

IPhO 1991

Theory problems and solutions · Havana, Cuba · 3 theory problems with complete solutions

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Problem 1: Spinning ball bouncing on the floor

The figure 1.1 shows a solid, homogeneous ball radius R . Before falling to the floor its center of mass is at rest, but the ball is spinning with angular velocity ω_0 about a horizontal axis through its center. The lowest point of the ball is at a height h above the floor.

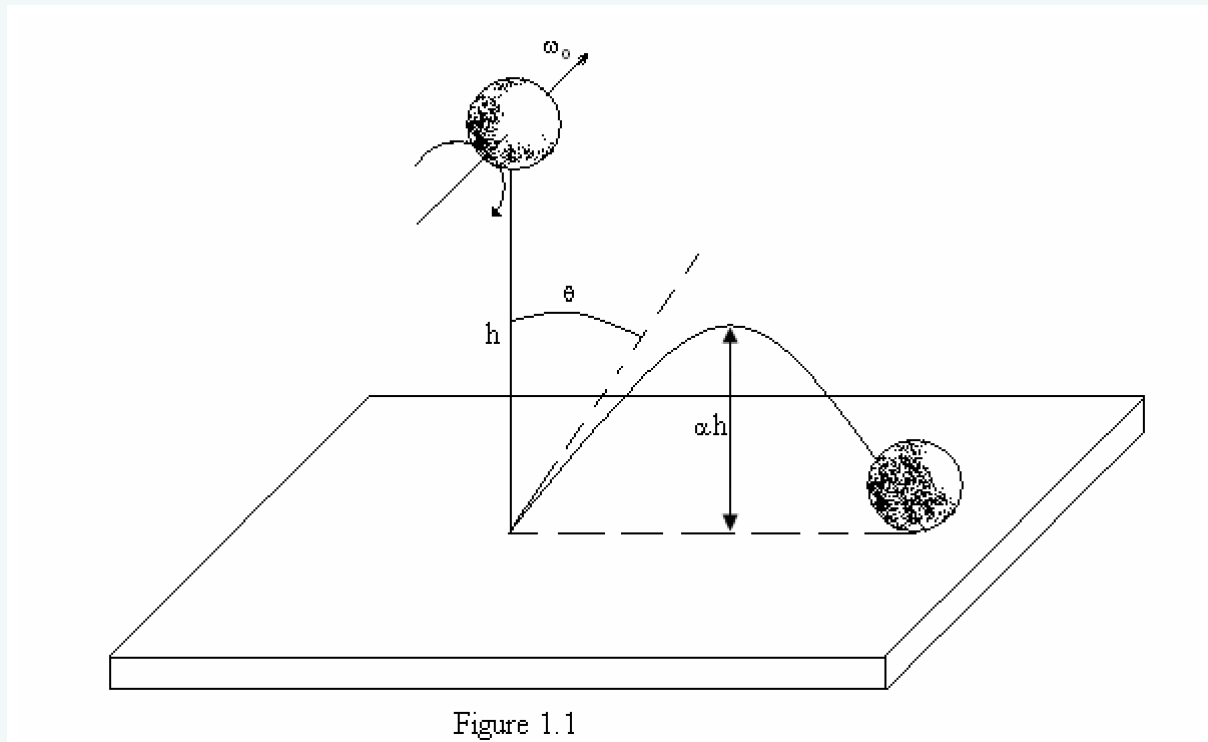


Figure 1.1

Figure 1.1

When released, the ball falls under gravity, and rebounds to a new height such that its lowest point is now αh above the floor. The deformation of the ball and the floor on impact may be considered negligible. Ignore the presence of the air. The impact time, although, is finite.

The mass of the ball is m , the acceleration due the gravity is g , the dynamic coefficient of friction between the ball and the floor is μ_k , and the moment of inertia of the ball about the given axis is:

$$I = \frac{2mR^2}{5}$$

You are required to consider two situations, in the first, the ball slips during the entire impact time, and in the second the slipping stops before the end of the impact time.

Situation I: slipping throughout the impact.

Find:

a) $\tan \theta$, where θ is the rebound angle indicated in the diagram;

- b) the horizontal distance traveled in flight between the first and second impacts;
- c) the minimum value of ω_0 for this situations.

Situation II: slipping for part of the impacts.

Find, again:

- a) $\tan \theta$;
- b) the horizontal distance traveled in flight between the first and second impacts.

Taking both of the above situations into account, sketch the variation of $\tan \theta$ with ω_0 .

Official English text of the problem.

