $a$ . Neglect the dynamics and viscosity of the water in this case. The angle of swing is assumed to be small.



(d) (5 marks)

The buoy contains sensitive accelerometers which can measure the vertical and swinging motions and can relay this information by radio to shore. In relatively calm waters it is recorded that the vertical oscillation period is about 1 second and the swinging oscillation period is about 1.5 seconds. From this information, show that the floating angle  $\alpha$  is about  $90^\circ$  and thereby estimate the radius of the buoy and its total mass, given that the cylinder length  $l$  equals  $a$ .

[You may take it that  $\rho \simeq 1000 \text{ kg m}^{-3}$  and  $g \simeq 9.8 \text{ m s}^{-2}$ .]

Official English text of the problem.

## Answers

- (a) With  $\alpha$  in radians,

$$\frac{d}{\rho} = \frac{\alpha - \sin \alpha \cos \alpha}{2\pi}.$$

- (b) The vertical oscillation frequency is

$$f_z = \frac{1}{2\pi} \sqrt{\frac{3g \sin \alpha}{2a(\alpha - \sin \alpha \cos \alpha)}}.$$

- (c) The swinging frequency is

$$f_\theta = \frac{1}{2\pi} \sqrt{\frac{12g}{29a}}.$$

- (d)

$$\alpha \simeq 90^\circ, \quad a \simeq 0.23 \text{ m}, \quad M \simeq 20 \text{ kg}.$$

Using the nominal periods exactly gives  $\alpha \simeq 91.1^\circ$ .

## Solution

(a)

Let the mass of the cylinder be  $m$ . The rod has the same mass, so

$$m = d\pi a^2 l, \quad M = 2m.$$

The submerged cross-section is a circular segment subtending an angle  $2\alpha$  at the cylinder's centre. Its area is the sector area minus the triangle area:

$$A_{\text{sub}} = \frac{1}{2}a^2(2\alpha) - \frac{1}{2}a^2 \sin 2\alpha = a^2(\alpha - \sin \alpha \cos \alpha).$$

Here and below, angles in these expressions are in radians.

Define

$$S(\alpha) = \alpha - \sin \alpha \cos \alpha.$$

Neglecting the rod's volume, the displaced volume is  $a^2 l S(\alpha)$ . By Archimedes' principle, equilibrium requires

$$\rho g a^2 l S(\alpha) = 2d\pi a^2 l g.$$

Consequently,

$$\boxed{\frac{d}{\rho} = \frac{S(\alpha)}{2\pi} = \frac{2\alpha - \sin 2\alpha}{4\pi}}.$$

(b)

At equilibrium, the width of the cylinder's cross-section at the water surface is

$$w = 2a \sin \alpha.$$

Thus the horizontal area cut by the water surface is  $wl = 2al \sin \alpha$ .

Take the displacement  $z$  as positive downwards. To first order in  $z$ , the increase in displaced volume is

$$\Delta V = 2al \sin \alpha z.$$

The equilibrium buoyancy already balances the weight. The additional buoyancy therefore gives the restoring force

$$F_z = -\rho g \Delta V = -2\rho g al \sin \alpha z.$$

The stated one-third increase in effective mass gives

$$M_{\text{eff}} = M + \frac{1}{3}M = \frac{4}{3}M.$$

From part (a), the actual mass satisfies

$$M = \rho a^2 l S(\alpha).$$

Hence the equation of motion is

$$\frac{4}{3}\rho a^2 l S(\alpha) \ddot{z} + 2\rho g al \sin \alpha z = 0.$$

This is simple harmonic motion, with

$$\omega_z^2 = \frac{3g \sin \alpha}{2a S(\alpha)}.$$

The frequency in cycles per second is therefore

$$f_z = \frac{1}{2\pi} \sqrt{\frac{3g \sin \alpha}{2a(\alpha - \sin \alpha \cos \alpha)}}.$$

Only the vertical displacement has been linearized; no small- $\alpha$  approximation has been made.

**(c)**

Use the prescribed approximation that the cylinder rotates about its fixed horizontal central axis.

Because the cylinder has a circular cross-section, rotation about this axis leaves the submerged geometry unchanged. The buoyancy force acts vertically through the central axis and produces no torque about it. The cylinder's own weight also acts through this axis. The restoring torque therefore comes entirely from the rod's weight.

The rod extends from a distance  $a$  to a distance  $3a$  from the axis. Its centre of mass is consequently at distance  $2a$ . For a displacement  $\theta$  from the downward vertical,

$$\tau = -2mga \sin \theta \simeq -2mga \theta,$$

where the last approximation uses  $|\theta| \ll 1$ .

The cylinder's moment of inertia about its central axis is

$$I_{\text{cyl}} = \frac{1}{2}ma^2.$$

For the thin uniform rod, the parallel-axis theorem gives

$$I_{\text{rod}} = \frac{1}{12}m(2a)^2 + m(2a)^2 = \frac{13}{3}ma^2.$$

Thus

$$I = I_{\text{cyl}} + I_{\text{rod}} = \frac{29}{6}ma^2.$$

Neglecting water dynamics and viscosity as instructed,

$$I\ddot{\theta} = -2mga \theta,$$

or

$$\ddot{\theta} + \frac{12g}{29a} \theta = 0.$$

Therefore,

$$\omega_\theta = \sqrt{\frac{12g}{29a}}, \quad f_\theta = \frac{1}{2\pi} \sqrt{\frac{12g}{29a}}.$$

**(d)**

Let the vertical and swinging periods be  $T_z$  and  $T_\theta$ . Taking the ratio of the squared frequencies eliminates  $a$  and  $g$ :

$$\left(\frac{T_\theta}{T_z}\right)^2 = \frac{\omega_z^2}{\omega_\theta^2} = \frac{29 \sin \alpha}{8S(\alpha)}.$$

Using  $T_z \simeq 1$  s and  $T_\theta \simeq 1.5$  s,

$$\frac{S(\alpha)}{\sin \alpha} \simeq \frac{29}{8(1.5)^2} = \frac{29}{18} \simeq 1.611.$$

At  $\alpha = 90^\circ = \pi/2$ ,

$$\frac{S(\alpha)}{\sin \alpha} = \frac{\pi}{2} \simeq 1.571,$$

and the predicted period ratio is

$$\frac{T_\theta}{T_z} = \sqrt{\frac{29}{4\pi}} \simeq 1.519,$$

consistent with the approximate measured ratio 1.5. Solving with the nominal periods gives  $\alpha \simeq 91.15^\circ$ . Thus

$$\boxed{\alpha \simeq 90^\circ},$$

so approximately half the cylinder is submerged.

The swinging period gives the radius directly:

$$T_\theta = 2\pi \sqrt{\frac{29a}{12g}},$$

hence

$$a = \frac{3gT_\theta^2}{29\pi^2} = \frac{3(9.8)(1.5)^2}{29\pi^2} \text{ m} \simeq 0.231 \text{ m}.$$

Therefore,

$$\boxed{a \simeq 0.23 \text{ m}}.$$

Since  $l = a$ , equilibrium gives the total mass as

$$M = \rho a^3 S(\alpha).$$

For  $\alpha \simeq \pi/2$ ,

$$M \simeq \frac{\pi}{2} \rho a^3 \simeq \frac{\pi}{2} (1000)(0.231)^3 \text{ kg} \simeq 19.4 \text{ kg}.$$

Using the nominal-period value  $\alpha \simeq 91.15^\circ$  gives  $M \simeq 19.9 \text{ kg}$ . At the accuracy warranted by the stated periods,

$$\boxed{M \simeq 20 \text{ kg}}.$$

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The solution was checked against the official solution.