

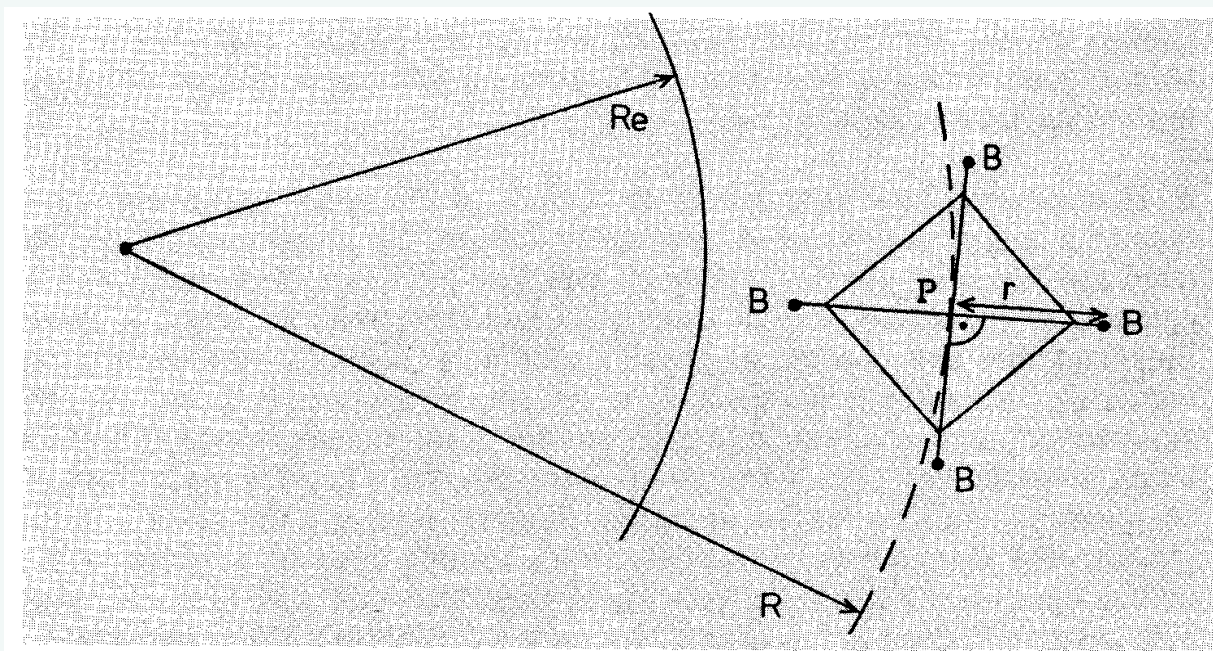
IPhO 1992

Theory problems and solutions · Espoo (Helsinki), Finland · 3 theory problems with complete solutions

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Problem 1: A Rotating Satellite

The figure shows a satellite which is circling the Earth in an approximately circular orbit in the Earth's equatorial plane. The satellite consists of a massless central body P and four small peripheral bodies B . The four bodies B each have mass m : they are fastened to P by means of long thin wires of length r that do not stretch. All these five bodies, P and the four bodies B , are coplanar with the equatorial plane, and they can rotate within this plane. The four radial wires are linked to each other by further thin wires which keep the angles between the radial wires constant at 90° .



The link wires are included in the system in order to prevent oscillatory movement of the individual bodies B which would otherwise make the analysis of the movements extremely complicated. All the bodies B rotate around P at the same angular velocity, which is ω with respect to the fixed stars. Thus, the satellite behaves as a rigid body.

Analyze the following questions for the general case, considering all possible situations, including both senses of rotation of the bodies B . Also obtain numerical values for certain of the quantities found in questions (1) and (2)—the quantities required and the necessary numerical data are listed at the end of the problem.

1) The drawing shows the satellite in the position where for the various wires, $\mathbf{r}$ is parallel, anti-parallel or perpendicular to $\mathbf{R}$. (The vector $\mathbf{r}$ runs from body P to a body B and has length r ; the vector $\mathbf{R}$ runs from the centre of mass of the Earth to the body P.)

Determine the force exerted by a radial wire on one of the bodies B in each of these four positions. These positions correspond approximately to the maximum and minimum forces.

2) Inside the four bodies B there are four identical machines, powered by solar energy, connected to the radial wires. Each machine pulls the wire in, towards B, for a short time whenever there is near maximum force in the wire (as indicated in the previous question), and lets the same length of wire out again when the tension is at a minimum. The length of wire that is pulled in and let out is 1% of the mean length of the radial wire. The mean length does not change with time.

What is the net power converted by one machine, averaged over one rotation of the satellite?

The net power is defined as $\frac{W_1 - W_2}{T}$, where W_1 is the work that the machine performs on the wire when pulling it in, W_2 is the work that the wire performs on the machine when it is reeled out and T is the period of rotation.

3) Discuss the changes in the motion of the satellite that are caused by the action of the machines. In particular, analyze any changes that may occur in each of the situations listed in the table overleaf.

Fill in the table with your results and comments, and don't forget to hand it in.

Data:

Numerical answers are required in the following situation:

The radius of the orbit of the central body is given by $R = R_E + 500 \text{ km}$.

The mean length of the radial wires is $r = 100 \text{ km}$.

Thus the diameter of the satellite system is 200 km .

The bodies B have masses $m = 1000 \text{ kg}$.

Initially the four bodies B rotate, as referred to the stars, around the central body P at 10 revolutions/hour.

The masses of the wires are negligible and the central body P is massless.

Advice:

Consider both senses of rotation for ω .

Exact solutions are not expected. Results with 5% accuracy are fully acceptable.

Ignore the gravitational effect of the moon and the sun.

Useful data:

Quantity	Value
Mass of Earth	$M_E = 5.97 \times 10^{24} \text{ kg}$
Gravitational constant	$G = 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Radius of Earth at the equator	$R_E = 6378 \text{ km}$
Denote the product $M_E G$ by K .	$K = 3.983 \times 10^{14} \text{ m}^3 \text{ s}^{-2}$

Country code:

ANSWER TABLE

Fill in this table as part of your answer. Write down equalities or inequalities and/or short explanations where necessary.