Answers

Let D be the horizontal distance traveled during the rebound flight. In the usual short-impact approximation, with gravity's impulse during contact neglected:

- **Situation I(a):**

$$\tan \theta = \mu_k \frac{1 + \sqrt{a}}{\sqrt{a}}$$

- **Situation I(b):**

$$D = 4\mu_k h \sqrt{a} (1 + \sqrt{a})$$

- **Situation I(c):**

$$\omega_{0,\min} = \omega_c = \frac{7\mu_k}{2R} \sqrt{2gh} (1 + \sqrt{a})$$

At equality, slipping stops exactly at separation. Nonzero slip at separation requires $\omega_0 > \omega_c$.

- **Situation II(a):**

$$\tan \theta = \frac{2R\omega_0}{7\sqrt{2agh}}$$

- **Situation II(b):**

$$D = \frac{4R\omega_0}{7} \sqrt{\frac{2ah}{g}}$$

• Requested sketch:

$$\tan \theta = \begin{cases} \frac{2R\omega_0}{7\sqrt{2agh}}, & 0 \leq \omega_0 < \omega_c, \\ \mu_k \frac{1 + \sqrt{a}}{\sqrt{a}}, & \omega_0 \geq \omega_c. \end{cases}$$

The graph rises linearly from the origin to a corner at

$$\left(\omega_c, \mu_k \frac{1 + \sqrt{a}}{\sqrt{a}} \right),$$

then remains horizontal. It is continuous at the corner.

Solution

Flight speeds and impact equations

Let u be the downward speed immediately before impact and w the upward speed immediately after impact. Mechanical energy is conserved during each free flight, so

$$u = \sqrt{2gh}, \quad w = \sqrt{2gah} = \sqrt{a}u.$$

The centre of mass falls by h and rises by ah , because its height always exceeds that of the lowest point by R .

Choose the positive horizontal direction to be the direction of the friction force on the initially spinning ball. Write v for the horizontal centre-of-mass velocity, and take positive ω in the initial spin direction. The horizontal velocity of the lowest point is then

$$v - R\omega.$$

Initially this is $-R\omega_0$, so friction accelerates the centre of mass and reduces the spin.

Let the normal impulse be

$$P = \int_{\text{impact}} N dt.$$

For contact duration τ , vertical impulse–momentum gives

$$P - mg\tau = m(u + w).$$

In the standard short-impact approximation,

$$g\tau \ll u + w,$$

and hence

$$P \simeq m(u + w).$$

The impact can have a finite duration while satisfying this approximation.

Let J be the accumulated horizontal frictional impulse. Linear and angular impulse–momentum give

$$mv = J, \quad I(\omega - \omega_0) = -RJ.$$

Using $I = 2mR^2/5$,

$$v = \frac{J}{m}, \quad \omega = \omega_0 - \frac{5J}{2mR}.$$

Consequently,

$$R\omega - v = R\omega_0 - \frac{7J}{2m}.$$

This quantity decreases monotonically while the ball slips. Slipping stops when it reaches zero.

Situation I(a): rebound angle

If slipping continues throughout impact, the friction force has magnitude $\mu_k N$ throughout. Therefore

$$J = \mu_k P = \mu_k m(u + w),$$

and the outgoing horizontal speed is

$$v = \mu_k(u + w).$$

The angle θ in the figure is measured from the vertical, so

$$\tan \theta = \frac{v}{w} = \mu_k \frac{u + w}{w}.$$

Since $w = \sqrt{a}u$,

$$\tan \theta = \mu_k \frac{1 + \sqrt{a}}{\sqrt{a}}.$$

Situation I(b): horizontal distance

The ball leaves and returns to the same centre-of-mass height, R . Its flight time T therefore satisfies

$$0 = wT - \frac{1}{2}gT^2,$$

giving

$$T = \frac{2w}{g}.$$

The horizontal velocity remains constant in flight, so

$$D = vT = \frac{2\mu_k(u + w)w}{g}.$$

Substituting $u^2 = 2gh$ and $w = \sqrt{a}u$,

$$D = 4\mu_k h \sqrt{a}(1 + \sqrt{a}).$$

Situation I(c): minimum initial angular velocity

Because $R\omega - v$ decreases monotonically during slipping, checking its value at the end of impact is sufficient. At the boundary between the two situations, it becomes zero exactly at separation:

$$R\omega_0 - \frac{7J}{2m} = 0.$$

With $J = \mu_k m(u + w)$, this gives

$$\omega_c = \frac{7\mu_k}{2R}(u + w).$$

Thus

$$\omega_{0,\min} = \omega_c = \frac{7\mu_k}{2R} \sqrt{2gh}(1 + \sqrt{a}).$$

For $\omega_0 > \omega_c$, slipping persists even at separation. For $\omega_0 = \omega_c$, it persists until the final instant of contact.

