

IPhO 1990

Theory problems and solutions · Groningen, Netherlands · 3 theory problems with complete solutions

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Problem 1: X-ray Diffraction from a crystal

We wish to study X-ray diffraction by a cubic crystal lattice. To do this we start with the diffraction of a plane, monochromatic wave that falls perpendicularly on a 2-dimensional grid that consists of $N_1 \times N_2$ slits with separations d_1 and d_2 . The diffraction pattern is observed on a screen at a distance L from the grid. The screen is parallel to the grid and L is much larger than d_1 and d_2 .

a - Determine the positions and widths of the principal maximum on the screen. The width is defined as the distance between the minima on either side of the maxima.

We consider now a cubic crystal, with lattice spacing a and size $N_0 \cdot a \times N_0 \cdot a \times N_1 \cdot a$. N_1 is much smaller than N_0 . The crystal is placed in a parallel X-ray beam along the z-axis at an angle Θ (see Fig. 1). The diffraction pattern is again observed on a screen at great distance from the crystal.

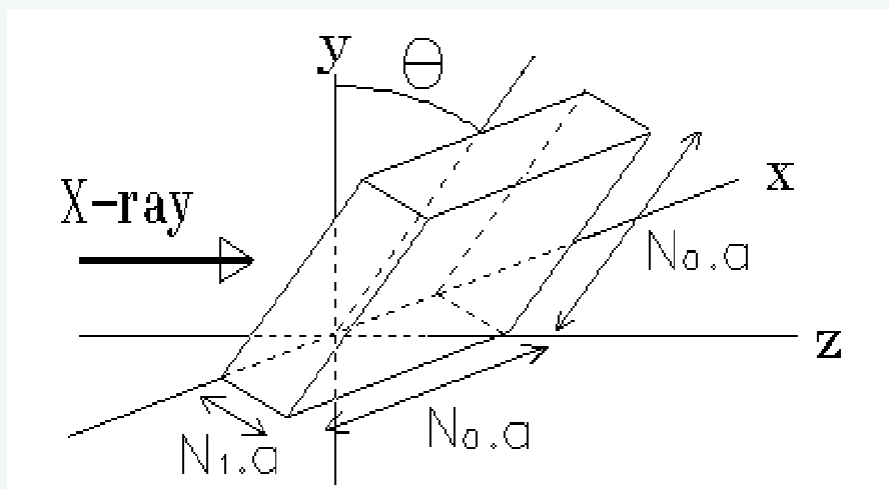


Figure 1 Diffraction of a parallel X-ray beam along the z-axis. The angle between the crystal and the y-axis is Θ .

b - Calculate the position and width of the maxima as a function of the angle Θ (for small Θ).

- What in particular are the consequences of the fact that $N_1 \ll N_0$.

The diffraction pattern can also be derived by means of Bragg's theory, in which it is assumed that the X-rays are reflected from atomic planes in the lattice. The diffraction pattern then arises from interference of these reflected rays with each other.

c - Show that this so-called Bragg reflection yields the same conditions for the maxima as those that you found in b.

In some measurements the so-called powder method is employed. A beam of X-rays is scattered by a powder of very many, small crystals. (Of course the sizes of the crystals are much larger than the lattice spacing, a .)

Scattering of X-rays of wavelength 0.15 nm by Potassium Chloride [KCl] (which has a cubic lattice, see Fig.2) results in the production of concentric dark circles on a photographic plate. The distance between the crystals and the plate is 0.10 m , and the radius of the smallest circle is 0.053 m (see Fig. 3). K^+ and Cl^- ions have almost the same size, and they may be treated as identical scattering centres.

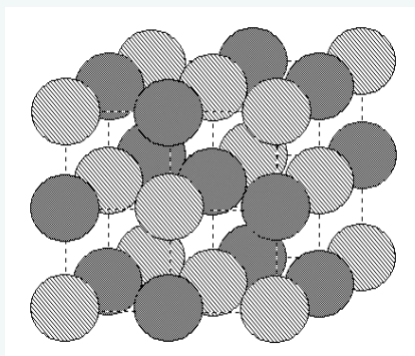


Figure 2. The cubic lattice of Potassium Chloride in which the K^+ and Cl^- ions have almost the same size.

