

IPhO 1993

Theory problems and solutions · Williamsburg, Virginia, USA · 3 theory problems with complete solutions

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Problem 1: Atmospheric Electricity

General Tabulated Information

Quantity	Symbol	Value
Earth's mean radius	R_E	$6.4 \times 10^6 \text{ m}$.
acceleration due to gravity	g	9.8 m s^{-2} .
Newtonian gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$.
permittivity of vacuum	ϵ_0	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$.
permeability of vacuum	μ_0	$8.85 \times 10^{-7} \text{ N A}^{-2}$ [sic; correct value $4\pi \times 10^{-7} \text{ N A}^{-2}$].
speed of light in vacuum (or air)	c	$3.00 \times 10^8 \text{ m s}^{-1}$.
elementary charge	e	$1.60 \times 10^{-19} \text{ C}$.
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg}$.
mass of proton	m_p	$1.67 \times 10^{-27} \text{ kg}$.
Planck constant	h	$6.63 \times 10^{-34} \text{ J s}$.
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$.
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$.
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$.

Theoretical Problem 1

ATMOSPHERIC ELECTRICITY

From the standpoint of electrostatics, the surface of the Earth can be considered to be a good conductor. It carries a certain total charge Q_0 and an average surface charge density σ_0 .

1) Under fair-weather conditions, there is a downward electric field, E_0 , at the Earth's surface equal to about 150 V/m . Deduce the magnitude of the Earth's surface charge density and the total charge carried on the Earth's surface.

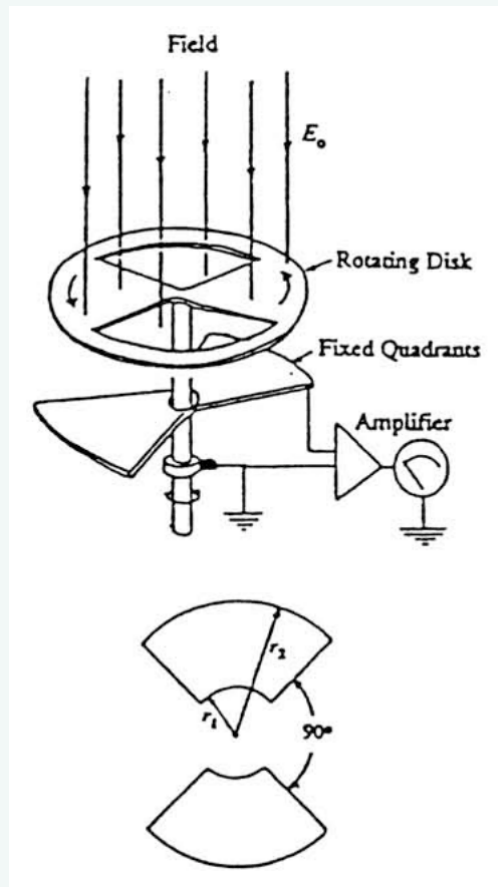
2) The magnitude of the downward electric field decreases with height, and is about 100 V/m at a height of 100 m . Calculate the average amount of net charge per m^3 of the atmosphere between the Earth's surface and 100 m altitude.

3) The net charge density you have calculated in (2) is actually the result of having almost equal numbers of positive and negative singly-charged ions per unit volume (n_+ and n_-). Near the Earth's surface, under fair-weather conditions, $n_+ \approx n_- \approx 6 \times 10^8 \text{ m}^{-3}$. These ions move under the action of the vertical electric field. Their speed is proportional to the field strength:

$$v \approx 1.5 \times 10^{-4} \cdot E,$$

where v is in m/s and E in V/m. How long would it take for the motion of the atmospheric ions to neutralize half of the Earth's surface charge, if no other processes (e.g. lightning) occurred to maintain it?

4) One way of measuring the atmospheric electric field, and hence σ_0 , is with the system shown in the diagram. A pair of metal quadrants, insulated from ground but connected to each other, are mounted just underneath a grounded uniformly rotating disk with two quadrant-shaped holes cut in it. (In the diagram, the spacing has been exaggerated in order to show the arrangement.) Twice in each revolution the insulated quadrants are completely exposed to the field, and then ($1/4$ of a period later) are completely shielded from it. Let T be the period of revolution, and let the inner and outer radii of the insulated quadrants be r_1 and r_2 as shown.



Take $t = 0$ to be an instant when the insulated quadrants are completely shielded.

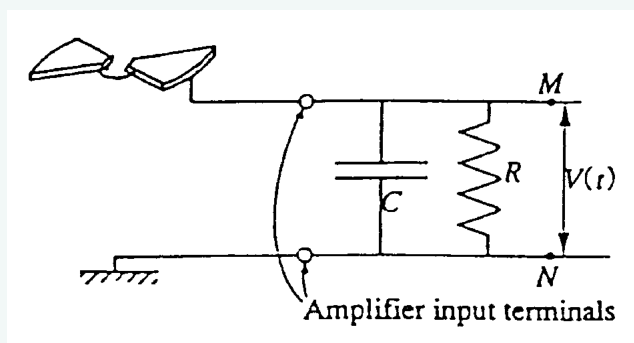
Obtain expressions that give the total charge $q(t)$ induced on the upper surface of the insulated quadrants as a function of time between $t = 0$ and $t = T/2$, and sketch a graph of this variation.

[The effects of the atmospheric ion current can be ignored in this situation.]

(5) The system described in (4) is connected to an amplifier whose input circuit is equivalent to a capacitor C and a resistor R in parallel. (You can assume that the capacitance of the quadrant system is negligible compared to C .) Sketch graphs of the form of the voltage difference V between the points M and N as a function of t during one revolution of the disk, just after it has been set into rotation with period of revolution T , if:

a) $T = T_a \ll CR$;

b) $T = T_b \gg CR$.



[Assume that C and R have fixed values; only T changes between situations (a) and (b).]

Obtain an expression for the approximate ratio, V_a/V_b , of the largest values of $V(t)$ in cases (a) and (b).

6) Assume that $E_0 = 150 \text{ V/m}$, $r_1 = 1 \text{ cm}$, $r_2 = 7 \text{ cm}$, $C = 0.01 \mu\text{F}$, $R = 20 \text{ M}\Omega$, and suppose the disk is set into rotation at 50 revolutions per second.

Approximately, what is the largest value of V during one revolution in this case?

Official English text of the problem.