Situation II(a): rebound angle

When $0 < \omega_0 < \omega_c$, slipping stops before the normal impulse is complete. Let J_* be the frictional impulse accumulated at that instant. The condition $v = R\omega$ gives

$$R\omega_0 - \frac{7J_*}{2m} = 0,$$

so

$$J_* = \frac{2mR\omega_0}{7}.$$

Therefore, at the instant slipping stops,

$$v_* = \frac{2R\omega_0}{7}, \quad \omega_* = \frac{2\omega_0}{7}.$$

These tangential velocities remain unchanged during the rest of contact. To see this, suppose a static friction force f acts while $v = R\omega$. Then

$$m\dot{v} = f, \quad I\dot{\omega} = -Rf, \quad \dot{v} = R\dot{\omega}.$$

Together these imply

$$f \left(\frac{1}{m} + \frac{R^2}{I} \right) = 0,$$

and hence $f = 0$. No further tangential impulse is required.

The outgoing horizontal speed is therefore

$$v = \frac{2R\omega_0}{7},$$

and

$$\tan \theta = \frac{v}{w} = \frac{2R\omega_0}{7\sqrt{2agh}}.$$

The result also includes the limiting case $\omega_0 = 0$, when there is no slipping and the rebound is vertical.

Situation II(b): horizontal distance

Using the same flight time $T = 2w/g$,

$$D = \frac{2R\omega_0}{7} \frac{2w}{g}.$$

Thus

$$D = \frac{4R\omega_0}{7g} \sqrt{2agh} = \frac{4R\omega_0}{7} \sqrt{\frac{2ah}{g}}.$$

Requested sketch

For fixed h, R, a, μ_k , the graph has two portions:

- From $\omega_0 = 0$ to ω_c , it is a straight line through the origin with slope

$$\frac{2R}{7\sqrt{2agh}}.$$

- At ω_c , it joins a horizontal line of height

$$\mu_k \frac{1 + \sqrt{a}}{\sqrt{a}}.$$

The joining values agree because

$$\frac{2R\omega_c}{7\sqrt{2agh}} = \mu_k \frac{1 + \sqrt{a}}{\sqrt{a}}.$$

At larger initial spins, friction already acts throughout contact, so increasing the spin cannot increase the available tangential impulse.

The solution was checked against the official solution.

Problem 2: Charged balls moving along a square loop in an electric field (relativity)

In a square loop with a side length L , a large number of balls of negligible radius and each with a charge q are moving at a speed u with a constant separation a between them, as seen from a frame of reference that is fixed with respect to the loop. The balls are arranged on the loop like the beads on a necklace, L being much greater than a , as indicated in the figure 2.1. The nonconducting wire forming the loop has a homogeneous charge density per unit length in the frame of the loop. Its total charge is equal and opposite to the total charge of the balls in that frame.

Consider the situation in which the loop moves with velocity v parallel to its side AB (fig. 2.1) through a homogeneous electric field of strength E which is perpendicular to the loop velocity and makes an angle θ with the plane of the loop.

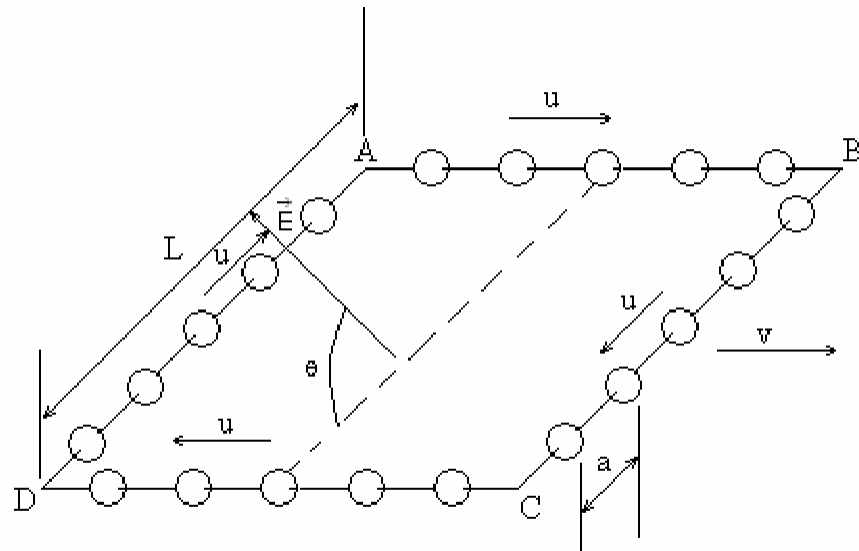


Figure 2.1

Figure 2.1

Taking into account relativistic effects, calculate the following magnitudes in the frame of reference of an observer who sees the loop moving with velocity v :

- The spacing between the balls on each of the side of the loop, a_{AB} , a_{BC} , a_{CD} , y a_{DA} .
- The value of the net charge of the loop plus balls on each of the side of the loop: Q_{AB} , Q_{BC} , Q_{CD} y, Q_{DA}
- The modulus M of the electrically produced torque tending to rotate the system of the loop and the balls.
- The energy W due to the interaction of the system, consisting of the loop and the balls with the electric field.