The quantity indicated below ...	increases if ...	decreases if ...	stays unchanged if ...	stays unchanged in all situations.
orbital velocity of the satellite				Yes ☐ No ☐
radius R of the orbit of the satellite				Yes ☐ No ☐
angular velocity ω of the satellite				Yes ☐ No ☐
gravitational potential energy of the satellite				Yes ☐ No ☐

Could the satellite reach a higher orbit as a result of the work done by the machines?

Yes ☐

No ☐

Could the satellite reach an arbitrarily high orbit, practically leaving the gravitational influence of Earth? Why?

Answer:

Official English text of the problem.

Answers

Take the orbital direction as positive, and define

$$\Omega = \sqrt{\frac{K}{R^3}} > 0, \quad \epsilon = \frac{\Delta r}{r} = 0.01.$$

The spin ω is signed: positive for prograde rotation and negative for retrograde rotation.

1. To leading order in r/R , the wire forces, directed towards P, have magnitudes

$$F_{\parallel} = F_{\text{anti}} \simeq mr(\omega^2 + 2\Omega^2)$$

and

$$F_{\perp,1} = F_{\perp,2} \simeq mr(\omega^2 - \Omega^2).$$

For either specified spin direction,

$$F_{\max} \simeq 30.7 \text{ kN}, \quad F_{\min} \simeq 30.3 \text{ kN}, \quad F_{\max} - F_{\min} \simeq 367 \text{ N}.$$

A complete rotation with taut flexible wires requires $|\omega| \geq \Omega$, to this accuracy.

2. The mean net power supplied by **one machine** is

$$\overline{P}_1 = \frac{3\epsilon m r^2 \Omega^2}{\pi} |\omega - \Omega|.$$

Numerically,

$$\overline{P}_1 = 1.91 \text{ kW (prograde)}, \quad \overline{P}_1 = 2.17 \text{ kW (retrograde)}.$$

At synchronous rotation, $\omega = \Omega$, there is no repeated pumping cycle and the mean power is zero.

3. The completed table for the secular rigid-body model is:

Quantity	Increases if	Decreases if	Stays unchanged if	Unchanged in all situations?
Orbital speed v	$\omega > \Omega$	$\omega < \Omega$	$\omega = \Omega$	No
Orbital radius R	$\omega < \Omega$	$\omega > \Omega$	$\omega = \Omega$	No
Signed spin angular velocity ω	$\omega > \Omega$	$\omega < \Omega$	$\omega = \Omega$	No
Gravitational potential energy U	$\omega < \Omega$	$\omega > \Omega$	$\omega = \Omega$	No

Thus retrograde rotation becomes faster in magnitude. Formally, a prograde spin with $0 < \omega < \Omega$ slows and subsequently reverses; flexible-wire feasibility in this interval is qualified below.

Higher orbit: Yes, on the ascending branch, in particular with sufficiently rapid retrograde rotation.

Arbitrarily high orbit: Yes within the ideal Newtonian model. Increasingly negative spin angular momentum compensates the increasing orbital angular momentum. There is no finite upper radius, although reaching infinity would require infinite time and unbounded spin energy.

Solution

We use the small parameters

$$\frac{r}{R} = \frac{100}{6878} \simeq 0.0145, \quad \epsilon = 0.01.$$

We retain the leading gravitational-gradient forces and the leading work in the small wire stroke. The orbit is treated as circular on each pumping cycle, with its radius and the mean spin changing slowly. The mean moment of inertia is

$$I \simeq 4mr^2.$$

1) Forces in the four radial wires

Let $\mathbf{e}_R$ point from Earth towards P, and let $\boldsymbol{\rho} = r\mathbf{e}$ locate a peripheral body relative to P. Expanding Earth's gravitational acceleration about P gives

$$\mathbf{g}(\mathbf{R} + \boldsymbol{\rho}) - \mathbf{g}(\mathbf{R}) \simeq \frac{K}{R^3} [3(\boldsymbol{\rho} \cdot \mathbf{e}_R)\mathbf{e}_R - \boldsymbol{\rho}].$$

This is the tidal acceleration.

At the four specified positions, this acceleration is radial relative to P, to the retained order. If θ is the angle between $\mathbf{e}$ and $\mathbf{e}_R$, its radial component is

$$\delta g_r = \Omega^2 r (3 \cos^2 \theta - 1).$$

The peripheral body needs radial acceleration $-\omega^2 r$ relative to P. Taking outward from P as positive, Newton's second law gives

$$-m\omega^2 r = m\delta g_r - F,$$

where F is the inward wire force. Hence

$$F = mr [\omega^2 + \Omega^2 (3 \cos^2 \theta - 1)].$$

For the outward and inward bodies, $\cos^2 \theta = 1$:

$$F_{\parallel} = F_{\text{anti}} \simeq mr(\omega^2 + 2\Omega^2).$$

Earth's differential attraction tends to separate these bodies from P, increasing the required tension.

For the two transverse bodies, $\cos^2 \theta = 0$:

$$F_{\perp,1} = F_{\perp,2} \simeq mr(\omega^2 - \Omega^2).$$

Here the tidal force points towards P and supplies some of the centripetal force.

Consequently,

$$\Delta F = F_{\text{max}} - F_{\text{min}} = 3mr\Omega^2 = \frac{3Kmr}{R^3}.$$

The small difference between the nearer and farther bodies is beyond this first-order approximation.

Using the supplied data,

$$R = 6.878 \times 10^6 \text{ m}, \quad \Omega = \sqrt{\frac{3.983 \times 10^{14}}{(6.878 \times 10^6)^3}} = 1.1064 \times 10^{-3} \text{ rad s}^{-1},$$

and

$$|\omega| = \frac{10(2\pi)}{3600} = 1.74533 \times 10^{-2} \text{ rad s}^{-1}.$$

The two force scales are

$$mr\omega^2 = 3.04617 \times 10^4 \text{ N}, \quad mr\Omega^2 = 122.412 \text{ N}.$$

Therefore,

$$F_{\text{max}} \simeq 3.07066 \times 10^4 \text{ N}, \quad F_{\text{min}} \simeq 3.03393 \times 10^4 \text{ N},$$

giving the rounded answers 30.7 kN and 30.3 kN.

The sense of rotation does not affect these forces, since they depend on ω^2 .

