

IPhO 1994

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Problem 1: Relativistic Particle

In the theory of special relativity the relation between energy E and momentum P of a free particle with rest mass m_0 is

$$E = \sqrt{p^2 c^2 + m_0^2 c^4} = mc^2$$

When such a particle is subject to a conservative force, the total energy of the particle, which is the sum of $\sqrt{p^2 c^2 + m_0^2 c^4}$ and the potential energy, is conserved. If the energy of the particle is very high, the rest energy of the particle can be ignored (such a particle is called an ultra relativistic particle).

1) consider the one dimensional motion of a very high energy particle (in which rest energy can be neglected) subject to an attractive central force of constant magnitude f . Suppose the particle is located at the centre of force with initial momentum p_0 at time $t = 0$. Describe the motion of the particle by separately plotting, for at least one period of the motion: x against time t , and momentum p against space coordinate x . Specify the coordinates of the "turning points" in terms of given parameters p_0 and f . Indicate, with arrows, the direction of the progress of the motion in the (p, x) diagram. There may be short intervals of time during which the particle is not ultrarelativistic. However, these should be neglected.

Use Answer Sheet 1.

2) A meson is a particle made up of two quarks. The rest mass M of the meson is equal to the total energy of the two-quark system divided by c^2 .

Consider a one--dimensional model for a meson at rest, in which the two quarks are assumed to move along the x -axis and attract each other with a force of constant magnitude f . It is assumed they can pass through each other freely. For analysis of the high energy motion of the quarks the rest mass of the quarks can be neglected. At time $t = 0$ the two quarks are both at $x = 0$. Show separately the motion of the two quarks graphically by a (x, t) diagram and a (p, x) diagram, specify the coordinates of the "turning points" in terms of M and f , indicate the direction of the process in your (p, x) diagram, and determine the maximum distance between the two quarks.

Use Answer Sheet 2.

3) The reference frame used in part 2 will be referred to as frame S , the Lab frame, referred to as S' , moves in the negative x -direction with a constant velocity $v = 0.6c$. the coordinates in the two reference frames are so chosen that the point $x = 0$ in S coincides with the point $x' = 0$ in S' at

time $t = t' = 0$. Plot the motion of the two quarks graphically in a (x', t') diagram. Specify the coordinates of the turning points in terms of M , f and c , and determine the maximum distance between the two quarks observed in Lab frame S' .

Use Answer Sheet 3.

The coordinates of particle observed in reference frames S and S'' are related by the Lorentz transformation

$$\begin{cases} x' = \gamma(x + \beta ct), \\ t' = \gamma\left(t + \beta \frac{x}{c}\right) \end{cases}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1 - \beta^2}$ and v is the velocity of frame S moving relative to the frame S'' .

4) For a meson with rest energy $Mc^2 = 140 \text{ MeV}$ and velocity $0.60c$ relative to the Lab frame S'' , determine its energy E' in the Lab Frame S'' .

Official English text of the problem.

Answers

1. Choose $p_0 > 0$, and define

$$\tau_0 = \frac{p_0}{f}, \quad A_0 = \frac{cp_0}{f}.$$

The position–time graph consists of straight segments through

$$(t, x) : (0, 0) \rightarrow (\tau_0, A_0) \rightarrow (2\tau_0, 0) \rightarrow (3\tau_0, -A_0) \rightarrow (4\tau_0, 0).$$

The turning points are (τ_0, A_0) and $(3\tau_0, -A_0)$, and

$$T_1 = \frac{4p_0}{f}.$$

With x horizontal and p vertical, the phase diagram is the clockwise diamond

$$(x, p) : (0, p_0) \rightarrow (A_0, 0) \rightarrow (0, -p_0) \rightarrow (-A_0, 0) \rightarrow (0, p_0).$$

2. Define

$$p_* = \frac{Mc}{2}, \quad \tau = \frac{Mc}{2f}, \quad a = \frac{Mc^2}{2f}.$$

Quark 1 follows the motion in part 1 with $p_0 = p_*$; quark 2 has

$$x_2 = -x_1, \quad p_2 = -p_1.$$

Their positions at the turning times are

$$t = \tau : (x_1, x_2) = (a, -a), \quad t = 3\tau : (x_1, x_2) = (-a, a).$$

Both phase trajectories follow the same diamond clockwise, starting at opposite momentum vertices. Thus

$$T = \frac{2Mc}{f}, \quad d_{\max} = \frac{Mc^2}{f}.$$

3. Here $\gamma = 5/4$. In coordinates $(t'/\tau, x'/a)$, join the following vertices by straight lines:

Quark 1: $(0, 0) \rightarrow (2, 2) \rightarrow (3, 1) \rightarrow (5, 3)$,

Quark 2: $(0, 0) \rightarrow (\frac{1}{2}, -\frac{1}{2}) \rightarrow (\frac{9}{2}, \frac{7}{2}) \rightarrow (5, 3)$.

The two interior vertices of each trajectory are its turning points. Subsequent cycles repeat after a translation $(\Delta t', \Delta x') = (5\tau, 3a)$. The maximum simultaneous laboratory separation is

$$d'_{\max} = a = \frac{Mc^2}{2f}.$$

4. The meson's laboratory energy is

$$E' = \frac{Mc^2}{\sqrt{1 - 0.60^2}} = 175 \text{ MeV}.$$

Solution

Position–time plots below use time horizontally and position vertically. Phase diagrams use x horizontally and p vertically. The laboratory frame is denoted consistently by S' .