All the answers should be given in terms of quantities specified in the problem.

Note. The electric charge of an isolated object is independent of the frame of reference in which the measurements takes place. Any electromagnetic radiation effects should be ignored.

Some formulae of special relativity

Consider a reference frame S' moving with velocity V with reference to another reference frame S . The axes of the frames are parallel, and their origins coincide a $t = 0$. V is directed along the positive direction of the x axis.

Relativistic sum of velocities

If a particle is moving with velocity u' in the x' direction , as measured in S' , the velocity of the particle measured in S is given by:

$$u = \frac{u' + V}{1 + \frac{u'V}{c^2}}$$

Relativistic Contraction

If an object at rest in frame S has length L_0 in the x -direction, an observer in frame S' (moving at velocity V in the x -direction) will measure its length to be:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

Official English text of the problem.

Answers

The results use the continuous charge approximation justified by $L \gg a$, with the directions shown in Fig. 2.1. Here c is the speed of light.

• a)

$$a_{AB} = \frac{a\sqrt{1 - v^2/c^2}}{1 + uv/c^2}, \quad a_{BC} = a, \quad a_{CD} = \frac{a\sqrt{1 - v^2/c^2}}{1 - uv/c^2}, \quad a_{DA} = a.$$

• b)

$$Q_{AB} = \frac{qLuv}{ac^2}, \quad Q_{BC} = 0, \quad Q_{CD} = -\frac{qLuv}{ac^2}, \quad Q_{DA} = 0.$$

• c)

$$M = \frac{|q|L^2uvE}{ac^2} \sin \theta.$$

• d) For the field direction shown, whose in-plane component points from AB toward CD ,

$$W = \frac{qL^2uvE}{ac^2} \cos \theta.$$

Solution

Let S' be the rest frame of the loop and S the observer's frame. Take the x -axis along $A \rightarrow B$, so that the loop moves with velocity $v\hat{x}$. Take $\hat{y}$ along $A \rightarrow D$, perpendicular to AB within the loop's plane.

Define

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \kappa = \frac{uv}{c^2}.$$

Since $L \gg a$, we replace the discrete balls by a continuous line charge on each straight side, neglecting fluctuations associated with individual balls entering or leaving a corner. In S' , the ball and wire charge densities are respectively

$$\lambda'_b = \frac{q}{a}, \quad \lambda'_w = -\frac{q}{a}.$$

a) Spacing between the balls

The spacing must be measured between positions recorded **simultaneously in S** . Such events are generally not simultaneous in S' .

On either side parallel to the motion, let the ball velocity in S' be w' , where

$$w' = +u \quad \text{on } AB, \quad w' = -u \quad \text{on } CD.$$

Choose two neighbouring balls ordered along increasing x' . For events on their trajectories,

$$\Delta x' = a + w' \Delta t'.$$

The Lorentz transformation gives

$$\Delta t = \gamma \left(\Delta t' + \frac{v \Delta x'}{c^2} \right).$$

Requiring $\Delta t = 0$ yields

$$\Delta t' = -\frac{v \Delta x'}{c^2}.$$

Consequently,

$$\Delta x' = a - \frac{vw'}{c^2} \Delta x', \quad \Delta x' = \frac{a}{1 + vw'/c^2}.$$

The spatial transformation then gives

$$\Delta x = \gamma(\Delta x' + v \Delta t') = \gamma \Delta x' \left(1 - \frac{v^2}{c^2} \right) = \frac{\Delta x'}{\gamma}.$$

Thus

$$a_{\parallel} = \frac{a}{\gamma(1 + vw'/c^2)}.$$

Substituting the two velocities,

$$\boxed{a_{AB} = \frac{a\sqrt{1 - v^2/c^2}}{1 + uv/c^2}, \quad a_{CD} = \frac{a\sqrt{1 - v^2/c^2}}{1 - uv/c^2}.$$

On BC or DA , all balls on a given side have the same x' . Therefore $\Delta x' = 0$, and simultaneous observations in S also have $\Delta t' = 0$. Transverse distances are unchanged by the Lorentz transformation, so

$$\boxed{a_{BC} = a_{DA} = a.}$$

b) Net charge on each side

In S , the sides parallel to the motion are contracted:

$$\ell_{AB} = \ell_{CD} = \frac{L}{\gamma}, \quad \ell_{BC} = \ell_{DA} = L.$$

Each wire side contains a fixed charge

$$Q_w = -\frac{qL}{a}.$$

This is the same in both frames because charge is invariant and the wire charges do not move between sides.