Answers

Take $V = V_M - V_N$. For parts 4–6, define the fully exposed area and the magnitude of its induced charge by

$$A_m = \frac{\pi}{2}(r_2^2 - r_1^2), \quad Q_m = \varepsilon_0 E_0 A_m > 0.$$

1. The Earth's surface charge is negative:

$$\sigma_0 \simeq -1.3 \times 10^{-9} \text{ C m}^{-2}, \quad Q_0 \simeq -6.8 \times 10^5 \text{ C}.$$

The requested magnitudes are the absolute values.

2. The average atmospheric charge density is

$$\bar{\rho} \simeq +4.4 \times 10^{-12} \text{ C m}^{-3}.$$

3. Counting only positive ions arriving at the surface, and taking their concentration as approximately constant,

$$t_{1/2} = \frac{\varepsilon_0 \ln 2}{en_+(1.5 \times 10^{-4})} \simeq 4.3 \times 10^2 \text{ s} \simeq 7.1 \text{ min}.$$

4. The induced upper-surface charge is

$$q(t) = \begin{cases} -Q_m \frac{4t}{T}, & 0 \leq t \leq T/4, \\ -Q_m \left(2 - \frac{4t}{T}\right), & T/4 \leq t \leq T/2. \end{cases}$$

Its graph joins $(0, 0)$, $(T/4, -Q_m)$, and $(T/2, 0)$ by straight lines and repeats every $T/2$.

5. For the first revolution, starting with $V(0) = 0$: - **(a)** $V \simeq -q/C$: two positive triangular pulses, each of height

$$V_a \simeq \frac{Q_m}{C}.$$

- **(b)** V alternates between approximately $+V_b$ and $-V_b$, with exponential transitions of time constant RC , where

$$V_b \simeq \frac{4RQ_m}{T_b}.$$

Thus

$$\frac{V_a}{V_b} \simeq \frac{T_b}{4RC}.$$

6. The largest voltage during the first revolution is

$$V_{\max} \simeq 1.0 \times 10^{-3} \text{ V} = 1.0 \text{ mV}.$$

Including the finite resistance gives 0.989 mV, reached at $t = T/4 = 5.0 \text{ ms}$.

Solution

1) Earth's surface charge

The electric field inside a conductor in electrostatic equilibrium is zero. Gauss's law applied to a small pillbox across its surface gives

$$E_{\perp, \text{outside}} = \frac{\sigma_0}{\varepsilon_0},$$

where the normal points outward from the conductor. Since the field points downward into the Earth,

$$\sigma_0 = -\varepsilon_0 E_0.$$

Hence

$$\sigma_0 = -(8.85 \times 10^{-12})(150) = -1.3275 \times 10^{-9} \text{ C m}^{-2}.$$

Approximating the Earth as a sphere and using this as its average surface charge density,

$$Q_0 = 4\pi R_E^2 \sigma_0 = -4\pi(6.4 \times 10^6)^2(1.3275 \times 10^{-9}) \simeq -6.8 \times 10^5 \text{ C}.$$

Thus the surface charge density has magnitude approximately 1.3 nC m^{-2} , and the total charge has magnitude approximately $6.8 \times 10^5 \text{ C}$.

2) Atmospheric space charge

Let z increase upward. The signed vertical fields at the two heights are

$$E_z(0) = -150 \text{ V/m}, \quad E_z(100 \text{ m}) = -100 \text{ V/m}.$$

Over a height of 100 m , the Earth's curvature is negligible because $100/R_E \ll 1$. Gauss's law for a vertical column of horizontal cross-sectional area A therefore gives

$$\bar{\rho}Ah = \varepsilon_0 A [E_z(h) - E_z(0)].$$

Consequently,

$$\bar{\rho} = \varepsilon_0 \frac{-100 - (-150)}{100} = (8.85 \times 10^{-12})(0.50) = 4.425 \times 10^{-12} \text{ C m}^{-3}.$$

The charge is **positive**, consistent with the downward field becoming weaker as height increases.

3) Time to neutralize half the surface charge

The Earth is negatively charged. Positive ions drift downward and can reach its surface; negative ions drift upward. Thus the surface-neutralization current contains the contribution of the **positive ions only**.

Write the ion mobility as

$$\mu = 1.5 \times 10^{-4} \text{ m}^2 \text{ V}^{-1} \text{ s}^{-1}, \quad v = \mu E.$$

Taking the supplied near-surface concentration n_+ to remain approximately constant in this drift estimate, the positive charge arriving per unit area per unit time is

$$j_+ = en_+v = en_+\mu E.$$

Let $S(t) = |\sigma(t)|$ denote the magnitude of the remaining negative surface charge density. The conductor boundary condition gives

$$E(t) = \frac{S(t)}{\varepsilon_0}.$$

Since each arriving positive ion reduces S ,

$$\frac{dS}{dt} = -j_+ = -\frac{en_+\mu}{\varepsilon_0} S.$$

The discharge is therefore exponential:

$$S(t) = S(0)e^{-t/\tau_+}, \quad \tau_+ = \frac{\varepsilon_0}{en_+\mu}.$$

Numerically,

$$en_+\mu = (1.60 \times 10^{-19})(6 \times 10^8)(1.5 \times 10^{-4}) = 1.44 \times 10^{-14} \text{ S m}^{-1},$$

and

$$\tau_+ = \frac{8.85 \times 10^{-12}}{1.44 \times 10^{-14}} \simeq 615 \text{ s}.$$

Setting $S(t_{1/2}) = S(0)/2$,

$$t_{1/2} = \tau_+ \ln 2 \simeq 426 \text{ s} \simeq 7.1 \text{ min}.$$

Adding the conductivity of the upward-moving negative ions would incorrectly count a current that does not enter the Earth's surface.