A physical restriction follows from $F_{\text{min}} \geq 0$:

$$|\omega| \geq \Omega.$$

If $|\omega| < \Omega$, the transverse position would require a compressive radial force. A flexible wire cannot provide it, so the assumed rigid rotation cannot persist through a full cycle.

2) Net power of one machine

The stroke is

$$\Delta r = \epsilon r = 1000 \text{ m.}$$

To first order in this stroke, the work supplied during one inward stroke and recovered during the following outward stroke are

$$W_{\text{in}} \simeq F_{\text{max}} \Delta r, \quad W_{\text{out}} \simeq F_{\text{min}} \Delta r.$$

Thus one in–out cycle supplies

$$\Delta W = (F_{\text{max}} - F_{\text{min}}) \Delta r = 3\epsilon m r^2 \Omega^2.$$

The centrifugal contributions cancel to this order. Numerically,

$$\Delta W \simeq 3.67236 \times 10^5 \text{ J.}$$

The Earth–satellite line rotates at Ω relative to the stars. Therefore the wire's orientation relative to that line changes at

$$\dot{\theta} = \omega - \Omega.$$

Each wire encounters **two maxima and two minima per relative revolution**. The relative-revolution period is

$$T_{\text{rel}} = \frac{2\pi}{|\omega - \Omega|},$$

so the net work of one machine during that period is $2\Delta W$. Hence

$$\boxed{\bar{P}_1 = \frac{2\Delta W}{T_{\text{rel}}} = \frac{3\epsilon m r^2 \Omega^2}{\pi} |\omega - \Omega|}.$$

For prograde rotation, $\omega = |\omega|$:

$$T_{\text{rel}} = 384.37 \text{ s}, \quad \bar{P}_1 = \frac{2(3.67236 \times 10^5)}{384.37} \simeq \boxed{1.91 \text{ kW}}.$$

For retrograde rotation, $\omega = -|\omega|$:

$$T_{\text{rel}} = \frac{2\pi}{|\omega| + \Omega} = 338.54 \text{ s},$$

$$\bar{P}_1 \simeq \boxed{2.17 \text{ kW}}.$$

At $\omega = \Omega$, the orientation relative to Earth is fixed. There are no repeated passages through tension maxima and minima, so the sustained mean power is zero.

3) Changes in the satellite's motion

Write the total satellite mass as $M = 4m$. To the retained accuracy, its orbital speed, angular momentum, mechanical energy and gravitational potential energy are

$$v = \sqrt{\frac{K}{R}},$$

$$L = M\sqrt{KR} + I\omega,$$

$$E = -\frac{KM}{2R} + \frac{1}{2}I\omega^2, \quad U = -\frac{KM}{R}.$$

Each gravitational force points towards Earth's centre and therefore exerts no torque about that centre. The machines exert internal forces. Thus, neglecting radiation pressure,

$$\dot{L} = 0.$$

Differentiating,

$$\frac{M}{2} \sqrt{\frac{K}{R}} \dot{R} + I\dot{\omega} = 0,$$

or

$$\dot{R} = -\frac{2I}{MR\Omega} \dot{\omega} = -\frac{2r^2}{R\Omega} \dot{\omega}.$$

An increase in signed spin angular momentum therefore reduces the orbital radius.

The four machines supply total mechanical power $P_{\text{tot}} = 4\bar{P}_1$. Differentiating the energy and substituting angular-momentum conservation gives

$$P_{\text{tot}} = \frac{KM}{2R^2} \dot{R} + I\omega\dot{\omega} = I(\omega - \Omega)\dot{\omega}.$$

Thus

$$P_{\text{tot}} = I(\omega - \Omega)\dot{\omega}.$$

Since the operating machines supply positive net power,

$$\text{sgn}(\dot{\omega}) = \text{sgn}(\omega - \Omega).$$

More explicitly, the cycle-averaged rates are

$$\dot{\omega} = \frac{3\epsilon\Omega^2}{\pi} \text{sgn}(\omega - \Omega)$$

and

$$\dot{R} = -\frac{6\epsilon r^2 \Omega}{\pi R} \text{sgn}(\omega - \Omega).$$

These relations establish every entry in the answer table:

- **Faster prograde rotation**, $\omega > \Omega$: the spin increases, the orbit contracts, the orbital speed increases, and the gravitational potential energy decreases.
- **Retrograde rotation**, $\omega < 0$: the signed spin decreases, so its retrograde magnitude increases. The orbit expands, the orbital speed decreases, and the gravitational potential energy increases.
- **Formally**, $0 < \omega < \Omega$: the prograde spin slows while the orbit expands. Continued operation would reverse the spin. This interval is subject to the flexible-wire limitation found in part (1).
- **Synchronous rotation**, $\omega = \Omega$: there is no repeated actuation, hence no secular change caused by the machines.

In particular, the orbital speed and potential-energy signs follow directly from

$$\dot{v} = -\frac{v}{2R} \dot{R}, \quad \dot{U} = \frac{KM}{R^2} \dot{R}.$$

To interpret the table's angular-velocity row as an unsigned rotation rate: $|\omega|$ increases for both faster prograde rotation and retrograde rotation; it decreases for the formal slower-prograde case until the spin reverses.

Possibility of reaching higher and arbitrarily high orbits

A sufficiently rapidly counter-rotating satellite has $\omega < -\Omega$, so its wires remain in tension and $\dot{R} > 0$. It can therefore reach a higher orbit.

Angular-momentum conservation does not impose a finite upper radius. Between the initial state and a later circular orbit,

$$M\sqrt{KR} + I\omega = M\sqrt{KR_0} + I\omega_0.$$

Since $I/M = r^2$,

$$\omega(R) = \omega_0 - \frac{\sqrt{K}}{r^2} (\sqrt{R} - \sqrt{R_0}).$$

As R grows, the increasingly positive orbital angular momentum is compensated by increasingly negative spin angular momentum. The solar-powered machines supply the required energy.

The rate of ascent becomes progressively smaller. On the ascending branch,

$$\dot{R} = \frac{6\epsilon r^2 \sqrt{K}}{\pi} R^{-5/2},$$

which integrates to

$$R(t)^{7/2} = R_0^{7/2} + \frac{21\epsilon r^2 \sqrt{K}}{\pi} t.$$

Every finite target radius is reached after a finite time in this ideal model, but $R \rightarrow \infty$ requires $t \rightarrow \infty$. Moreover, the required spin energy grows without bound:

$$\frac{1}{2}I\omega^2 \sim \frac{2mK}{r^2}R \quad (R \rightarrow \infty).$$

Thus **arbitrarily high finite orbits are possible under the stated idealizations**, while infinite-distance escape is only an asymptotic limit of this circular-orbit pumping process.