The number of balls on each side is its observed length divided by the corresponding observed spacing:

$$N_{AB} = \frac{L/\gamma}{a/[\gamma(1+\kappa)]} = \frac{L}{a}(1+\kappa),$$

$$N_{CD} = \frac{L/\gamma}{a/[\gamma(1-\kappa)]} = \frac{L}{a}(1-\kappa),$$

and

$$N_{BC} = N_{DA} = \frac{L}{a}.$$

Adding the wire charge to the ball charge gives

$$Q_{AB} = q\frac{L}{a}(1+\kappa) - \frac{qL}{a} = \frac{qL}{a}\kappa,$$

$$Q_{CD} = q\frac{L}{a}(1-\kappa) - \frac{qL}{a} = -\frac{qL}{a}\kappa.$$

The two transverse sides remain neutral. Hence

$$Q_{AB} = \frac{qLuv}{ac^2}, \quad Q_{BC} = 0, \quad Q_{CD} = -\frac{qLuv}{ac^2}, \quad Q_{DA} = 0.$$

Their sum is zero, as required by invariance of the total charge. The charge of an individual side need not remain zero because the balls occupying that side at a simultaneous instant differ between frames.

c) Electrically produced torque

Write

$$Q = \frac{qLuv}{ac^2},$$

so that the net charges on AB and CD are Q and $-Q$.

In the continuous approximation these charges are uniformly distributed along their respective sides. Their resultants act at the side midpoints. The separation between those midpoints is $L\hat{y}$, since this distance is perpendicular to the loop's motion and is not contracted.

The electric dipole moment is therefore

$$\mathbf{p} = Q\mathbf{r}_{AB} - Q\mathbf{r}_{CD} = -QL\hat{y}.$$

In a uniform electric field the total force is zero, while the torque is

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}.$$

The field component within the loop's plane is parallel to $\hat{y}$ and produces no torque. The component perpendicular to the plane is $E \sin \theta$, giving a torque about an axis parallel to AB , of magnitude

$$M = |Q|LE \sin \theta.$$

Thus

$$M = \frac{|q|L^2uvE}{ac^2} \sin \theta.$$

This has units of N m. If q is understood as a positive charge magnitude, $|q|$ can simply be written q .

d) Interaction energy

For a uniform external electric field,

$$\phi_{\text{ext}}(\mathbf{r}) = \phi_0 - \mathbf{E} \cdot \mathbf{r}.$$

The interaction energy is

$$W = \int \phi_{\text{ext}} dq.$$

Since the system's total charge is zero, the constant ϕ_0 cancels:

$$W = -\mathbf{E} \cdot \int \mathbf{r} dq = -\mathbf{p} \cdot \mathbf{E}.$$

The angle marker in Fig. 2.1 places the in-plane component of $\mathbf{E}$ along $A \rightarrow D$, so

$$\mathbf{E} \cdot \hat{\mathbf{y}} = E \cos \theta.$$

Using $\mathbf{p} = -QL\hat{\mathbf{y}}$,

$$W = QLE \cos \theta.$$

Therefore

$$W = \frac{qL^2 uv E}{ac^2} \cos \theta.$$

This has units of joules and vanishes when the field is perpendicular to the loop. For positive q , the positive sign follows because the electric dipole moment opposes the field's in-plane component.

The solution was checked against the official solution.

Problem 3: Cooling Atoms by laser

To study the properties of isolated atoms with a high degree of precision they must be kept almost at rest for a length of time. A method has recently been developed to do this. It is called "laser cooling" and is illustrated by the problem below.

In a vacuum chamber a well collimated beam of Na^{23} atoms (coming from the evaporation of a sample at 10^3 K) is illuminated head-on with a high intensity laser beam (fig. 3.1). The frequency of laser is chosen so there will be resonant absorption of a photon by those atoms whose velocity is v_0 . When the light is absorbed, these atoms are excited to the first energy level, which has a mean value E above the ground state and uncertainty of Γ (fig. 3.2).

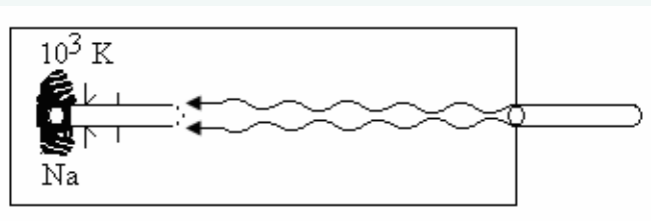


Figure 3.1

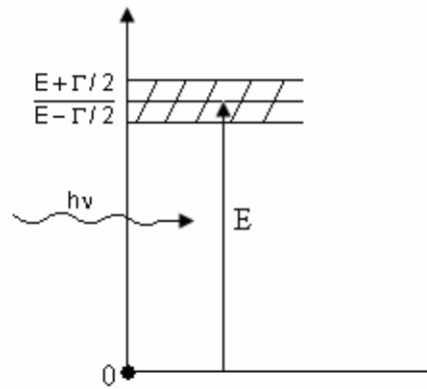


Figure 3.2

For this process, the atom's decrease in velocity Δv_1 is given by $\Delta v_1 = v_1 - v_0$. Light is then emitted by the atom as it returns to its ground state. The atom's velocity changes by $\Delta v' = v'_1 - v_1$ and its direction of motion changes by an angle φ (fig. 3.3).