1) Motion in the constant-magnitude central force

Choose the positive x -direction along the initial momentum, so $p_0 > 0$. In the ultrarelativistic approximation,

$$E_{\text{particle}} \simeq c|p|, \quad \dot{x} = \frac{pc^2}{E_{\text{particle}}} \simeq c \operatorname{sgn}(p).$$

Thus the particle moves at speed c between reversals. The brief non-ultrarelativistic intervals at the turning points are neglected as instructed.

The attractive force and a corresponding potential are

$$\dot{p} = -f \operatorname{sgn}(x), \quad U(x) = f|x|.$$

Energy conservation gives

$$c|p| + f|x| = cp_0.$$

At a spatial turning point, $p = 0$, so the maximum displacement is

$$A_0 = \frac{cp_0}{f}.$$

Initially $x > 0$, hence $p = p_0 - ft$. The first reversal occurs after

$$\tau_0 = \frac{p_0}{f},$$

at $x = c\tau_0 = A_0$. The force continues to point toward the origin, so the particle returns with negative momentum. After crossing the origin, the force changes sign.

One complete cycle is:

t	x	p
0	0	p_0

t	x	p
τ_0	A_0	0
$2\tau_0$	0	$-p_0$
$3\tau_0$	$-A_0$	0
$4\tau_0$	0	p_0

Join the position entries by straight lines. The slopes are $+c$ up to τ_0 , $-c$ from τ_0 to $3\tau_0$, and $+c$ thereafter. The origin crossing at $2\tau_0$ is not a reversal.

Consequently,

$$T_1 = 4\tau_0 = \frac{4p_0}{f}.$$

The phase curve follows directly from energy conservation:

$$\frac{|x|}{A_0} + \frac{|p|}{p_0} = 1.$$

It is a diamond with vertices $(0, p_0)$, $(A_0, 0)$, $(0, -p_0)$, and $(-A_0, 0)$. Starting at the upper vertex, the motion proceeds to the right vertex, then the lower vertex, then the left vertex, and back to the upper vertex: clockwise.

For an initially negative signed momentum, reflect both x and p ; the amplitude and period use $|p_0|$.

2) The two quarks in the meson rest frame

Take the interaction potential to vanish when the quarks coincide. For separation $r = |x_1 - x_2|$,

$$U(r) = fr.$$

Because the meson is at rest, the initial quark momenta are equal and opposite. Label the initially right-moving quark as quark 1:

$$p_1(0) = p_*, \quad p_2(0) = -p_*.$$

At $t = 0$, the potential energy is zero. The total energy therefore gives

$$Mc^2 = 2cp_*, \quad p_* = \frac{Mc}{2}.$$

Symmetry preserves

$$x_2 = -x_1, \quad p_2 = -p_1.$$

Each quark experiences a force of magnitude f directed toward the other quark, which is also toward the origin while they are separated. Therefore each follows the motion found in part 1, with

$$\tau = \frac{Mc}{2f}, \quad a = c\tau = \frac{Mc^2}{2f}.$$

The complete energy equation confirms this:

$$Mc^2 = 2c|p_1| + f|x_1 - x_2| = 2(c|p_1| + f|x_1|).$$

The following table specifies both plots over one period:

t/τ	x_1/a	p_1/p_*	x_2/a	p_2/p_*
0	0	1	0	-1
1	1	0	-1	0
2	0	-1	0	1
3	-1	0	1	0
4	0	1	0	-1

For the position–time diagrams, join each quark’s position entries by straight lines. The two triangular curves are reflections of each other about $x = 0$, and the quarks pass through one another at $t = 2\tau$.

For either phase diagram,

$$\frac{|x_i|}{a} + \frac{|p_i|}{p_*} = 1.$$

Quark 1 starts at $(x, p) = (0, p_*)$, while quark 2 starts at $(0, -p_*)$. Both traverse the diamond clockwise.

At the turning points, the quarks are at opposite extremes, so

$$d_{\max} = 2a = \frac{Mc^2}{f}.$$

Their individual period is

$$T = 4\tau = \frac{2Mc}{f}.$$

3) Motion and maximum separation in the laboratory frame

The given transformation has

$$\beta = \frac{3}{5}, \quad \gamma = \frac{5}{4},$$

so

$$\gamma(1 + \beta) = 2, \quad \gamma(1 - \beta) = \frac{1}{2}.$$

A segment with velocity $u = \pm c$ transforms to

$$u' = \frac{u + \beta c}{1 + \beta u/c} = \pm c.$$

Thus all laboratory trajectory segments also have slopes $\pm c$ or $-c$. We obtain their endpoints by transforming the turning events.

For example, quark 1’s first turn occurs at $(t, x) = (\tau, a)$, with $a = c\tau$. Therefore

$$t' = \gamma\tau(1 + \beta) = 2\tau, \quad x' = \gamma a(1 + \beta) = 2a.$$

All four turning events in one cycle are:

Turning event	t'	x'
Quark 2, first turn	$\frac{\tau}{2} = \frac{Mc}{4f}$	$-\frac{a}{2} = -\frac{Mc^2}{4f}$

Turning event	t'	x'
Quark 1, first turn	$2\tau = \frac{Mc}{f}$	$2a = \frac{Mc^2}{f}$
Quark 1, second turn	$3\tau = \frac{3Mc}{2f}$	$a = \frac{Mc^2}{2f}$
Quark 2, second turn	$\frac{9\tau}{2} = \frac{9Mc}{4f}$	$\frac{7a}{2} = \frac{7Mc^2}{4f}$

Both trajectories begin at $(t', x') = (0, 0)$. The rest-frame crossing at $(t, x) = (2\tau, 0)$ becomes

$$(t', x') = \left(\frac{5\tau}{2}, \frac{3a}{2} \right).$$

The end of the rest-frame cycle, $(4\tau, 0)$, becomes

$$(t', x') = (5\tau, 3a).$$

These points and the turning-point table completely specify the laboratory plots: connect the successive points for each quark by straight lines. Subsequent cycles repeat after translating both curves by

$$\Delta t' = 5\tau = \gamma T, \quad \Delta x' = 3a.$$

To find the observed separation, the two positions must be evaluated at the **same laboratory time**:

$$d'(t') = |x'_1(t') - x'_2(t')|.$$

Initially,

$$x'_1 = ct', \quad x'_2 = -ct',$$

so the separation grows as $2ct'$. Quark 2 reverses first, at $t' = \tau/2$. At that instant,

$$x'_1 = \frac{a}{2}, \quad x'_2 = -\frac{a}{2},$$

and hence $d' = a$.