The solution was checked against the official solution.

Problem 2: The Longitudinal Motion of a Linear Molecule

In this problem you will analyze the longitudinal motion of a linear molecule, i.e., the motion along the molecular axis. The rotational motion and the bending of the molecule are not considered. The molecule is assumed to consist of N atoms of mass $m_1, m_2, \dots, m_N$, respectively. Each atom is assumed to be connected to its neighbors by a chemical bond. Each bond is approximated by a massless spring which obeys Hooke's law with spring constants $k_1, k_2, \dots, k_{N-1}$. The molecule is shown in Fig. 1.

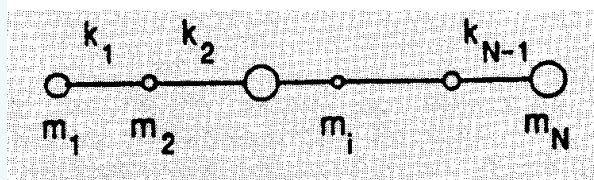


Fig.1. A linear molecule with N atoms.

Use the following facts when solving this problem: The longitudinal vibrational motion of a linear molecule consists of a superposition of separate vibrational motions called normal vibrations, or normal modes. In a normal mode all atoms vibrate in simple harmonic motion with the same frequency and pass through their equilibrium positions simultaneously.

Questions

1) Let x_i be the displacement of atom i from its equilibrium position. Express the force F_i acting on each atom as a function of the displacements $x_1, x_2, \dots, x_N$ and the spring constants $k_1, k_2, \dots, k_{N-1}$. What relationship is there among the forces $F_1, F_2, \dots, F_N$? Using this relationship, derive a relationship between the displacements $x_1, x_2, \dots, x_N$ and give a physical interpretation of this relationship.

2) Analyze the motion of a diatomic molecule AB (Fig. 2). The value of the spring constant is k . Derive an expression for the forces acting on atoms A and B. Determine the possible types of motion of the molecule. Determine the corresponding vibrational frequencies and interpret the result. In particular, how is it possible for the atoms to vibrate with the same frequency even though their masses are not the same?

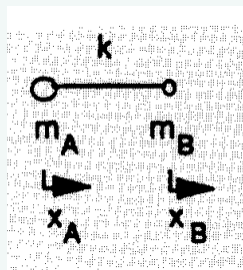


Fig.2. The diatomic molecule AB

3) Analyze the motion of the triatomic molecule BA_2 (Fig. 3)

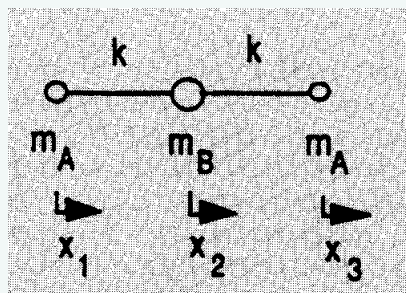


Fig.3. The triatomic molecule BA_2 .

Express the net force on each atom as a function of its displacement only. Deduce the possible motions of the molecule and the corresponding vibrational frequencies.

4) The frequencies of the two longitudinal modes of vibration of the CO_2 molecule are $3.998 \times 10^{13} \text{ Hz}$ and $7.042 \times 10^{13} \text{ Hz}$, respectively. Determine a numerical value for the spring constant of the CO bond.

How well do you think this approximation for the bond structure of the molecule describes the vibrational motion of the real molecule?

The atomic mass of the carbon atom = 12 amu and that of the oxygen atom = 16 amu. The atomic mass unit = $1.660 \times 10^{-27} \text{ kg}$.

Official English text of the problem.

Answers

1. The forces are

$$\begin{aligned} F_1 &= k_1(x_2 - x_1), \\ F_i &= k_i(x_{i+1} - x_i) - k_{i-1}(x_i - x_{i-1}), \quad 2 \leq i \leq N-1, \\ F_N &= -k_{N-1}(x_N - x_{N-1}). \end{aligned}$$

They satisfy $\sum_i F_i = 0$, hence

$$\sum_i m_i x_i = C_0 + C_1 t.$$

The center of mass moves uniformly. For internal vibrations measured in its rest frame,

$$\sum_i m_i x_i = 0.$$

2. For AB,

$$F_A = k(x_B - x_A), \quad F_B = k(x_A - x_B).$$

There is rigid translation with zero frequency and one internal vibrational mode:

$$f = \frac{1}{2\pi} \sqrt{k \left(\frac{1}{m_A} + \frac{1}{m_B} \right)} = \frac{1}{2\pi} \sqrt{\frac{k}{\mu}}, \quad \mu = \frac{m_A m_B}{m_A + m_B}.$$

In this vibration, $m_A x_A = -m_B x_B$: the atoms move in opposite directions, with amplitudes inversely proportional to their masses.

3. For BA_2 , the normal modes are:

Motion	Displacement ratios $x_1 : x_2 : x_3$	Frequency
Rigid translation	1 : 1 : 1	0
Symmetric stretch	1 : 0 : -1	$f_s = \frac{1}{2\pi} \sqrt{\frac{k}{m_A}}$
Asymmetric stretch	$1 : -\frac{2m_A}{m_B} : 1$	$f_a = \frac{1}{2\pi} \sqrt{k \left(\frac{1}{m_A} + \frac{2}{m_B} \right)}$

In the symmetric stretch,

$$F_1 = -kx_1, \quad F_2 = 0, \quad F_3 = -kx_3.$$

In the asymmetric stretch,

$$\begin{aligned} F_1 &= -k \left(1 + \frac{2m_A}{m_B} \right) x_1, \\ F_2 &= -k \left(2 + \frac{m_B}{m_A} \right) x_2, \\ F_3 &= -k \left(1 + \frac{2m_A}{m_B} \right) x_3. \end{aligned}$$

General motion is a superposition of the two vibrations and uniform translation.