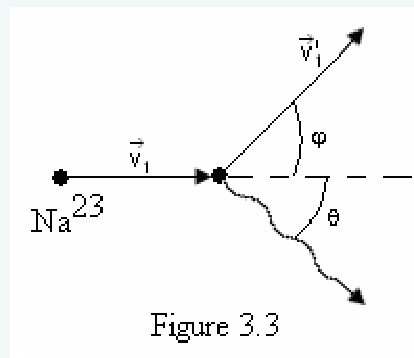


Figure 3.3

This sequence of absorption and emission takes place many times until the velocity of the atoms has decreased by a given amount Δv such that resonant absorption of light at frequency ν no longer occurs. It is then necessary to change the frequency of laser so as to maintain resonant absorption. The atoms moving at the new velocity are further slowed down until some of them have a velocity close to zero.

As first approximation we may ignore any atomic interaction processes apart from the spontaneous absorption and emission light described above.

Furthermore, we may assume the laser to be so intense that the atoms spend practically no time in the ground state.

Questions

- Find the laser frequency needed ensure the resonant absorption of the light by those atoms whose kinetic energy of the atoms inside the region behind the collimator. Also find the reduction in the velocity of these atoms, Δv_1 , after the absorption process.
- Light of the frequency calculated in question a) is absorbed by atoms which velocities lie within a range Δv_0 . Calculate this velocity range.
- When an atom emits light, its direction of motion changes by φ from initial direction. Calculate φ .

- d) Find the maximum possible velocity decrease Δv for a given frequency.
- e) What is the approximate number N of absorption-emission events necessary to reduce the velocity of an atom from its initial value v_0 -found in question a) above- almost to zero? Assume the atom travels in a straight line.
- f) Find the time t that the process in question e takes. Calculate the distance ΔS an atom travels in this time.

Data

$$\begin{aligned} E &= 3,36 \cdot 10^{-19} \text{ J} \\ \Gamma &= 7,0 \cdot 10^{-27} \text{ J} \\ c &= 3 \cdot 10^8 \text{ ms}^{-1} \\ m_p &= 1,67 \cdot 10^{-27} \text{ kg} \\ h &= 6,62 \cdot 10^{-34} \text{ Js} \\ k &= 1,38 \cdot 10^{-23} \text{ JK}^{-1} \end{aligned}$$

where c is speed of light, h is Planck's constant, k is the Boltzmann constant, and m_p is the mass of proton.

Official English text of the problem.

Answers

The numerical results use the assumed intended condition in (a), $\frac{1}{2}mv_0^2 = \frac{3}{2}kT$, giving $v_0 = 1.04 \times 10^3 \text{ m s}^{-1}$.

- a)

$$\nu \simeq \frac{E}{h(1 + v_0/c)} \simeq 5.08 \times 10^{14} \text{ Hz}$$

This is approximately $1.76 \times 10^9 \text{ Hz}$ below the stationary-atom resonance.

$$\Delta v_1 \simeq -2.92 \times 10^{-2} \text{ m s}^{-1}$$

The positive magnitude of the speed reduction is $2.92 \times 10^{-2} \text{ m s}^{-1}$.

- b) The full resonant velocity range is

$$\Delta v_0 = \frac{c\Gamma}{h\nu} \simeq 6.3 \text{ m s}^{-1},$$

extending approximately from $v_0 - 3.1 \text{ m s}^{-1}$ to $v_0 + 3.1 \text{ m s}^{-1}$.

- c) For the emission angle θ shown in figure 3.3, let $v_r = E/(mc)$. Then

$$\varphi \simeq \frac{v_r}{v_1} \sin \theta \simeq 2.8 \times 10^{-5} \sin \theta \text{ rad}.$$

The maximum initial deflection is approximately $2.8 \times 10^{-5} \text{ rad}$.

- **d)** Starting at the central resonant velocity v_0 ,

$$\Delta v \simeq \frac{c\Gamma}{2h\nu} \simeq 3.1 \text{ m s}^{-1}.$$

An atom starting at the upper edge of the resonant range can traverse the full range, approximately 6.3 m s^{-1} .