From $t' = \tau/2$ to $t' = 2\tau$, both quarks move to the right at speed c :

$$x'_1 = ct', \quad x'_2 = ct' - a.$$

Their separation therefore stays equal to a . It then decreases to zero at $5\tau/2$, increases back to a at 3τ , stays at a until $9\tau/2$, and decreases to zero at 5τ . This pattern repeats.

Thus

$$d'_{\max} = a = \frac{Mc^2}{2f} = \frac{1}{2} d_{\max}.$$

The different laboratory reversal times are essential: the two simultaneous rest-frame turning events are not simultaneous in the laboratory.

4) Laboratory energy of the meson

For the complete meson, the total rest-frame momentum is zero and its total energy is Mc^2 .

Transforming its energy gives

$$E' = \gamma(E + \beta c P_x) = \gamma Mc^2.$$

With the supplied values,

$$\gamma = \frac{1}{\sqrt{1 - 0.60^2}} = \frac{1}{0.80} = 1.25,$$

and therefore

$$E' = 1.25(140 \text{ MeV}) = 175 \text{ MeV}.$$

The solution was checked against the official solution.

Problem 2: Superconducting Magnet

Super conducting magnets are widely used in laboratories. The most common form of super conducting magnets is a solenoid made of super conducting wire. The wonderful thing about a superconducting magnet is that it produces high magnetic fields without any energy dissipation due to Joule heating, since the electrical resistance of the superconducting wire becomes zero when the magnet is immersed in liquid helium at a temperature of 4.2 K. Usually, the magnet is provided with a specially designed superconducting switch, as shown in Fig. 1. The resistance r of the switch can be controlled: either $r = 0$ in the superconducting state, or $r = r_n$ in the normal state. When the persistent mode, with a current circulating through the magnet and superconducting switch indefinitely. The persistent mode allows a steady magnetic field to be maintained for long periods with the external source cut off.

The details of the superconducting switch are not given in Fig. 1. It is usually a small length of superconducting wire wrapped with a heater wire and suitably thermally insulated from the liquid helium bath. On being heated, the temperature of the superconducting wire increases and it reverts to the resistive normal state. The typical value of r_n is a few ohms. Here we assume it to be 5Ω . The inductance of a superconducting magnet depends on its size; assume it be 10 H for the magnet in Fig. 1. The total current I can be changed by adjusting the resistance R .

This problem will be graded by the plots only!

The arrows denote the positive direction of I , I_1 and I_2 .

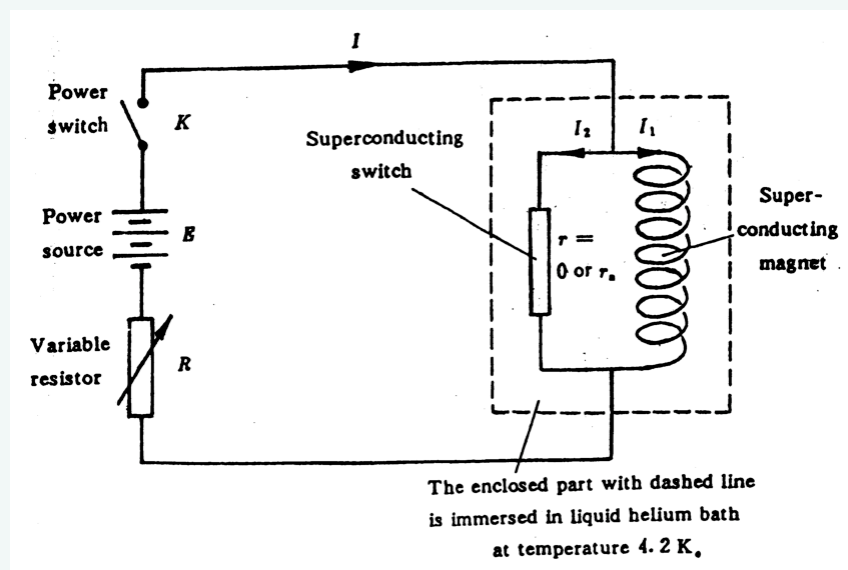


Fig. 1

1) If the total current I and the resistance r of the superconducting switch are controlled to vary with time in the way shown in Figs. 2a and 2b respectively, and assuming the currents I_1 and I_2 flowing through the magnet and the switch respectively are equal at the beginning (Fig. 2c and Fig. 2d), how do they vary with time from t_1 to t_4 ? Plot your answer in Fig. 2c and Fig. 2d

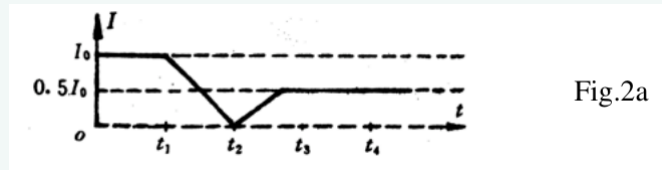
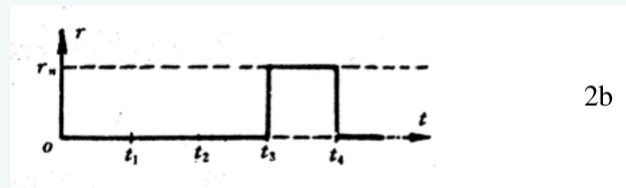