4. The two measured frequencies separately give

$$k_s = 1.676 \times 10^3 \text{ N m}^{-1}, \quad k_a = 1.418 \times 10^3 \text{ N m}^{-1}.$$

Thus a representative approximate bond stiffness is $k \approx 1.5 \times 10^3 \text{ N m}^{-1}$. No single k fits both frequencies exactly: the predicted frequency ratio is 1.915, whereas the observed ratio is 1.761. The model gives a reasonable approximation, with a discrepancy of about 9% in this ratio.

Solution

1) Forces and the center of mass

Take the same positive direction along the molecular axis for every displacement. The change in length of the spring joining atoms i and $i + 1$ is

$$\Delta \ell_i = x_{i+1} - x_i.$$

The potential energy of the ideal Hookean springs is therefore

$$U = \frac{1}{2} \sum_{i=1}^{N-1} k_i (x_{i+1} - x_i)^2.$$

Using $F_i = -\partial U / \partial x_i$ gives

$$\begin{aligned} F_1 &= k_1(x_2 - x_1), \\ F_i &= k_i(x_{i+1} - x_i) - k_{i-1}(x_i - x_{i-1}), \quad 2 \leq i \leq N-1, \end{aligned}$$

and

$$F_N = -k_{N-1}(x_N - x_{N-1}).$$

Each spring exerts equal and opposite forces on its two ends. Consequently,

$$\sum_{i=1}^N F_i = 0.$$

Newton's second law then gives

$$\frac{d^2}{dt^2} \left(\sum_i m_i x_i \right) = 0,$$

so

$$\sum_i m_i x_i = C_0 + C_1 t.$$

Writing $M_{\text{tot}} = \sum_i m_i$, the center-of-mass displacement is

$$X_{\text{CM}} = \frac{\sum_i m_i x_i}{M_{\text{tot}}}.$$

Thus the center of mass has constant velocity, as expected for an isolated system with no external force.

For a nonzero-frequency normal mode,

$$x_i = a_i \cos(\omega t + \phi).$$

Since $\ddot{x}_i = -\omega^2 x_i$,

$$0 = \sum_i m_i \ddot{x}_i = -\omega^2 \sum_i m_i x_i.$$

Hence

$$\boxed{\sum_i m_i x_i = 0 \quad (\omega \neq 0).}$$

Internal vibration leaves the center of mass fixed. Uniform translation can be superposed independently.

2) The diatomic molecule AB

The spring extension is $x_B - x_A$, so

$$m_A \ddot{x}_A = k(x_B - x_A), \quad m_B \ddot{x}_B = -k(x_B - x_A).$$

Introduce the center-of-mass displacement and the relative displacement:

$$X = \frac{m_A x_A + m_B x_B}{m_A + m_B}, \quad q = x_B - x_A.$$

Adding the equations of motion gives

$$\ddot{X} = 0.$$

Subtracting the accelerations gives

$$\ddot{q} = -k \left(\frac{1}{m_A} + \frac{1}{m_B} \right) q.$$

Therefore

$$X = X_0 + Vt, \quad q = Q \cos(\omega t + \phi),$$

where

$$\boxed{\omega^2 = k \left(\frac{1}{m_A} + \frac{1}{m_B} \right) = \frac{k}{\mu}}, \quad \mu = \frac{m_A m_B}{m_A + m_B}.$$

The complete motion is

$$\begin{aligned} x_A &= X_0 + Vt - \frac{m_B}{m_A + m_B} Q \cos(\omega t + \phi), \\ x_B &= X_0 + Vt + \frac{m_A}{m_A + m_B} Q \cos(\omega t + \phi). \end{aligned}$$

There are two independent types of motion:

- **Rigid translation:** $q = 0$, so $x_A = x_B = X_0 + Vt$. The spring is undeformed and both forces vanish. This is the zero-frequency mode.
- **Internal vibration:** in the center-of-mass rest frame,

$$m_A x_A + m_B x_B = 0.$$

The atoms oscillate in opposite phase with frequency

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{\mu}}.$$

To understand the equal frequencies despite unequal masses, eliminate the other atom's displacement:

$$x_B = -\frac{m_A}{m_B} x_A, \quad x_A = -\frac{m_B}{m_A} x_B.$$

Then

$$F_A = -k \left(1 + \frac{m_A}{m_B} \right) x_A, \quad F_B = -k \left(1 + \frac{m_B}{m_A} \right) x_B.$$

Dividing each restoring-force coefficient by the corresponding mass gives the same value:

$$\frac{k(1 + m_A/m_B)}{m_A} = \frac{k(1 + m_B/m_A)}{m_B} = k \left(\frac{1}{m_A} + \frac{1}{m_B} \right).$$

Thus both atoms have the same angular frequency. Their unequal masses are accommodated by unequal amplitudes:

$$\frac{|x_A|_{\max}}{|x_B|_{\max}} = \frac{m_B}{m_A}.$$

The heavier atom has both the smaller displacement amplitude and the smaller acceleration amplitude.

3) The triatomic molecule BA₂

Write $m = m_A$ and $M = m_B$. Hooke's law gives

$$\begin{aligned} m\ddot{x}_1 &= k(x_2 - x_1), \\ M\ddot{x}_2 &= k(x_1 + x_3 - 2x_2), \\ m\ddot{x}_3 &= k(x_2 - x_3). \end{aligned}$$

Rigid translation, $x_1 = x_2 = x_3 = X_0 + Vt$, makes all three forces zero.

For internal vibration, work in the center-of-mass rest frame:

$$m(x_1 + x_3) + Mx_2 = 0.$$

Define the average and half-difference of the endpoint displacements:

$$s = \frac{x_1 + x_3}{2}, \quad d = \frac{x_1 - x_3}{2}.$$

Then

$$x_1 = s + d, \quad x_3 = s - d, \quad x_2 = -\frac{2m}{M}s.$$

Adding and subtracting the endpoint equations gives

$$m\ddot{s} = k(x_2 - s) = -k\left(1 + \frac{2m}{M}\right)s,$$

$$m\ddot{d} = -kd.$$

These are two independent harmonic oscillators, so they identify all internal normal modes.

Symmetric stretching mode. Set $s = 0$. Then

$$x_2 = 0, \quad x_1 = d, \quad x_3 = -d.$$

The middle atom remains stationary, while the outer atoms move with equal amplitudes in opposite directions. Both bonds lengthen together and shorten together.