- **e)** With the laser retuned during slowing,

$$N \simeq \frac{mc v_0}{E} \simeq 3.6 \times 10^4.$$

- **f)** Taking Γ as the full natural energy linewidth,

$$t \simeq 5.4 \times 10^{-4} \text{ s} = 0.54 \text{ ms}, \quad \Delta S \simeq 0.28 \text{ m}.$$

Solution

a) Resonant frequency and absorption recoil

The sentence specifying the initial kinetic energy is incomplete in the supplied text. We use the likely intended condition that the selected atoms have the mean thermal kinetic energy:

$$\frac{1}{2}mv_0^2 = \frac{3}{2}kT.$$

Approximating the sodium atom's mass by $23m_p$,

$$m = 23(1.67 \times 10^{-27}) = 3.841 \times 10^{-26} \text{ kg},$$

and therefore

$$v_0 = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23})(10^3)}{3.841 \times 10^{-26}}} = 1.038 \times 10^3 \text{ m s}^{-1}.$$

Thus $v_0/c \simeq 3.46 \times 10^{-6}$, so nonrelativistic mechanics and the first-order Doppler formula suffice.

Choose the positive direction along the incident atomic beam. The counterpropagating photon has momentum $-h\nu/c$. Conservation of momentum during absorption gives

$$mv_1 = mv_0 - \frac{h\nu}{c},$$

hence

$$\Delta v_1 = v_1 - v_0 = -\frac{h\nu}{mc}.$$

Energy conservation gives

$$\frac{1}{2}mv_0^2 + h\nu = \frac{1}{2}mv_1^2 + E.$$

Substituting the momentum relation,

$$h\nu \left(1 + \frac{v_0}{c}\right) = E + \frac{(h\nu)^2}{2mc^2}.$$

The final term is the recoil energy. Its value is approximately

$$E_r \simeq \frac{E^2}{2mc^2} = 1.6 \times 10^{-29} \text{ J},$$

which is small even compared with $\Gamma = 7.0 \times 10^{-27} \text{ J}$. Neglecting it,

$$\nu \simeq \frac{E}{h(1 + v_0/c)}.$$

This is also the Doppler resonance condition: the approaching atom sees a frequency $\nu(1 + v_0/c)$, which must equal E/h .

For clarity, retaining extra digits temporarily,

$$\begin{aligned} \nu_{\text{rest}} &= \frac{E}{h} = 5.0755287 \times 10^{14} \text{ Hz}, \\ \nu &\simeq 5.0755111 \times 10^{14} \text{ Hz}. \end{aligned}$$

The required laser frequency is therefore below the stationary-atom resonance by

$$\nu_{\text{rest}} - \nu \simeq \nu_{\text{rest}} \frac{v_0}{c} = 1.76 \times 10^9 \text{ Hz}.$$

The absorption recoil is

$$\Delta v_1 = -\frac{h\nu}{mc} \simeq -2.92 \times 10^{-2} \text{ m s}^{-1}.$$

Its sign is negative under the definition printed in the question.

b) Resonant velocity range

Following the energy interval shown in figure 3.2, we treat resonance as occurring when the Doppler-shifted photon energy lies between $E - \Gamma/2$ and $E + \Gamma/2$:

$$E - \frac{\Gamma}{2} \leq h\nu \left(1 + \frac{v}{c}\right) \leq E + \frac{\Gamma}{2}.$$

The limiting velocities are consequently

$$v_{\pm} = c \left(\frac{E \pm \Gamma/2}{h\nu} - 1 \right).$$

Since the laser is tuned to the centre of the line for velocity v_0 ,

$$v_{\pm} = v_0 \pm \frac{c\Gamma}{2h\nu}.$$

Thus the full velocity range is

$$\Delta v_0 = v_+ - v_- = \frac{c\Gamma}{h\nu}.$$

Using $h\nu \simeq E$,

$$\Delta v_0 \simeq \frac{(3.0 \times 10^8)(7.0 \times 10^{-27})}{3.36 \times 10^{-19}} = 6.25 \text{ m s}^{-1} \simeq 6.3 \text{ m s}^{-1}.$$

The half-width is approximately 3.1 m s^{-1} .

c) Deflection caused by spontaneous emission

The emitted photon has momentum approximately E/c . Define the corresponding recoil speed

$$v_r = \frac{E}{mc} \simeq 2.92 \times 10^{-2} \text{ m s}^{-1}.$$

The small Doppler and recoil corrections to the emitted photon's energy are negligible here.