4) Induced charge on the quadrants

Neglect fringing at the edges and treat the shielding as complete wherever the grounded disk covers the quadrants. On an exposed upper surface, the downward field requires surface charge density

$$\sigma_{\text{upper}} = -\varepsilon_0 E_0.$$

The total area of the two annular quadrants is

$$A_m = 2 \left[\frac{\pi}{4} (r_2^2 - r_1^2) \right] = \frac{\pi}{2} (r_2^2 - r_1^2).$$

The disk's angular speed is

$$\omega = \frac{2\pi}{T}.$$

During the first quarter-revolution, each opening exposes an angular width $\theta = \omega t$. The total exposed area is therefore

$$A(t) = 2 \left[\frac{1}{2} (r_2^2 - r_1^2) \theta \right] = \frac{2\pi(r_2^2 - r_1^2)}{T} t, \quad 0 \leq t \leq T/4.$$

It reaches A_m at $T/4$, then decreases at the same rate to zero at $T/2$. Thus

$$A(t) = \begin{cases} A_m \frac{4t}{T}, & 0 \leq t \leq T/4, \\ A_m \left(2 - \frac{4t}{T} \right), & T/4 \leq t \leq T/2. \end{cases}$$

Since $q(t) = -\varepsilon_0 E_0 A(t)$,

$$q(t) = \begin{cases} -\frac{2\pi\varepsilon_0 E_0 (r_2^2 - r_1^2)}{T} t, & 0 \leq t \leq T/4, \\ -\frac{2\pi\varepsilon_0 E_0 (r_2^2 - r_1^2)}{T} \left(\frac{T}{2} - t \right), & T/4 \leq t \leq T/2. \end{cases}$$

The sketch is a downward triangular pulse: it starts at zero, decreases linearly to $-Q_m$ at $T/4$, then increases linearly to zero at $T/2$. There are two identical pulses per revolution.

Here q is the charge on the **upper surface**. An isolated quadrant system can acquire this surface charge by redistributing its existing charge.

5) Voltage across the amplifier input

Take $V = V_M - V_N$, with N grounded. Before rotation begins, the quadrants are shielded and the resistor has discharged the capacitor, so

$$q(0) = 0, \quad V(0) = 0.$$

Neglecting the quadrant system's capacitance compared with C , the charge associated with the connected input conductor is $q + CV$. Positive charge leaves this conductor through the resistor at rate V/R , so charge conservation gives

$$\frac{d}{dt}(q + CV) = -\frac{V}{R},$$

or

$$\boxed{C \frac{dV}{dt} + \frac{V}{R} = -\frac{dq}{dt}}.$$

As the quadrants become exposed, their upper surfaces acquire negative charge. This leaves positive charge on the capacitor's upper plate, so the voltage initially rises positive.

From part 4,

$$-\frac{dq}{dt} = \begin{cases} +I_0, & 0 < t < T/4, \\ -I_0, & T/4 < t < T/2, \\ +I_0, & T/2 < t < 3T/4, \\ -I_0, & 3T/4 < t < T, \end{cases} \quad I_0 = \frac{4Q_m}{T}.$$

Thus the circuit is driven by a square-wave current.

For completeness, put $\tau = RC$, $B = RI_0$, and $t_k = kT/4$. On quarter $k = 0, 1, 2, 3$, the voltage is

$$V(t) = (-1)^k B + [V(t_k) - (-1)^k B] e^{-(t-t_k)/\tau}.$$

This expression, together with $V(0) = 0$, specifies the continuous startup waveform.

(a) Fast rotation: $T_a \ll RC$

During the first revolution, little charge leaks through R . To leading order,

$$C \frac{dV}{dt} \simeq -\frac{dq}{dt}.$$

Using the initial conditions,

$$V(t) \simeq -\frac{q(t)}{C}.$$

The voltage therefore consists of two positive triangular pulses:

t/T_a	0	1/4	1/2	3/4	1
$V/(Q_m/C)$, to leading order	0	1	0	1	0

Join successive entries by straight lines. The largest voltage is approximately

$$\boxed{V_a \simeq \frac{Q_m}{C}}.$$

For finite T_a/RC , the segments are slightly curved, the minima become slightly negative, and the second positive peak is slightly lower than the first. The positive triangles above are the leading approximation for the requested **first revolution**.

(b) Slow rotation: $T_b \gg RC$

Except during short transitions, the capacitor current is small and

$$\frac{V}{R} \simeq -\frac{dq}{dt}.$$

Hence the voltage approaches alternating levels

$$V \simeq \pm V_b, \quad V_b = \frac{4RQ_m}{T_b}.$$

The sketch has the following features:

- It starts at zero and rises exponentially toward $+V_b$, with time constant RC .
- After $t = T_b/4$, it falls exponentially toward $-V_b$.
- After $t = T_b/2$, it rises toward $+V_b$.
- After $t = 3T_b/4$, it falls toward $-V_b$.

Each quarter is long compared with RC , so most of it is an almost horizontal plateau. The voltage remains continuous at the switching times. For example, immediately after the first switch,

$$V(t) \simeq -V_b + 2V_b e^{-(t-T_b/4)/(RC)},$$

so it crosses zero approximately $RC \ln 2$ after that switch.

The requested ratio is therefore

$$\frac{V_a}{V_b} \simeq \frac{Q_m/C}{4RQ_m/T_b} = \frac{T_b}{4RC}.$$

6) Numerical voltage

The fully exposed area is

$$A_m = \frac{\pi}{2} (0.07^2 - 0.01^2) = 7.54 \times 10^{-3} \text{ m}^2.$$

Thus

$$Q_m = \varepsilon_0 E_0 A_m = (8.85 \times 10^{-12})(150)(7.54 \times 10^{-3}) \simeq 1.00 \times 10^{-11} \text{ C}.$$

The relevant times are

$$T = \frac{1}{50} = 0.020 \text{ s},$$

$$RC = (20 \times 10^6)(0.01 \times 10^{-6}) = 0.20 \text{ s}.$$

In particular, the time to the first peak is only

$$\frac{T}{4} = 0.0050 \text{ s} = 0.025 RC.$$

Consequently, the fast-rotation approximation gives

$$V_{\max} \simeq \frac{Q_m}{C} = \frac{1.00 \times 10^{-11}}{1.00 \times 10^{-8}} = 1.0 \times 10^{-3} \text{ V}.$$

The finite-resistance correction follows from the first-quarter solution:

$$V_{\max} = \frac{4RQ_m}{T} \left(1 - e^{-T/(4RC)}\right) = \frac{Q_m}{C} \frac{1 - e^{-0.025}}{0.025} \simeq 0.989 \text{ mV}.$$

It occurs at $t = T/4 = 5.0 \text{ ms}$; the second positive peak is smaller.

The solution was checked against the official solution.

Problem 2: Laser Forces on a Transparent Prism

General Tabulated Information

Quantity	Symbol	Value
Earth's mean radius	R_E	$6.4 \times 10^6 \text{ m}$
acceleration due to gravity	g	9.8 m s^{-2}
Newtonian gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
permittivity of vacuum	ϵ_0	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$
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Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}$

LASER FORCES ON A TRANSPARENT PRISM

By means of refraction a strong laser beam can exert appreciable forces on small transparent objects. To see that this is so, consider a small glass triangular prism with an apex angle $A = \pi - 2\alpha$, a base of length $2h$ and a width w . The prism has an index of refraction n and a mass density ρ .