The forces, expressed using each atom's own displacement, are

$$F_1 = -kx_1, \quad F_2 = 0, \quad F_3 = -kx_3.$$

Therefore

$$\boxed{\omega_s^2 = \frac{k}{m_A}, \quad f_s = \frac{1}{2\pi} \sqrt{\frac{k}{m_A}}.}$$

The central mass does not enter this frequency because the two spring forces on it cancel throughout the motion.

Asymmetric stretching mode. Set $d = 0$. Then

$$x_1 = x_3 = s, \quad x_2 = -\frac{2m}{M}s.$$

The outer atoms move together, and the middle atom moves oppositely. One bond lengthens while the other shortens.

For either outer atom,

$$F_1 = k(x_2 - x_1) = -k\left(1 + \frac{2m}{M}\right)x_1,$$

$$F_3 = -k\left(1 + \frac{2m}{M}\right)x_3.$$

For the middle atom, use $x_1 + x_3 = -(M/m)x_2$:

$$F_2 = k(x_1 + x_3 - 2x_2) = -k\left(2 + \frac{M}{m}\right)x_2.$$

All three equations give

$$\boxed{\omega_a^2 = k\left(\frac{1}{m_A} + \frac{2}{m_B}\right), \quad f_a = \frac{1}{2\pi} \sqrt{k\left(\frac{1}{m_A} + \frac{2}{m_B}\right)}.$$

These single-displacement force laws apply to the indicated normal modes. A general superposition obeys the original coupled force equations.

The complete motion can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (X_0 + Vt) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + A_s \cos(\omega_s t + \phi_s) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + A_a \cos(\omega_a t + \phi_a) \begin{pmatrix} 1 \\ -2m_A/m_B \\ 1 \end{pmatrix}.$$

Thus there are two longitudinal vibrational modes and one translational mode.

4) Numerical bond stiffness and assessment of the model

For CO₂, the outer atoms are oxygen and the central atom is carbon:

$$m_A = m_O = 16(1.660 \times 10^{-27}) = 2.656 \times 10^{-26} \text{ kg},$$

$$m_B = m_C = 12(1.660 \times 10^{-27}) = 1.992 \times 10^{-26} \text{ kg}.$$

Since $f_a > f_s$, assign

$$f_s = 3.998 \times 10^{13} \text{ Hz}, \quad f_a = 7.042 \times 10^{13} \text{ Hz}.$$

From the symmetric mode,

$$k_s = m_O(2\pi f_s)^2,$$

so

$$k_s = (2.656 \times 10^{-26}) [2\pi(3.998 \times 10^{13})]^2 = 1.676 \times 10^3 \text{ N m}^{-1}.$$

From the asymmetric mode,

$$k_a = \frac{(2\pi f_a)^2}{1/m_O + 2/m_C},$$

so

$$k_a = 1.418 \times 10^3 \text{ N m}^{-1}.$$

These estimates place the bond stiffness near

$$k \approx 1.5 \times 10^3 \text{ N m}^{-1}.$$

Their difference shows that the model cannot reproduce both measured frequencies with a single spring constant.

A direct consistency test, independent of k , is the frequency ratio. The model predicts

$$\frac{f_a}{f_s} = \sqrt{1 + \frac{2m_O}{m_C}} = \sqrt{1 + \frac{32}{12}} = \sqrt{\frac{11}{3}} = 1.915.$$

The measured ratio is

$$\left(\frac{f_a}{f_s} \right)_{\text{measured}} = \frac{7.042}{3.998} = 1.761.$$

For example, fitting the lower frequency predicts

$$f_a = 1.91485(3.998 \times 10^{13}) = 7.656 \times 10^{13} \text{ Hz},$$

which is 8.71% above the measured upper frequency.

The ideal-spring model therefore gives a reasonable approximate description of the two stretching motions and their frequency scale. Its quantitative accuracy is limited: the real molecular restoring forces can include coupling between the two bond extensions, and departures from Hooke's law, which this model omits.

The solution was checked against the official solution.

Problem 3: A Satellite in Sunshine

In this problem you will calculate the temperature of a space satellite. The satellite is assumed to be a sphere with a diameter of 1 m. All of the satellite remains at a uniform temperature. All of the spherical surface of the satellite is coated with the same kind of coating. The satellite is located near the earth but is not in the earth's shadow.

The surface temperature of the sun (its blackbody surface temperature) $T_{\text{sun}} = 6000 \text{ K}$ and its radius is $6.96 \times 10^8 \text{ m}$. The distance between the sun and the earth is $1.5 \times 10^{11} \text{ m}$. The sunlight heats the satellite to a temperature at which the blackbody emission from the satellite equals the power absorbed from the sunlight. The power per unit area emitted by a black body is given by Stefan-Boltzmann's law $P = \sigma T^4$ where σ is the universal constant $5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$. In the first approximation, you can assume that both the sun and the satellite absorb all electromagnetic radiation incident upon them.

- 1) Find an expression for the temperature T of the satellite. What is the numerical value of this temperature?
- 2) The blackbody radiation spectrum $u(T, f)$ of a body at temperature T obeys Planck's radiation law

$$u(T, f) df = \frac{8\pi k^4 T^4}{c^3 h^3} \frac{\eta^3 d\eta}{e^\eta - 1}$$

where $\eta = hf/kT$ and $u(T, f)df$ is the energy density of the electromagnetic radiation in a frequency interval $[f, f + df]$. In the equation $h = 6.6 \times 10^{-34} \text{ J s}$ is Planck's constant, $k = 1.4 \times 10^{-23} \text{ J K}^{-1}$ is Boltzmann's constant and $c = 3.0 \times 10^8 \text{ m s}^{-1}$ is the speed of light.

The blackbody spectrum, integrated over all frequencies f and directions of emission, gives the total radiated power per unit area $P = \sigma T^4$ as expressed in the Stefan-Boltzmann law given above.

$$\sigma = \frac{2\pi^5 k^4}{15c^2 h^3}$$

The figure shows the normalized spectrum

$$\frac{c^3 h^3}{8\pi k^4} \frac{u(T, f)}{T^4}$$

as a function of η .

In many applications it is necessary to keep the satellite as cool as possible. To cool the satellite, engineers use a reflective coating that reflects light above a cut-off frequency but does not prevent heat radiation at lower frequency from escaping. Assume that this (sharp) cut-off frequency corresponds to $hf/k = 1200$ K.