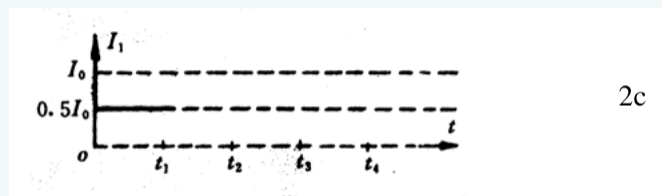
Fig.2a

Fig.2a



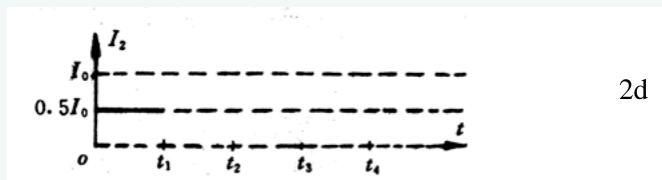
2b

2b



2c

2c



2d

2d

2) Suppose the power switch K is turned on at time $t = 0$ when $r = 0$, $I_1 = 0$ and $R = 7.5 \Omega$, and the total current I is 0.5 A . With K kept closed, the resistance r of the superconducting switch is varied in the way shown in Fig. 3b. Plot the corresponding time dependences of I , I_1 and I_2 in Figs. 3a, 3c and 3d respectively.

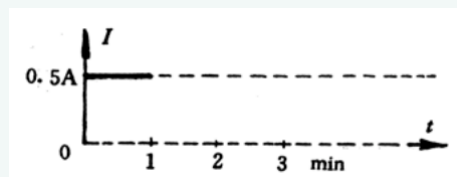
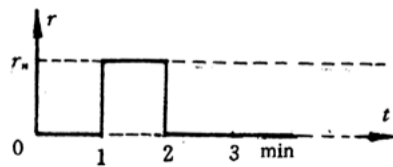
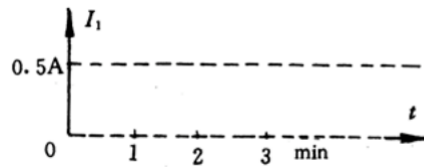


Fig. 3a



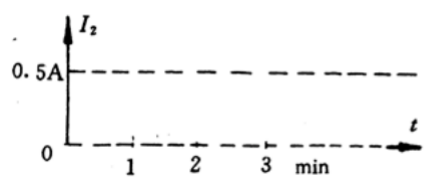
3b

3b



3c

3c



3d

3d

3) Only small currents, less than 0.5 A, are allowed to flow through the superconducting switch when it is in the normal state, with larger currents the switch will be burnt out. Suppose the superconducting magnet is operated in a persistent mode, i. e. $I = 0$, and $I_1 = i_1$ (e. g. 20 A), $I_2 = -i_1$, as shown in Fig. 4, from $t = 0$ to $t = 3 \text{ min}$. If the experiment is to be stopped by reducing the current through the magnet to zero, how would you do it? This has to be done in several operation steps. Plot the corresponding changes of I , r , I_1 and I_2 in Fig. 4

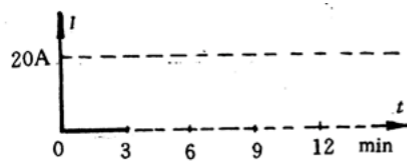
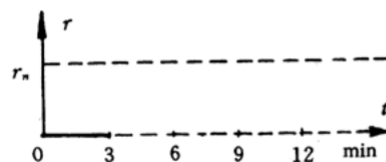


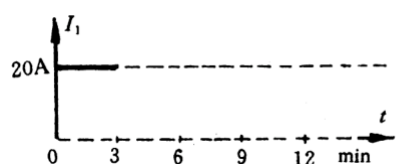
Fig. 4a

Fig. 4a



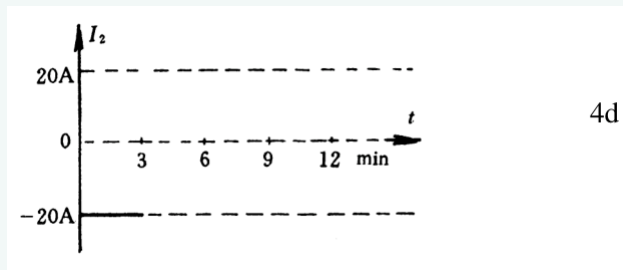
4b

4b



4c

4c



4d

4d

4) Suppose the magnet is operated in a persistent mode with a persistent current of 20 A ($t = 0$ to $t = 3\text{min}$. See Fig. 5). How would you change it to a persistent mode with a current of 30a? plot your answer in Fig. 5.