Suppose that this prism is placed in a laser beam travelling horizontally in the x direction. (Throughout this problem assume that the prism does not rotate, i.e., its apex always points opposite to the direction of the laser beam, its triangular faces are parallel to the xy plane, and its base is parallel to the yz plane, as shown in Fig. 1.) Take the index of refraction of the surrounding air to be $n_{\text{air}} = 1$. Assume that the faces of the prism are coated with an anti-reflection coating so that no reflection occurs.

Fig. 1.

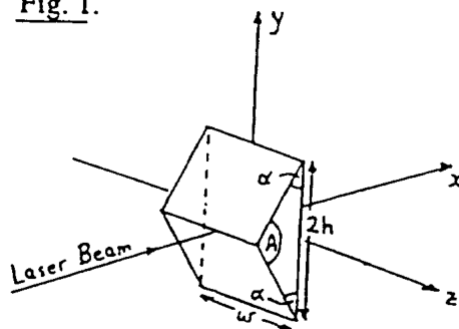


Fig. 1.

The laser beam has an intensity that is uniform across its width in the z direction but falls off linearly with distance y from the x axis such that it has a maximum value of I_0 at $y = 0$ and falls to zero at $y = \pm 4h$ (Fig. 2). [Intensity is power per unit area, e.g. expressed in W m^{-2} .]

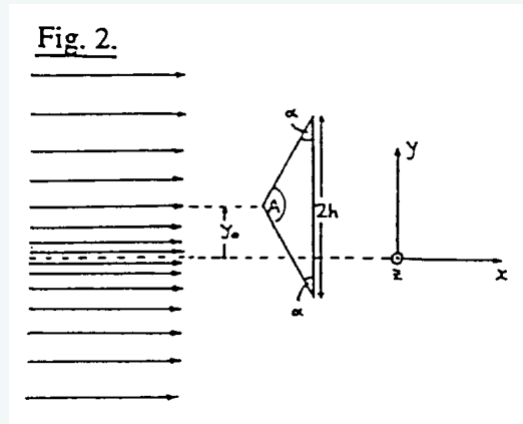


Fig. 2.

1) Write equations from which the angle θ (see Fig. 3) may be determined (in terms of α and n) in the case when laser light strikes the upper face of the prism.

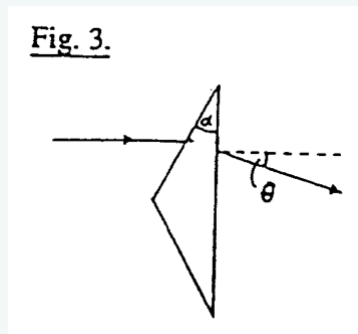


Fig. 3.

2) Express, in terms of I_0 , θ , h , w and y_0 , the x and y components of the net force exerted on the prism by the laser light when the apex of the prism is displaced a distance y_0 from the x axis where $|y_0| \leq 3h$.

Plot graphs of the values of the horizontal and vertical components of force as functions of vertical displacement y_0 .

3) Suppose that the laser beam is 1 mm wide in the z direction and $80 \mu\text{m}$ thick (in the y direction). The prism has $\alpha = 30^\circ$, $h = 10 \mu\text{m}$, $n = 1.5$, $w = 1 \text{ mm}$ and $\rho = 2.5 \text{ g cm}^{-3}$.

How many watts of laser power would be required to balance this prism against the pull of gravity (in the $-y$ direction) when the apex of the prism is at a distance $y_0 = -h/2$ ($= -5 \mu\text{m}$) below the axis of the laser beam?

4) Suppose this experiment is done in the absence of gravity with the same prism and a laser beam with the same dimensions as in (3), but with $I_0 = 10^8 \text{ W m}^{-2}$.

What would be the period of the oscillations that occur when the prism is displaced and released a distance $y = h/20$ from the center line of the laser beam?

Answers

1. If β is the angle of refraction at the upper face and γ is the incidence angle at the base,

$$\sin \alpha = n \sin \beta, \quad \gamma = \alpha - \beta, \quad \sin \theta = n \sin \gamma.$$

Equivalently,

$$\theta = \arcsin \left[\sin \alpha \left(\sqrt{n^2 - \sin^2 \alpha} - \cos \alpha \right) \right].$$

2. Define

$$u = \frac{y_0}{h}, \quad C_x = \frac{I_0 w h}{4c} (1 - \cos \theta), \quad C_y = \frac{I_0 w h}{4c} \sin \theta.$$

Then

$$F_x = C_x \begin{cases} 7 - u^2, & |u| \leq 1, \\ 8 - 2|u|, & 1 \leq |u| \leq 3, \end{cases}$$

$$F_y = -C_y \begin{cases} u(2 - |u|), & |u| \leq 1, \\ \operatorname{sgn}(u), & 1 \leq |u| \leq 3. \end{cases}$$

The horizontal-force graph is even, with a central parabolic maximum and straight outer segments. The vertical-force graph is odd and restoring, with central quadratic segments and constant outer segments.

3. The required **total laser power** is

$$P \simeq 33 \text{ W.}$$

The corresponding peak intensity is $I_0 \simeq 8.3 \times 10^8 \text{ W m}^{-2}$.

4. The small-oscillation approximation gives

$$T_0 = 2\pi \sqrt{\frac{2\rho c h^2 \tan \alpha}{I_0 \sin \theta}} \simeq 1.12 \times 10^{-2} \text{ s.}$$

Retaining the weak nonlinearity at the specified amplitude $h/20$ gives

$$T \simeq 1.13 \times 10^{-2} \text{ s,}$$

about 1.08% longer.

Solution

We use geometrical optics, neglect absorption as appropriate for the transparent prism, and apply the stipulated absence of reflection. Forces are calculated to leading order in the prism's speed divided by c .

1) Refraction and the outgoing angle

The upper face makes an angle α with the vertical base. Its inward normal therefore points at an angle α below the horizontal. The horizontal incident ray has incidence angle α .

Let β be the refracted angle measured from this normal. Snell's law gives

$$\sin \alpha = n \sin \beta.$$

The ray inside the prism consequently travels at an angle

$$\gamma = \alpha - \beta$$

below the horizontal.