What is the new equilibrium temperature of the satellite? The exact answer is not needed. Therefore, do not perform any tedious integrations; make approximations where necessary. The integral over the entire range is

$$\int_0^\infty \frac{\eta^3 d\eta}{e^\eta - 1} = \frac{\pi^4}{15}$$

and the maximum of $\eta^3/(e^\eta - 1)$ occurs at $\eta = 2.82$. For small η you can expand the exponential function as $e^\eta \simeq 1 + \eta$.

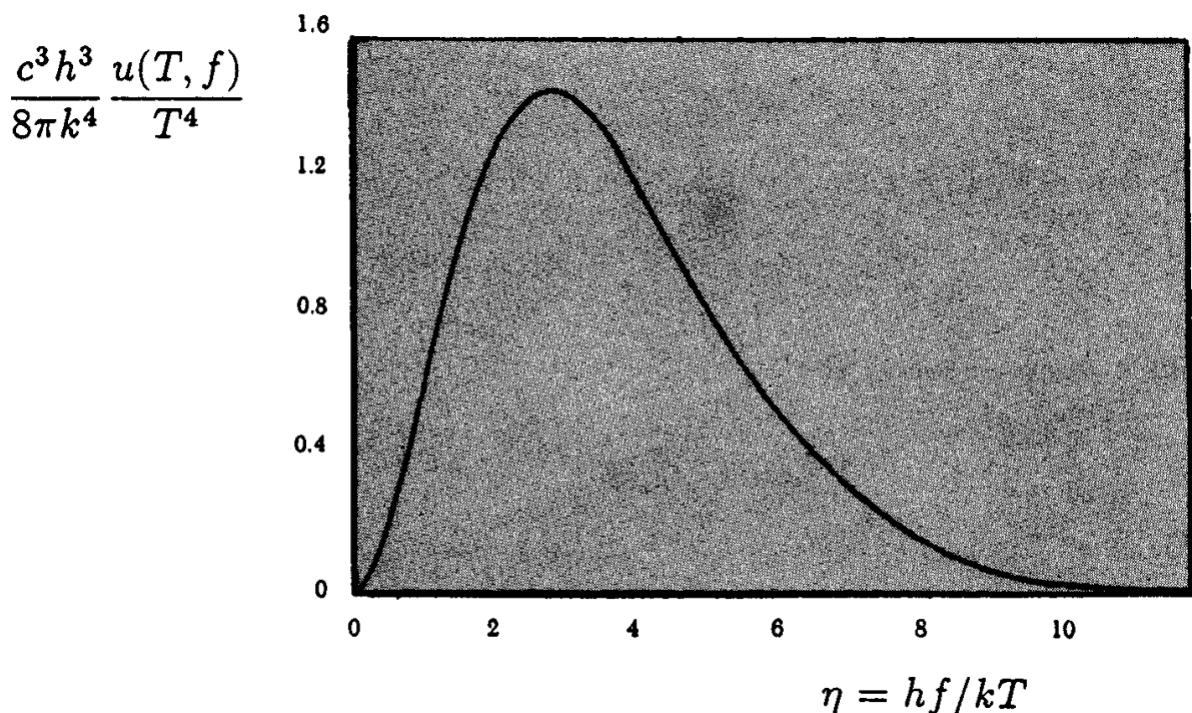
3) If we now have a real satellite, with extending solar panels that generate electricity, the dissipated heat in the electronics inside the satellite acts as an extra source of heat. Assuming that the power of the internal heat source is 1 kW, what is the equilibrium temperature of the satellite in case 2 above?

4) A manufacturer advertises a special paint in the following way:

"This paint will reflect more than 90% of all incoming radiation (both visible light and infrared) but it will radiate at all frequencies (visible light and infrared) as a black body, thus removing lots of heat from the satellite. Thus the paint will help keep the satellite as cool as possible."

Can such paint exist? Why or why not?

5) What properties should a coating have in order to raise the temperature of a spherical body similar to that of the satellite considered here above the temperature calculated in 1)?



Answers

1. With solar radius $R_{\odot}$ and Sun–satellite distance D ,

$$T_1 = T_{\text{sun}} \sqrt{\frac{R_{\odot}}{2D}} \simeq 289 \text{ K.}$$

2. Using the specified small- η approximation,

$$T_2 \simeq T_1 \left[\frac{5}{\pi^4} \left(\frac{1200 \text{ K}}{T_{\text{sun}}} \right)^3 \right]^{1/4} \simeq 41 \text{ K.}$$

3. Accounting for the coating's suppression of high-frequency thermal emission, $T_3 \simeq 3.2 \times 10^2 \text{ K}$ (approximately 317 K).
4. **No.** Kirchhoff's law requires equal absorptivity and emissivity at each frequency. A surface reflecting more than 90% cannot simultaneously emit as a blackbody at those frequencies.
5. The coating should **absorb solar radiation strongly and emit thermal infrared radiation weakly**: high absorptivity at solar frequencies and high reflectivity at the satellite's lower thermal frequencies.

Solution

Let the satellite radius be $a = 0.50 \text{ m}$, the solar radius be $R_{\odot}$, and the Sun–satellite distance be D . We use the stated radiation-balance model, neglecting radiation from Earth and other heat exchanges.

1) Blackbody satellite

The Sun's luminosity is

$$L_{\odot} = 4\pi R_{\odot}^2 \sigma T_{\text{sun}}^4.$$

Consequently, the solar irradiance at the satellite is

$$S = \frac{L_{\odot}}{4\pi D^2} = \sigma T_{\text{sun}}^4 \left(\frac{R_{\odot}}{D} \right)^2.$$

The satellite intercepts sunlight over its projected area πa^2 , while its entire surface $4\pi a^2$ emits radiation. Thermal equilibrium therefore requires

$$\pi a^2 S = 4\pi a^2 \sigma T_1^4.$$

Thus

$$T_1^4 = \frac{1}{4} T_{\text{sun}}^4 \left(\frac{R_{\odot}}{D} \right)^2, \quad T_1 = T_{\text{sun}} \sqrt{\frac{R_{\odot}}{2D}}.$$

Numerically,

$$T_1 = 6000 \sqrt{\frac{6.96 \times 10^8}{2(1.5 \times 10^{11})}} \text{ K} = \boxed{289 \text{ K}}.$$

The satellite's radius cancels because both absorbed and emitted powers scale as a^2 .