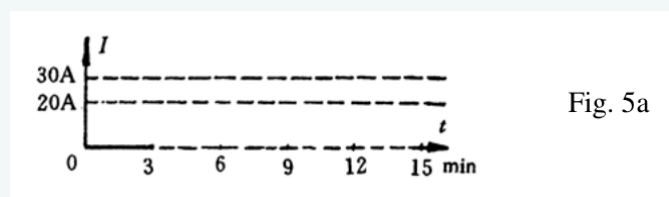
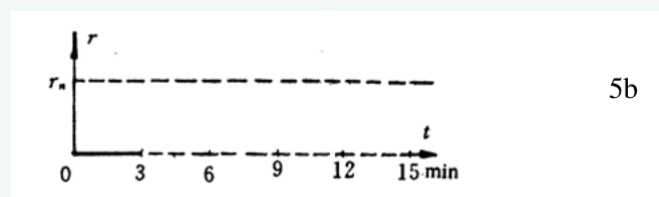


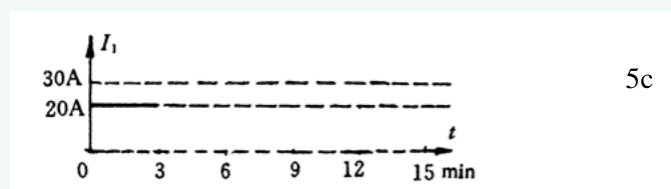
Fig. 5a

Fig. 5a



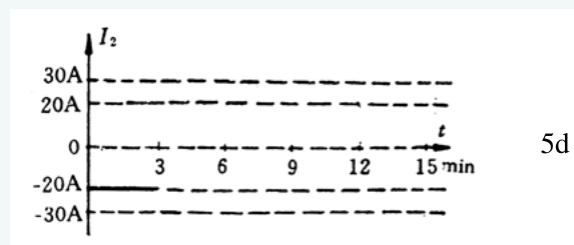
5b

5b



5c

5c



5d

5d

Official English text of the problem.

Answers

1. Throughout the prescribed sequence,

$$I_1 = \frac{I_0}{2}, \quad I_2 = I - \frac{I_0}{2}.$$

Thus I_2 falls linearly from $+I_0/2$ at t_1 to $-I_0/2$ at t_2 , rises linearly to zero as I reaches its plateau, and remains zero through t_3 and t_4 .

2. The source emf is 3.75 V. Initially,

$$(I, I_1, I_2) = (0.50, 0, 0.50) \text{ A}.$$

At 1 min, I and I_2 jump to 0.30 A, while I_1 remains zero. They then approach (0.50, 0.50, 0) A exponentially with time constant

$$\tau_R = \frac{10}{3} \text{ s} \approx 3.33 \text{ s}.$$

After 2 min, $I = 0.50$ A, I_1 is constant, and $I_2 \approx 0$. Exact expressions appear below.

3. With $r = 0$, first raise I from zero to 20 A, leaving $I_1 = 20$ A. Then set $r = 5 \Omega$, reduce I slowly to zero, and open K . The remaining magnet current decays exponentially with time constant 2.0 s. The explicit three-minute ramp below keeps

$$|I_2| < 0.223 \text{ A} < 0.5 \text{ A}$$

throughout the normal state.

4. First match $I = I_1 = 20$ A with $r = 0$. Set $r = 5 \Omega$ and increase I_1 to 30 A, using $I_2 = 1/9$ A during a three-minute ramp. Then set $I = 30$ A, cool the switch to $r = 0$, and reduce I to zero. The final persistent state is

$$I = 0, \quad I_1 = 30 \text{ A}, \quad I_2 = -30 \text{ A}, \quad r = 0.$$

Solution

We treat the magnet as an ideal inductor with constant inductance $L = 10$ H. We neglect lead resistance, stray inductance and switching time, as implicit in the problem.

With the indicated current directions, Kirchhoff's current law gives

$$I = I_1 + I_2.$$

The magnet and superconducting switch have the same terminal voltage:

$$V = L \frac{dI_1}{dt} = rI_2.$$

Therefore the central equation is

$$L \frac{dI_1}{dt} = r(I - I_1).$$

When $r = 0$, the magnet current is constant. Changes in the source current then pass entirely through the superconducting switch:

$$\frac{dI_1}{dt} = 0, \quad I_2 = I - I_1.$$

When $r = r_n$ and the total current is prescribed, the equation becomes

$$\tau \frac{dI_1}{dt} + I_1 = I, \quad \boxed{\tau = \frac{L}{r_n} = 2.0 \text{ s.}}$$

The inductor current remains continuous at switching instants; I and I_2 may jump.

1) Prescribed total current and switch resistance

Initially the two branch currents are equal, so

$$I_1 = I_2 = \frac{I_0}{2}.$$

Until t_3 , the switch is superconducting. Hence

$$I_1 = \frac{I_0}{2}, \quad I_2 = I - \frac{I_0}{2}.$$

By t_3 , the imposed current has reached $I_0/2$. Thus $I_2 = 0$ when the switch becomes resistive. During $t_3 < t < t_4$,

$$L \frac{dI_1}{dt} = r_n \left(\frac{I_0}{2} - I_1 \right) = 0.$$

The currents consequently remain unchanged. Returning the switch to $r = 0$ at t_4 also causes no change.

The requested plots are:

- **Fig. 2c:** a horizontal line at $I_1 = I_0/2$ throughout.
- **Fig. 2d:** a horizontal line at $I_0/2$ until t_1 , followed by a straight decline to $-I_0/2$ at t_2 . It crosses zero halfway between t_1 and t_2 . It then rises linearly to zero when the prescribed I reaches $I_0/2$, and stays at zero through t_3 , t_4 , and afterward.

The negative portion of Fig. 2d must be drawn below the original zero axis.