The normal to the exit face is horizontal, so γ is also the incidence angle there. Applying Snell's law again,

$$n \sin \gamma = \sin \theta.$$

Thus

$$\theta = \arcsin \left[n \sin \left(\alpha - \arcsin \frac{\sin \alpha}{n} \right) \right].$$

An equivalent expression follows by expanding the sine:

$$\begin{aligned} \sin \theta &= n(\sin \alpha \cos \beta - \cos \alpha \sin \beta) \\ &= \sin \alpha \sqrt{n^2 - \sin^2 \alpha} - \sin \alpha \cos \alpha. \end{aligned}$$

Hence

$$\boxed{\sin \theta = \sin \alpha \left(\sqrt{n^2 - \sin^2 \alpha} - \cos \alpha \right)}.$$

The transmitted path shown in Fig. 3 requires $n \sin \gamma \leq 1$, which is satisfied by the numerical data in the problem. By symmetry, light entering the lower face exits at angle θ above the horizontal.

2) Force components and graphs

The intensity profile is

$$I(y) = I_0 \left(1 - \frac{|y|}{4h} \right), \quad |y| \leq 4h.$$

Let P_+ and P_- be the powers entering the upper and lower faces. When the apex is at y_0 , their projected intervals perpendicular to the incident beam are respectively

$$[y_0, y_0 + h], \quad [y_0 - h, y_0].$$

Therefore

$$P_+ = w \int_{y_0}^{y_0+h} I(y) dy, \quad P_- = w \int_{y_0-h}^{y_0} I(y) dy.$$

The area element here is $w dy$, the area projected perpendicular to the incident light. No additional incidence-angle factor is needed.

Light of power P in air carries momentum at a rate P/c . For light entering the upper face, the incident and outgoing directions are

$$(1, 0), \quad (\cos \theta, -\sin \theta).$$

The force on the prism equals the incoming momentum flux minus the outgoing momentum flux:

$$\mathbf{F}_+ = \frac{P_+}{c}(1 - \cos \theta, \sin \theta).$$

Likewise,

$$\mathbf{F}_- = \frac{P_-}{c}(1 - \cos \theta, -\sin \theta).$$

Adding,

$$\boxed{F_x = \frac{1 - \cos \theta}{c}(P_+ + P_-), \quad F_y = \frac{\sin \theta}{c}(P_+ - P_-).}$$

To evaluate the powers, put $u = y_0/h$. An antiderivative of the dimensionless intensity profile is

$$J(v) = v - \frac{v|v|}{8}, \quad J'(v) = 1 - \frac{|v|}{4}.$$

Consequently,

$$\frac{P_+}{I_0wh} = J(u+1) - J(u), \quad \frac{P_-}{I_0wh} = J(u) - J(u-1).$$

For $|u| \leq 1$, the prism straddles the beam axis. Evaluating these expressions gives

$$\begin{aligned} P_+ + P_- &= \frac{I_0wh}{4}(7 - u^2), \\ P_+ - P_- &= -\frac{I_0wh}{4}u(2 - |u|). \end{aligned}$$

For $1 \leq |u| \leq 3$, the entire prism lies on one side of the axis, where intensity varies linearly:

$$\begin{aligned} P_+ + P_- &= \frac{I_0wh}{4}(8 - 2|u|), \\ P_+ - P_- &= -\frac{I_0wh}{4}\text{sgn}(u). \end{aligned}$$

Thus, with

$$C_x = \frac{I_0wh}{4c}(1 - \cos \theta), \quad C_y = \frac{I_0wh}{4c}\sin \theta,$$

the requested forces are

$$\boxed{F_x = C_x \begin{cases} 7 - u^2, & |u| \leq 1, \\ 8 - 2|u|, & 1 \leq |u| \leq 3, \end{cases} \quad F_y = -C_y \begin{cases} u(2 - |u|), & |u| \leq 1, \\ \text{sgn}(u), & 1 \leq |u| \leq 3. \end{cases}}$$

The graph values are:

$u = y_0/h$	F_x/C_x	F_y/C_y
-3	2	1
-2	4	1
-1	6	1
-1/2	6.75	0.75
0	7	0
1/2	6.75	-0.75

$u = y_0/h$	F_x/C_x	F_y/C_y
1	6	-1
2	4	-1
3	2	-1

For the horizontal-force graph, draw a downward-opening parabola between $u = -1$ and $u = 1$, with maximum $7C_x$ at the origin. Continue with straight segments to $2C_x$ at $u = \pm 3$. The force remains positive throughout the specified interval.

For the vertical-force graph, draw the constant value $+C_y$ from $u = -3$ to -1 , then the quadratic segment decreasing to zero at $u = 0$. Reflect this segment through the origin to reach $-C_y$ at $u = 1$, then continue horizontally to $u = 3$. Its slope at the origin is $-2C_y/h$. Both force graphs join with continuous slopes at $u = \pm 1$.

The vertical force is restoring: a prism below the axis receives more light on its upper face and is pushed upward; above the axis it is pushed downward.

3) Laser power required to balance gravity

The distance from the apex to the base is

$$d = h \tan \alpha.$$

The volume and mass are therefore

$$V = \frac{1}{2}(2h)d w = h^2 w \tan \alpha, \quad m = \rho h^2 w \tan \alpha.$$

Using SI units,

$$h = 10^{-5} \text{ m}, \quad w = 10^{-3} \text{ m}, \quad \rho = 2.5 \times 10^3 \text{ kg m}^{-3},$$

so

$$m = 1.4434 \times 10^{-10} \text{ kg}, \quad mg = 1.4145 \times 10^{-9} \text{ N}.$$

For $\alpha = 30^\circ$ and $n = 1.5$,

$$\beta = \arcsin \left(\frac{\sin 30^\circ}{1.5} \right) = 19.471^\circ, \\ \gamma = 10.529^\circ, \quad \theta = 15.908^\circ, \quad \sin \theta = 0.274094.$$

At $y_0 = -h/2$, we have $u = -1/2$, giving

$$F_y = \frac{3}{4}C_y = \frac{3I_0 w h \sin \theta}{16c}.$$

Vertical balance requires

$$\frac{3I_0 w h \sin \theta}{16c} = \rho h^2 w \tan \alpha g.$$

Hence

$$I_0 = \frac{16\rho g h c \tan \alpha}{3 \sin \theta} = 8.2571 \times 10^8 \text{ W m}^{-2}.$$

The required power is the power of the **whole beam**. Its thickness $80 \mu\text{m}$ equals $8h$, and the area under its triangular intensity profile is $4hI_0$. Thus

$$P = w \int_{-4h}^{4h} I(y) dy = 4hwI_0.$$

Substitution gives

$$P = 4(10^{-5})(10^{-3})(8.2571 \times 10^8) = 33.03 \text{ W}.$$

Therefore

$$\boxed{P \simeq 33 \text{ W}.}$$

This balances gravity in the y direction, as requested. The horizontal optical force remains positive.