2) Satellite with the frequency-selective coating

Write

$$\Theta = \frac{hf_c}{k} = 1200 \text{ K},$$

where f_c is the cutoff frequency. The ideal coated satellite absorbs and emits as a blackbody below f_c , while reflection suppresses both absorption and emission above f_c . This equality of spectral absorptivity and emissivity is Kirchhoff's law.

Define the fraction of blackbody power below a dimensionless cutoff x by

$$F(x) = \frac{15}{\pi^4} \int_0^x \frac{\eta^3}{e^\eta - 1} d\eta.$$

For a body at temperature T , the relevant cutoff is $x = \Theta/T$. The equilibrium equation is therefore

$$\pi a^2 \sigma T_{\text{sun}}^4 \left(\frac{R_{\odot}}{D} \right)^2 F\left(\frac{\Theta}{T_{\text{sun}}} \right) = 4\pi a^2 \sigma T_2^4 F\left(\frac{\Theta}{T_2} \right).$$

Using part 1,

$$T_2^4 F\left(\frac{\Theta}{T_2} \right) = T_1^4 F\left(\frac{\Theta}{T_{\text{sun}}} \right).$$

For sunlight,

$$x_{\odot} = \frac{1200}{6000} = 0.20.$$

Throughout this small interval, $e^\eta - 1 \simeq \eta$, so

$$F(0.20) \simeq \frac{15}{\pi^4} \int_0^{0.20} \eta^2 d\eta = \frac{5(0.20)^3}{\pi^4} \simeq 4.11 \times 10^{-4}.$$

Thus only about 0.041% of the incident solar power is absorbed.

We initially assume that the cooled satellite emits almost all its thermal radiation below the cutoff, so $F(\Theta/T_2) \simeq 1$. Then

$$T_2 \simeq T_1 [F(0.20)]^{1/4} = 289(4.11 \times 10^{-4})^{1/4} \text{ K},$$

giving

$$\boxed{T_2 \simeq 41 \text{ K}.}$$

This assumption is self-consistent: the satellite's cutoff is

$$\frac{\Theta}{T_2} \simeq \frac{1200}{41} \simeq 29,$$

far beyond the spectral maximum at $\eta = 2.82$. The excluded high-frequency thermal tail is negligible.

3) Addition of a 1 kW internal heat source

The emitting area is now needed explicitly:

$$A = 4\pi a^2 = \pi \text{ m}^2.$$

Before applying the coating, the absorbed solar power was

$$Q_{\odot,0} = A\sigma T_1^4 \simeq 1.24 \times 10^3 \text{ W}.$$

With the coating, it becomes

$$Q_{\odot} = Q_{\odot,0}F(0.20) \simeq (1.24 \times 10^3)(4.11 \times 10^{-4}) \simeq 0.51 \text{ W}.$$

This is negligible compared with the internal heat source $Q_{\text{int}} = 1000 \text{ W}$. The balance is

$$A\sigma T_3^4 F\left(\frac{1200 \text{ K}}{T_3}\right) = 1000 \text{ W} + Q_{\odot}.$$

Here we cannot set $F = 1$. Even an unrestricted blackbody would need the temperature

$$\left(\frac{1000 \text{ W}}{A\sigma}\right)^{1/4} \simeq 274 \text{ K},$$

for which the cutoff would already lie at $\eta_c \simeq 4.38$. A significant part of the thermal spectrum is above that cutoff, so the coated satellite must become hotter.

We can estimate the required fraction directly from the supplied graph. Near $\eta_c = 3.8$, the area under the curve is approximately

$$\int_0^{3.8} \frac{\eta^3}{e^{\eta} - 1} d\eta \simeq 3.6.$$

Since the total area is $\pi^4/15 \simeq 6.49$,

$$F(3.8) \simeq \frac{3.6}{6.49} \simeq 0.55.$$

Substitution gives

$$T_3 \simeq \left[\frac{1000.5 \text{ W}}{(\pi \text{ m}^2)(5.67 \times 10^{-8} \text{ W m}^{-2}\text{K}^{-4})(3.6/6.49)} \right]^{1/4} \simeq 317 \text{ K}.$$

Its corresponding cutoff is $1200/317 \simeq 3.79$, confirming the graphical estimate. Therefore,

$$T_3 \simeq 3.2 \times 10^2 \text{ K}.$$

4) The advertised paint

Kirchhoff's law of thermal radiation states that, at a given frequency and for corresponding directions,

$$\epsilon_f = \alpha_f,$$

where ϵ_f is emissivity and α_f is absorptivity.

Energy conservation gives

$$\alpha_f + \rho_f + \tau_f = 1,$$

where ρ_f and τ_f are reflectivity and transmissivity. Consequently, if $\rho_f > 0.90$,

$$\alpha_f < 0.10 \implies \epsilon_f < 0.10.$$

Blackbody emission would instead require $\epsilon_f = 1$.

Thus the claimed combination of high reflection and blackbody thermal emission at the same frequencies is impossible for a passive paint. It would violate Kirchhoff's law and detailed thermal balance.

5) A coating that raises the temperature

The coating should have the opposite spectral selectivity to that in part 2:

- Strong absorption at the relatively high frequencies carrying most solar radiation.
- Weak emission, and correspondingly strong reflection, at the lower infrared frequencies carrying most of the satellite's thermal radiation.

Let $\bar{\alpha}_{\odot}$ be absorptivity averaged over the solar spectrum, and let $\bar{\epsilon}_T$ be emissivity averaged over the satellite's blackbody spectrum at temperature T . The balance without internal heating is

$$\pi a^2 \bar{\alpha}_{\odot} S = 4\pi a^2 \bar{\epsilon}_T \sigma T^4.$$

Hence

$$T = T_1 \left(\frac{\bar{\alpha}_{\odot}}{\bar{\epsilon}_T} \right)^{1/4}.$$

The equilibrium temperature exceeds T_1 when

$$\bar{\alpha}_{\odot} > \bar{\epsilon}_T.$$

This is compatible with Kirchhoff's law because the two averages weight different frequency ranges. A frequency-independent grey coating would have equal absorptivity and emissivity and would leave the equilibrium temperature unchanged.

The solution was checked against the official solution.