2) Fixed source resistance

Let the source emf be $\mathcal{E}$. Initially $r = 0$, so the parallel combination has zero voltage:

$$\mathcal{E} = RI = (7.5 \Omega)(0.50 \text{ A}) = \boxed{3.75 \text{ V}}.$$

For $0 \leq t < 60 \text{ s}$, the magnet current stays at its initial value zero:

$$\boxed{I = 0.50 \text{ A}, \quad I_1 = 0, \quad I_2 = 0.50 \text{ A}.}$$

At $t = 60 \text{ s}$, the switch becomes resistive. Since I_1 cannot jump, its immediate value remains zero, giving

$$I(60^+) = I_2(60^+) = \frac{\mathcal{E}}{R + r_n} = \frac{3.75}{12.5} = \boxed{0.30 \text{ A}}.$$

During the normal-state interval, Kirchhoff's voltage law gives

$$\mathcal{E} = R(I_1 + I_2) + r_n I_2,$$

and therefore

$$I_2 = \frac{\mathcal{E} - RI_1}{R + r_n}.$$

Combining this with the inductor equation,

$$L \frac{dI_1}{dt} = \frac{r_n R}{R + r_n} \left(\frac{\mathcal{E}}{R} - I_1 \right).$$

Thus

$$\tau_R = \frac{L(R + r_n)}{Rr_n} = \frac{10(7.5 + 5)}{7.5(5)} \text{ s} = \frac{10}{3} \text{ s}.$$

For $60 \leq t < 120$ s, define

$$x = \frac{t - 60 \text{ s}}{\tau_R}.$$

The currents are

$$\begin{aligned} I_1(t) &= (0.50 \text{ A})(1 - e^{-x}), \\ I_2(t) &= (0.30 \text{ A})e^{-x}, \\ I(t) &= 0.50 \text{ A} - (0.20 \text{ A})e^{-x}. \end{aligned}$$

At $t = 120$ s, $x = 18$, and the switch returns to $r = 0$. The source current becomes $\mathcal{E}/R = 0.50$ A, while the magnet current freezes at its existing value. Hence, for all $t \geq 120$ s,

$$\begin{aligned} I &= 0.50 \text{ A}, \\ I_1 &= (0.50 \text{ A})(1 - e^{-18}), \\ I_2 &= (0.50 \text{ A})e^{-18} \approx 7.6 \text{ nA}. \end{aligned}$$

The plot features are:

- **Fig. 3a:** $I = 0.50$ A until one minute; a vertical drop to 0.30 A; then an exponential, concave-down rise toward 0.50 A.
- **Fig. 3c:** $I_1 = 0$ until one minute; then a continuous, concave-down exponential rise toward 0.50 A. It becomes exactly constant at two minutes.
- **Fig. 3d:** $I_2 = 0.50$ A until one minute; a vertical drop to 0.30 A; then exponential decay toward zero.

At two minutes, I and I_2 have upward jumps of only about 3.0 nA , invisible on the supplied scale. All visible transients occupy only a small fraction of a minute.

3) Reducing the persistent current to zero

The current restriction applies to the magnitude of the normal-state switch current:

$$|I_2| < 0.5 \text{ A}.$$

Since

$$I_2 = \frac{L}{r_n} \frac{dI_1}{dt},$$

it requires

$$\left| \frac{dI_1}{dt} \right| < \frac{r_n}{L} (0.5 \text{ A}) = 0.25 \text{ A s}^{-1}.$$

Heating the switch immediately in the initial persistent state would put 20 A through it. We must first transfer its current to the external source while it is still superconducting.

Here is one definite schedule for the 20 A example.

From 0 to 3 min: retain the given persistent state:

$$I = 0, \quad r = 0, \quad I_1 = 20 \text{ A}, \quad I_2 = -20 \text{ A}.$$

From 3 to 6 min: keep $r = 0$, close K with sufficiently large R_r and raise I linearly from zero to 20 A. Then

$$I_1 = 20 \text{ A}, \quad I_2 = I - 20 \text{ A}.$$

Thus I_2 rises linearly from -20 A to zero. At six minutes the source supplies the whole magnet current.

From 6 to 9 min: set $r = r_n = 5 \Omega$ and reduce the total current linearly from 20 A to zero.

Let

$$u = t - 360 \text{ s}, \quad T = 180 \text{ s}, \quad a = \frac{20 \text{ A}}{T} = \frac{1}{9} \text{ A s}^{-1}.$$

During this interval,

$$I = 20 \text{ A} - au.$$

Solving $\tau \dot{I}_1 + I_1 = I$, with $I_1(0) = 20 \text{ A}$, gives

$$\begin{aligned} I_1 &= 20 \text{ A} - au + a\tau(1 - e^{-u/\tau}), \\ I_2 &= -a\tau(1 - e^{-u/\tau}), \end{aligned} \quad \tau = 2.0 \text{ s}.$$

Consequently,

$$|I_2| \leq a\tau = \frac{2}{9} \text{ A} \approx 0.222 \text{ A} < 0.5 \text{ A}.$$

The magnet current follows slightly above the declining source current.

At 9 min and afterward: the source current has reached zero. Open K , leaving the switch normal. The remaining magnet current is

$$D = \frac{2}{9}(1 - e^{-90}) \text{ A} \approx 0.222 \text{ A}.$$

It circulates through the magnet and normal switch and decays as

$$I = 0, \quad I_1 = De^{-(t-540 \text{ s})/\tau}, \quad I_2 = -De^{-(t-540 \text{ s})/\tau}.$$

After another 10 s, $I_1 \approx 1.5 \text{ mA}$, and it continues toward zero.

The complete Fig. 4 plots are therefore:

Plot	Required shape
4a: I	Zero until 3 min; linear rise to 20 A at 6 min; linear fall to zero at 9 min; zero afterward.
4b: r	Zero until 6 min; step to 5Ω ; remain normal during the final discharge.
4c: I_1	20 A until 6 min; smoothly enters an almost linear decline, reaching approximately 0.222 A at 9 min; then decays exponentially to zero.