For the geometry in figure 3.3, the photon travels at angle θ below the initial direction. Conservation of momentum parallel and perpendicular to that direction gives

$$mv_1 = mv'_1 \cos \varphi + \frac{E}{c} \cos \theta,$$

$$mv'_1 \sin \varphi = \frac{E}{c} \sin \theta.$$

Dividing,

$$\tan \varphi = \frac{v_r \sin \theta}{v_1 - v_r \cos \theta}.$$

Initially $v_r \ll v_1$, so

$$\varphi \simeq \frac{v_r}{v_1} \sin \theta.$$

With $v_1 \simeq 1038.16 \text{ m s}^{-1}$,

$$\varphi \simeq 2.8 \times 10^{-5} \sin \theta \text{ rad}.$$

The largest deflection occurs for emission approximately perpendicular to the atomic velocity. More precisely, for fixed photon momentum,

$$\varphi_{\max} = \arcsin \left(\frac{v_r}{v_1} \right) \simeq 2.8 \times 10^{-5} \text{ rad}.$$

Thus one emission changes the initial direction only very slightly. A unique value of φ requires specifying the random emission angle θ .

d) Maximum slowing at fixed laser frequency

Consider first an atom starting at the central resonant velocity v_0 , as in part (a). As it slows, the photon energy in its rest frame decreases. The lower boundary of the resonance interval is reached when

$$h\nu \left(1 + \frac{v_0 - \Delta v}{c} \right) = E - \frac{\Gamma}{2}.$$

Subtracting this from

$$h\nu \left(1 + \frac{v_0}{c} \right) = E$$

gives

$$\frac{h\nu}{c} \Delta v = \frac{\Gamma}{2}.$$

Therefore, to the linewidth accuracy used in the problem,

$$\Delta v = \frac{c\Gamma}{2h\nu} \simeq 3.1 \text{ m s}^{-1}.$$

If "maximum possible" includes all initial velocities that can absorb this fixed laser frequency, an atom may begin at v_+ and slow to v_- . The maximum span is then

$$v_+ - v_- = \frac{c\Gamma}{h\nu} \simeq 6.3 \text{ m s}^{-1}.$$

Both values are much smaller than v_0 . Continued slowing therefore requires increasing the laser frequency toward E/h as the atoms slow.

e) Number of absorption–emission events

Every absorption gives the atom a directed momentum change approximately

$$\Delta p_{x,\text{abs}} = -\frac{E}{c}.$$

Spontaneous emission has no preferred forward or backward direction in the atom's rest frame. Consequently, to the present accuracy,

$$\langle \cos \theta \rangle = 0, \quad \langle \Delta p_{x,\text{em}} \rangle = 0.$$

The mean longitudinal speed change per complete cycle is therefore

$$\langle \Delta v_x \rangle = -v_r = -\frac{E}{mc}.$$

Using the straight-line approximation requested in the question and retuning the laser to maintain resonance,

$$Nv_r \simeq v_0.$$

Hence

$$N \simeq \frac{v_0}{v_r} = \frac{mc v_0}{E}.$$

Numerically,

$$N \simeq \frac{(3.841 \times 10^{-26})(3.0 \times 10^8)(1038.19)}{3.36 \times 10^{-19}} = 3.56 \times 10^4.$$

Thus approximately

$$N \simeq 3.6 \times 10^4$$

cycles are required. This estimates the reduction of the mean directed velocity; individual atoms undergo random emission recoils.

f) Cooling time and distance

For an excited state with mean lifetime τ , the full natural energy linewidth satisfies

$$\Gamma = \frac{\hbar}{\tau}.$$

Here Γ is taken as the full width, consistent with the boundaries $E \pm \Gamma/2$. Thus

$$\tau = \frac{\hbar}{\Gamma} = \frac{h}{2\pi\Gamma}.$$

Numerically,

$$\tau = \frac{6.62 \times 10^{-34}}{2\pi(7.0 \times 10^{-27})} = 1.51 \times 10^{-8} \text{ s}.$$

By the explicit assumption that the atoms spend practically no time in the ground state, the mean duration of one cycle is τ . Therefore

$$t = N\tau = (3.56 \times 10^4)(1.51 \times 10^{-8}) \simeq 5.36 \times 10^{-4} \text{ s},$$

or

$$\boxed{t \simeq 5.4 \times 10^{-4} \text{ s}}.$$

The mean recoil per cycle and mean cycle duration are approximately constant. Thus the mean deceleration has constant magnitude

$$a = \frac{v_r}{\tau} \simeq 1.94 \times 10^6 \text{ m s}^{-2}.$$

For slowing from v_0 almost to zero, the mean speed over the process is $v_0/2$, giving

$$\Delta S \simeq \frac{v_0 t}{2} = \frac{(1038.19)(5.36 \times 10^{-4})}{2} \simeq 0.278 \text{ m}.$$

Therefore

$$\boxed{\Delta S \simeq 0.28 \text{ m}}.$$

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.