4) Oscillation period without gravity

Throughout the stated oscillation, $|y_0| \leq h/20 < h$. The central force expression can be written

$$F_y = -\frac{I_0 w \sin \theta}{2c} y_0 \left(1 - \frac{|y_0|}{2h}\right).$$

Define

$$K = \frac{I_0 w \sin \theta}{2c}.$$

At amplitude $a = h/20$, the nonlinear factor differs from unity by at most

$$\frac{a}{2h} = \frac{1}{40} = 2.5\%.$$

The leading approximation is therefore simple harmonic motion:

$$m\ddot{y}_0 = -Ky_0, \quad \omega_0^2 = \frac{K}{m}.$$

Using the mass found above,

$$\omega_0^2 = \frac{I_0 \sin \theta}{2\rho ch^2 \tan \alpha},$$

and

$$\boxed{T_0 = 2\pi \sqrt{\frac{2\rho ch^2 \tan \alpha}{I_0 \sin \theta}}.$$

For $I_0 = 10^8 \text{ W m}^{-2}$,

$$K = 4.5682 \times 10^{-5} \text{ N m}^{-1}, \quad \omega_0 = 562.58 \text{ s}^{-1},$$

so

$$\boxed{T_0 = 0.0111685 \text{ s} \approx 11.2 \text{ ms}.}$$

The finite-amplitude correction can also be evaluated from the exact force. Taking the potential energy to vanish at the origin,

$$U(y_0) = \frac{K}{2}y_0^2 - \frac{K}{6h}|y_0|^3.$$

Release from rest at $y_0 = a$ gives

$$\frac{1}{2}m\dot{y}_0^2 = U(a) - U(y_0).$$

By symmetry, the period is four times the travel time from 0 to a :

$$T = 4\sqrt{\frac{m}{K}} \int_0^a \frac{dy}{\sqrt{a^2 - y^2 - (a^3 - y^3)/(3h)}}.$$

Setting $s = y/a$,

$$T = \frac{4}{\omega_0} \int_0^1 \frac{ds}{\sqrt{1 - s^2 - \frac{a}{3h}(1 - s^3)}}.$$

For $a/h = 1/20$, numerical evaluation yields

$$\frac{T}{T_0} = 1.01079,$$

and hence

$$T = 0.0112890 \text{ s} \approx 11.3 \text{ ms.}$$

The period is therefore about 0.011 s; the small-oscillation approximation is accurate to approximately 1.1% at the specified amplitude.

The solution was checked against the official solution.

Problem 3: Electron Beam

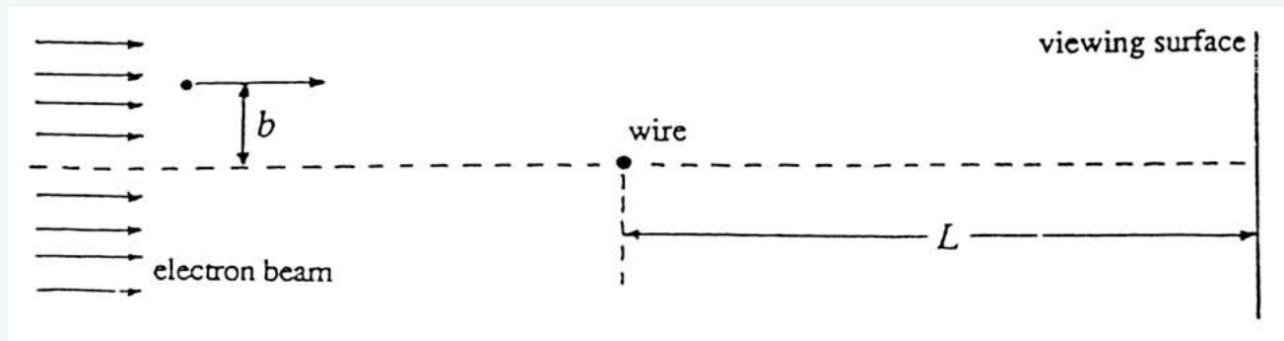
General Tabulated Information

Quantity	Symbol	Value
Earth's mean radius	R_E	$6.4 \times 10^6 \text{ m.}$
acceleration due to gravity	g	$9.8 \text{ m s}^{-2}.$
Newtonian gravitational constant	G	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}.$
permittivity of vacuum	ε_0	$8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}.$
permeability of vacuum	μ_0	$8.85 \times 10^{-7} \text{ N A}^{-2}$ [sic; correct value $4\pi \times 10^{-7} \text{ N A}^{-2}$].
speed of light in vacuum (or air)	c	$3.00 \times 10^8 \text{ m s}^{-1}.$
elementary charge	e	$1.60 \times 10^{-19} \text{ C.}$
mass of electron	m_e	$9.11 \times 10^{-31} \text{ kg.}$
mass of proton	m_p	$1.67 \times 10^{-27} \text{ kg}$
Planck constant	h	$6.63 \times 10^{-34} \text{ J s.}$
Avogadro constant	N_A	$6.02 \times 10^{23} \text{ mol}^{-1}.$
Boltzmann constant	k	$1.38 \times 10^{-23} \text{ J K}^{-1}.$
molar gas constant	R	$8.31 \text{ J mol}^{-1} \text{ K}^{-1}.$

Theoretical Problem 3

ELECTRON BEAM

An accelerating voltage V_0 produces a uniform, parallel beam of energetic electrons. The electrons pass a thin, long, positively charged copper wire stretched at right angles to the original direction of the beam, as shown in the figure. The symbol b denotes the distance at which an electron would pass the wire if the wire were uncharged. The electrons then proceed to a screen (viewing surface) a distance L ($\gg b$) beyond the wire, as shown. The beam initially extends to distances $\pm b_{\max}$ with respect to the axis of the wire. Both the width of the beam and the length of the wire may be considered infinite in the direction perpendicular to the paper.



The charged wire extends perpendicularly to the plane of the paper. The sketch is not to scale.