Plot	Required shape
4d: I_2	−20 A until 3 min; linear rise to zero at 6 min; becomes slightly negative, approaching −0.222 A; after 9 min returns exponentially toward zero from below.

As in any passive RL discharge, exact zero is approached asymptotically. On the minute-scale plot, the final tail is indistinguishable from zero. Returning the switch to $r = 0$ before that tail has decayed would trap the remaining current.

For a different initial i_1 , the same procedure applies with a sufficiently slow ramp to respect the 0.25 A s^{-1} bound.

4) Changing the persistent current from 20 A to 30 A

Again, first match the source current to the existing magnet current before heating the switch.

We can choose an exactly linear magnet-current ramp lasting three minutes:

$$\alpha = \frac{30 - 20}{180} \text{ A s}^{-1} = \frac{1}{18} \text{ A s}^{-1}.$$

The required switch current during this ramp is

$$\delta = \frac{L}{r_n} \alpha = \frac{1}{9} \text{ A} \approx 0.111 \text{ A},$$

well below the normal-state limit.

To produce this ramp exactly, prescribe

$$I = I_1 + \delta.$$

For $u = t - 360 \text{ s}$, $0 < u < 180 \text{ s}$,

$$I_1 = 20 \text{ A} + \alpha u, \quad I_2 = \delta, \quad I = 20 \text{ A} + \alpha u + \delta.$$

Indeed, $r_n I_2 = r_n \delta = L \alpha = L dI_1/dt$.

A complete schedule, specifying all four plots, is:

Time (min)	I (A), Fig. 5a	r (Ω), Fig. 5b	I_1 (A), Fig. 5c	I_2 (A), Fig. 5d
0–3	0	0	20	−20
3–6	Linear $0 \rightarrow 20$	0	20	Linear $-20 \rightarrow 0$
6–9	Linear $20 + \delta \rightarrow 30 + \delta$	5	Linear $20 \rightarrow 30$	$\delta = 1/9$
9–12	30	5	30	0
12–15	Linear $30 \rightarrow 0$	0	30	Linear $0 \rightarrow -30$
After 15	0	0	30	−30

Here the numerical value of δ in the table is in amperes.

The switching details matter:

- **At 6 min:** $I = I_1 = 20 \text{ A}$, so $I_2 = 0$. Make the switch normal, then increase I by the small amount δ . This starts the linear magnet-current ramp without any jump in I_1 .
- **At 9 min:** I_1 has reached exactly 30 A. Reduce I from $30 + \delta$ to 30 A. Then $I_2 = 0$, so the magnet current remains constant.

- **At 12 min:** return the switch to $r = 0$. Reduce I to zero over the next three minutes. The magnet current stays fixed at 30 A , while I_2 becomes increasingly negative.
- **At 15 min:** open K . The magnet and superconducting switch carry equal and opposite currents of magnitude 30 A .

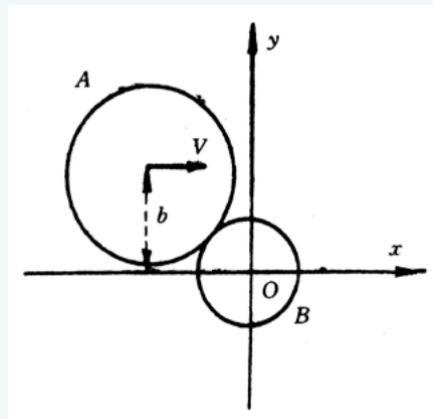
Thus Fig. 5a includes small 0.111 A steps at 6 and 9 min. Fig. 5d has a small positive rectangular pulse during 6–9 min. The large negative switch currents occur only while $r = 0$, so they do not violate the stated normal-state restriction.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 3: Collision of Discs with Surface Friction

A homogeneous disc A of mass m and radius R_A moves translationally on a smooth horizontal x - y plane in the x direction with a velocity V (see the figure on the next page). The center of the disk is at a distance b from the x -axis. It collides with a stationary homogeneous disc B whose center is initially located at the origin of the coordinate system. The disc B has the same mass and the same thickness as A, but its radius is R_B . It is assumed that the velocities of the discs at their point of contact, in the direction perpendicular to the line joining their centers, are equal after the collision. It is also assumed that the magnitudes of the relative velocities of the discs along the line joining their centers are the same before and after the collision.

- 1) For such a collision determine the X and Y components of the velocities of the two discs after the collision, i. e. V'_{AX} , V'_{AY} , V'_{BX} and V'_{BY} in terms of m , R_A , R_B , V and b .
- 2) Determine the kinetic energies E'_A for disc A and E'_B for disc B after the collision in terms of m , R_A , R_B , V and b .



Official English text of the problem.

Answers

Let $D = R_A + R_B$. For an impact, $0 \leq b < D$, with A above the x -axis as shown.

1. The velocity components are

$$\boxed{V'_{AX} = \frac{5Vb^2}{6D^2}}, \quad \boxed{V'_{AY} = \frac{5Vb\sqrt{D^2 - b^2}}{6D^2}},$$

$$\boxed{V'_{BX} = V \left(1 - \frac{5b^2}{6D^2} \right)}, \quad \boxed{V'_{BY} = -\frac{5Vb\sqrt{D^2 - b^2}}{6D^2}}.$$

2. Including both translation and rotation,

$$\boxed{E'_A = \frac{3mV^2b^2}{8D^2}}, \quad \boxed{E'_B = \frac{mV^2}{2} \left(1 - \frac{11b^2}{12D^2} \right)}.$$

Solution

1) Velocities after the collision

Treat the collision as instantaneous, so the line joining the centers remains fixed during the impact. The smooth horizontal plane supplies no horizontal impulse or torque about the vertical axis. Initially neither disc rotates.

Define

$$s = \frac{b}{D}, \quad c = \sqrt{1 - s^2}, \quad D = R_A + R_B.$$

At first contact, the center of A is at $(-Dc, b)$, while the center of B is at the origin. Thus the unit vectors along the line from A to B and perpendicular to it are

$$\mathbf{n} = (c, -s), \quad \mathbf{t} = (s, c).$$

The initial velocity of A is

$$\mathbf{v}_A = Vc\mathbf{n} + Vs\mathbf{t}.$$

Let the impulse exerted on B be

$$\mathbf{J} = J_n\mathbf{n} + J_t\mathbf{t}.$$

The impulse on A is $-\mathbf{J}$. Linear impulse–momentum therefore gives

$$\begin{aligned} v'_{An} &= Vc - \frac{J_n}{m}, & v'_{Bn} &= \frac{J_n}{m}, \\ v'_{At} &= Vs - \frac{J_t}{m}, & v'_{Bt} &= \frac{J_t}{m}. \end{aligned}$$

The normal relative velocity reverses direction while retaining its magnitude: the discs approach before impact and separate afterward. Consequently,

$$v'_{Bn} - v'_{An} = Vc,$$

so

$$\frac{2J_n}{m} - Vc = Vc \implies J_n = mVc.$$

Hence

$$\boxed{v'_{An} = 0, \quad v'_{Bn} = Vc}.$$

To determine the tangential impulse, we must include rotation. The moments of inertia are

$$I_A = \frac{1}{2}mR_A^2, \quad I_B = \frac{1}{2}mR_B^2.$$

Take counterclockwise angular velocity as positive. The contact points lie at $R_A\mathbf{n}$ relative to A's center and $-R_B\mathbf{n}$ relative to B's center. Angular impulse–momentum gives

$$I_A\omega'_A = -R_A J_t, \quad I_B\omega'_B = -R_B J_t.$$

Therefore,

$$\omega'_A = -\frac{2J_t}{mR_A}, \quad \omega'_B = -\frac{2J_t}{mR_B}.$$

The tangential velocities of the material points in contact are

$$u'_{At} = v'_{At} + R_A\omega'_A = Vs - \frac{3J_t}{m},$$

$$u'_{Bt} = v'_{Bt} - R_B\omega'_B = \frac{3J_t}{m}.$$

The stated equality of these velocities requires

$$Vs - \frac{3J_t}{m} = \frac{3J_t}{m},$$

which yields

$$J_t = \frac{mVs}{6}.$$

Thus

$$v'_{At} = \frac{5}{6}Vs, \quad v'_{Bt} = \frac{1}{6}Vs,$$

and the angular velocities, needed for part 2, are

$$\omega'_A = -\frac{Vs}{3R_A}, \quad \omega'_B = -\frac{Vs}{3R_B}.$$

Both discs rotate clockwise for the geometry shown.

Returning to Cartesian coordinates,

$$\mathbf{v}'_A = \frac{5}{6}Vs\mathbf{t} = \left(\frac{5}{6}Vs^2, \frac{5}{6}Vsc\right),$$

$$\mathbf{v}'_B = Vc\mathbf{n} + \frac{1}{6}Vs\mathbf{t} = \left(V\left(c^2 + \frac{s^2}{6}\right), -\frac{5}{6}Vsc\right).$$

Using $c^2 = 1 - s^2$, $s = b/D$, and $c = \sqrt{D^2 - b^2}/D$ gives the four components in the Answers section. Their sums satisfy conservation of linear momentum:

$$V'_{AX} + V'_{BX} = V, \quad V'_{AY} + V'_{BY} = 0.$$

2) Kinetic energies after the collision

Each disc's kinetic energy includes translation of its center and rotation about its center:

$$E'_i = \frac{1}{2}m|\mathbf{v}'_i|^2 + \frac{1}{2}I_i(\omega'_i)^2.$$

For A,

$$\begin{aligned}
 E'_A &= \frac{1}{2}m\left(\frac{5}{6}Vs\right)^2 + \frac{1}{2}\left(\frac{1}{2}mR_A^2\right)\left(\frac{Vs}{3R_A}\right)^2 \\
 &= \left(\frac{25}{72} + \frac{1}{36}\right)mV^2s^2 \\
 &= \frac{3}{8}mV^2s^2.
 \end{aligned}$$

Therefore,

$$E'_A = \frac{3mV^2b^2}{8(R_A + R_B)^2}.$$

For B, the normal and tangential velocity components are perpendicular, so

$$|\mathbf{v}'_B|^2 = V^2c^2 + \frac{V^2s^2}{36}.$$

It follows that

$$\begin{aligned}
 E'_B &= \frac{1}{2}mV^2\left(c^2 + \frac{s^2}{36}\right) + \frac{1}{2}\left(\frac{1}{2}mR_B^2\right)\left(\frac{Vs}{3R_B}\right)^2 \\
 &= \frac{1}{2}mV^2c^2 + \frac{1}{24}mV^2s^2 \\
 &= \frac{1}{2}mV^2\left(1 - \frac{11s^2}{12}\right).
 \end{aligned}$$

Thus

$$E'_B = \frac{mV^2}{2}\left(1 - \frac{11b^2}{12(R_A + R_B)^2}\right).$$

As a consistency check, the total kinetic-energy loss is

$$\frac{1}{2}mV^2 - (E'_A + E'_B) = \frac{mV^2b^2}{12(R_A + R_B)^2} \geq 0.$$

This loss results from tangential friction; the prescribed elastic condition applies only to normal relative motion. For a central collision, $b = 0$, A stops, B acquires velocity V , neither disc rotates, and no kinetic energy is lost.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.