Some numerical data are provided here; you will find other numerical data in the table at the front of the examination:

radius of wire	$= r_0 =$	10^{-6} m
maximum value of b	$= b_{\max} =$	10^{-4} m
electric charge per unit length of wire	$= q_{\text{linear}} =$	$4.4 \times 10^{-11} \text{ C m}^{-1}$
accelerating voltage	$= V_0 =$	$2 \times 10^4 \text{ V}$
length from wire to observing screen	$= L =$	0.3 m.

Note: For parts 2 - 4, make reasonable approximations that lead to analytical and numerical solutions.

- 1) Calculate the electric field $\mathbf{E}$ produced by the wire. Sketch the magnitude of $\mathbf{E}$ as a function of distance from the axis of the wire.
- 2) Use classical physics to calculate the angular deflection of an electron. Do this for values of the parameter b such that the electron does not strike the wire. Let θ_{final} denote the (small) angle between the initial velocity of the electron and the velocity when the electron reaches the viewing surface. Hence, calculate θ_{final} .
- 3) Calculate and sketch the pattern of impacts (i.e. the intensity distribution) on the viewing screen that classical physics predicts.
- 4) Quantum physics predicts a major difference in the intensity distribution (relative to what classical physics predicts). Sketch the pattern for the quantum prediction and provide quantitative detail.

Answers

Let y be the signed transverse coordinate on the screen, with $y = 0$ directly opposite the wire, and let I_0 be the incident beam intensity.

1. The field is radially outward:

$$\mathbf{E}(r) = \begin{cases} 0, & r < r_0, \\ \frac{q_{\text{linear}}}{2\pi\epsilon_0 r} \hat{\mathbf{r}}, & r > r_0. \end{cases}$$

Its magnitude jumps from zero to $7.9 \times 10^5 \text{ V m}^{-1}$ at the surface, then decreases as $1/r$.

2. Electrons passing on either side are deflected toward the axis by approximately the same angle:

$$\theta_{\text{final}} \simeq \frac{q_{\text{linear}}}{4\epsilon_0 V_0} = 6.2 \times 10^{-5} \text{ rad.}$$

3. Including the wire's finite radius, the classical pattern is

$$I_{\text{cl}}(y) \simeq \begin{cases} 2I_0, & |y| < 17.6 \text{ } \mu\text{m}, \\ I_0, & 17.6 \text{ } \mu\text{m} < |y| < 81.4 \text{ } \mu\text{m}, \\ 0, & |y| > 81.4 \text{ } \mu\text{m}. \end{cases}$$

Thus the central doubled-intensity band is about $35 \text{ } \mu\text{m}$ wide, and the whole illuminated region is about $163 \text{ } \mu\text{m}$ wide.

4. For coherent electrons, the central overlap region contains interference fringes:

$$\lambda_{\text{dB}} \simeq 8.7 \times 10^{-12} \text{ m}, \quad d \simeq \frac{\lambda_{\text{dB}}}{2\theta_{\text{final}}} = 7.0 \times 10^{-8} \text{ m} = 70 \text{ nm.}$$

Away from the envelope edges,

$$I_{\text{q}}(y) \simeq 4I_0 \cos^2\left(\frac{\pi y}{d}\right)$$

in the overlap region. The centre is bright; successive maxima reach $4I_0$, and minima reach zero. There are roughly 5.1×10^2 fringe periods across the overlap region. The wings remain approximately at I_0 .

Solution

1) Electric field

Copper is a conductor, so its electrostatic field vanishes inside the wire. Outside an effectively infinite wire, cylindrical symmetry requires a radial field depending only on the distance r from its axis.

Apply Gauss's law to a coaxial cylindrical surface of radius $r > r_0$ and length ℓ :

$$E(r)(2\pi r\ell) = \frac{q_{\text{linear}}\ell}{\epsilon_0}.$$

Therefore,

$$\mathbf{E}(r) = \frac{q_{\text{linear}}}{2\pi\epsilon_0 r} \hat{\mathbf{r}}.$$

The direction is outward because the wire is positively charged.

Numerically,

$$\frac{q_{\text{linear}}}{2\pi\epsilon_0} = \frac{4.4 \times 10^{-11}}{2\pi(8.85 \times 10^{-12})} = 0.791 \text{ V},$$

so

$$E(r_0^+) = \frac{0.791}{10^{-6}} \simeq 7.9 \times 10^5 \text{ V m}^{-1}.$$

The requested graph lies on the horizontal axis for $0 \leq r < r_0$, jumps upward at $r = r_0$, and follows a decreasing $1/r$ curve thereafter, approaching zero asymptotically. For example,

$$E(10r_0) = 7.9 \times 10^4 \text{ V m}^{-1}, \quad E(100r_0) = 7.9 \times 10^3 \text{ V m}^{-1}.$$

2) Angular deflection

Take the x -axis along the incident beam and the y -axis perpendicular to it in the plane of the figure. The wire passes through $x = y = 0$.

Use the nonrelativistic energy relation:

$$\frac{1}{2}m_e v^2 = eV_0, \quad v = \sqrt{\frac{2eV_0}{m_e}} \simeq 8.4 \times 10^7 \text{ m s}^{-1}.$$

Here $eV_0/(m_e c^2) \simeq 0.039$, so the nonrelativistic treatment is a reasonable approximation for this problem.

We calculate the small transverse impulse along the undeflected path $y = b$, taking v as constant. This is consistent with the weak wire potential: the potential difference between r_0 and L is only

$$\Phi(r_0) - \Phi(L) = \frac{q_{\text{linear}}}{2\pi\epsilon_0} \ln\left(\frac{L}{r_0}\right) \simeq 10 \text{ V},$$

much smaller than $V_0 = 2 \times 10^4 \text{ V}$.

For an electron passing above the wire, $b > 0$, the transverse force is

$$F_y = -eE_y = -\frac{eq_{\text{linear}}}{2\pi\epsilon_0} \frac{b}{x^2 + b^2}.$$

Since $dt = dx/v$, the transverse momentum change by the time the electron reaches $x = L$ is approximately

$$\begin{aligned} \Delta p_y &= \frac{1}{v} \int_{-\infty}^L F_y dx \\ &= -\frac{eq_{\text{linear}}}{2\pi\epsilon_0 v} \left[\frac{\pi}{2} + \tan^{-1}\left(\frac{L}{b}\right) \right]. \end{aligned}$$

The upstream limit assumes that the initial beam is specified sufficiently far from the wire compared with b .

For a small deflection,

$$\theta_{\text{final}} \simeq \frac{|\Delta p_y|}{m_e v}.$$

Using $m_e v^2 = 2eV_0$,

$$\theta_{\text{final}}(b) \simeq \frac{q_{\text{linear}}}{4\pi\epsilon_0 V_0} \left[\frac{\pi}{2} + \tan^{-1} \left(\frac{L}{b} \right) \right].$$

Because $L \gg b$, the bracket is approximately π , giving

$$\boxed{\theta_{\text{final}} \simeq \frac{q_{\text{linear}}}{4\epsilon_0 V_0}.$$

Thus, to this accuracy, the deflection is independent of the impact parameter:

$$\theta_{\text{final}} = \frac{4.4 \times 10^{-11}}{4(8.85 \times 10^{-12})(2 \times 10^4)} \simeq 6.2 \times 10^{-5} \text{ rad.}$$

Even at $b = b_{\text{max}}$, the finite-screen correction in the bracket is only about 10^{-4} in relative magnitude.

The upper half of the beam is deflected downward; the lower half is deflected upward. Within the same approximation, transmitted electrons have $b > r_0$. The small change in the collision cutoff caused by attraction is neglected.

3) Classical intensity distribution

Write

$$\alpha = \theta_{\text{final}}, \quad a = L\alpha \simeq 18.6 \mu\text{m}.$$

Since $L \gg b_{\text{max}}$, the deflection can be treated as occurring near the wire. An electron incident a distance b above the axis therefore reaches

$$y = b - a,$$

whereas one incident the same distance below reaches

$$y = -b + a.$$

Only $r_0 < b < b_{\text{max}}$ is transmitted. Consequently, the two transmitted portions occupy the following screen intervals:

$$\text{upper incident portion: } r_0 - a < y < b_{\text{max}} - a,$$

$$\text{lower incident portion: } -b_{\text{max}} + a < y < a - r_0.$$

Numerically, these are

$$-17.6 < y < 81.4 \mu\text{m}$$

and

$$-81.4 < y < 17.6 \mu\text{m},$$

respectively.

For each portion, the mapping has $|dy/db| = 1$. Conservation of electron flux therefore gives an intensity I_0 wherever one portion arrives, and $2I_0$ wherever both arrive.

Define

$$w = a - r_0 \simeq 17.6 \mu\text{m}, \quad Y = b_{\text{max}} - a \simeq 81.4 \mu\text{m}.$$

Then

$$I_{\text{cl}}(y) = \begin{cases} 2I_0, & |y| < w, \\ I_0, & w < |y| < Y, \\ 0, & |y| > Y. \end{cases}$$

The sketch is symmetric about $y = 0$. Starting from the left, the intensity jumps from zero to I_0 at $-81.4 \mu\text{m}$, rises to $2I_0$ at $-17.6 \mu\text{m}$, drops to I_0 at $+17.6 \mu\text{m}$, and returns to zero at $+81.4 \mu\text{m}$. The bands extend uniformly parallel to the wire.

Their widths are

$$\begin{aligned} \text{central overlap width} &= 2(a - r_0) \simeq 35.3 \mu\text{m}, \\ \text{each wing width} &= b_{\text{max}} - 2a + r_0 \simeq 63.7 \mu\text{m}. \end{aligned}$$

There is no central geometrical shadow: the two deflected portions overlap.

As a flux check,

$$\int I_{\text{cl}}(y) dy = 2I_0(b_{\text{max}} - r_0),$$

which is exactly the incident flux excluding the electrons intercepted by the wire in this approximation.

4) Quantum intensity distribution

For an ideal monoenergetic, coherent incident beam, the portions passing on opposite sides of the wire interfere where they overlap.

The electron de Broglie wavelength is

$$\lambda_{\text{dB}} = \frac{h}{p} = \frac{h}{\sqrt{2m_e e V_0}},$$

so

$$\lambda_{\text{dB}} = \frac{6.63 \times 10^{-34}}{\sqrt{2(9.11 \times 10^{-31})(1.60 \times 10^{-19})(2 \times 10^4)}} \simeq 8.7 \times 10^{-12} \text{ m}.$$

Within the overlap region, the two approximately plane waves propagate at angles $+\alpha$ and $-\alpha$ to the original direction. Their transverse wave-vector components are therefore $+k \sin \alpha$ and $-k \sin \alpha$, where

$$k = \frac{2\pi}{\lambda_{\text{dB}}}.$$

Reflection symmetry makes their phases equal at $y = 0$. Their phase difference at screen coordinate y is

$$\Delta\phi(y) = 2ky \sin \alpha.$$

Each wave separately contributes intensity I_0 , so

$$\begin{aligned} I_{\text{q}}(y) &= I_0 + I_0 + 2I_0 \cos \Delta\phi \\ &= 4I_0 \cos^2(ky \sin \alpha). \end{aligned}$$

The fringe spacing is thus

$$d = \frac{\lambda_{\text{dB}}}{2 \sin \alpha} \simeq \frac{\lambda_{\text{dB}}}{2\alpha},$$

giving

$$d \simeq 6.99 \times 10^{-8} \text{ m} \simeq 70 \text{ nm.}$$

To geometrical-envelope accuracy,

$$I_q(y) \simeq \begin{cases} 4I_0 \cos^2\left(\frac{\pi y}{d}\right), & |y| < w, \\ I_0, & w < |y| < Y, \\ 0, & |y| > Y. \end{cases}$$

The requested sketch has the same broad envelope as the classical pattern, but the central plateau is replaced by closely spaced fringes:

- Bright fringes occur at $y = nd$, with intensity $4I_0$; n is an integer.
- Dark fringes occur at $y = (n + \frac{1}{2})d$, with zero intensity.
- The central fringe at $y = 0$ is bright.
- The fringes are straight and parallel to the wire.
- Their mean intensity is $2I_0$, equal to the classical central intensity.

The approximate number of fringe periods across the overlap region is

$$N \simeq \frac{2w}{d} = \frac{35.3 \times 10^{-6}}{6.99 \times 10^{-8}} \simeq 5.1 \times 10^2.$$

Diffraction rounds the envelope boundaries, so the piecewise expression is intended away from those edges. The relevant transverse diffraction scale is of order

$$\sqrt{\lambda_{\text{dB}} L} \simeq 1.6 \text{ } \mu\text{m}.$$

A detector that cannot resolve the 70 nm fringes measures their average and approximately recovers the classical distribution.

The solution was checked against the official solution.