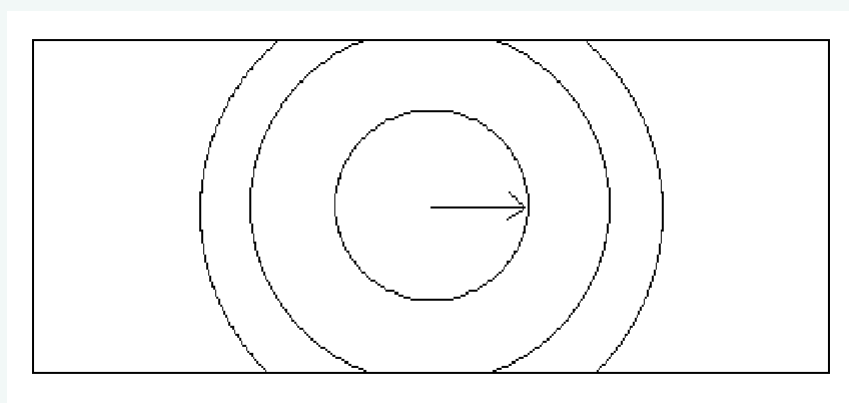


Figure 3. Scattering of X-rays by a powder of KCl crystals results in the production of concentric dark circles on a photographic plate.

d - Calculate the distance between two neighbouring K ions in the crystal.

Official English text of the problem.

Answers

Let λ denote the wavelength. Screen coordinates X, Y are measured from the undeflected beam; for the crystal, L denotes the crystal-to-screen distance.

- **a.** With integers m, n , put

$$u_m = \frac{m\lambda}{d_1}, \quad v_n = \frac{n\lambda}{d_2}.$$

The principal maxima are at

$$(X_{mn}, Y_{mn}) = \frac{L(u_m, v_n)}{\sqrt{1 - u_m^2 - v_n^2}}.$$

Their full first-zero widths in direction cosines are

$$\Delta u = \frac{2\lambda}{N_1 d_1}, \quad \Delta v = \frac{2\lambda}{N_2 d_2}.$$

For small diffraction angles, these become

$$X_m \simeq \frac{m\lambda L}{d_1}, \quad Y_n \simeq \frac{n\lambda L}{d_2}, \quad W_X \simeq \frac{2\lambda L}{N_1 d_1}, \quad W_Y \simeq \frac{2\lambda L}{N_2 d_2}.$$

- **b.** Write $r = \lambda/a$, and let h, k, ℓ be integer diffraction indices. Define

$$B = k \sin \Theta + \ell \cos \Theta, \quad C = k \cos \Theta - \ell \sin \Theta.$$

Fully constructive maxima satisfy

$$2B + r(h^2 + k^2 + \ell^2) = 0$$

and appear at

$$(X, Y) = \frac{Lr(h, C)}{1 + rB}.$$

For small Θ , use $B \simeq k\Theta + \ell$ and $C \simeq k - \ell\Theta$.

For forward diffraction at small scattering angles, the sharp spots have

$$(X, Y) \simeq Lr(h, k), \quad W_X \simeq W_Y \simeq \frac{2\lambda L}{N_0 a}.$$

Their thickness interference factor is maximal when

$$-k\Theta - \frac{\lambda}{2a}(h^2 + k^2) = \ell,$$

with adjacent zeros when the left-hand side equals $\ell \pm 1/N_1$. Thus $N_1 \ll N_0$ gives narrow spots with a much broader condition for their excitation. For $k \neq 0$, the corresponding full width in tilt is approximately

$$\Delta\Theta = \frac{2}{N_1 |k|}.$$

- **c.** Bragg's condition

$$2d \sin \theta_B = p\lambda$$

is equivalent to the lattice-interference conditions in **b**. In particular, reflection from planes normal to the tilted long crystal axis gives

$$2a \sin \Theta = p\lambda, \quad X = 0, \quad Y = -L \tan(2\Theta)$$

for the orientation chosen below.

- **d.** The distance between nearest neighbouring potassium ions is

$$d_{\text{K-K}} = 4.4 \times 10^{-10} \text{ m} = 0.44 \text{ nm.}$$

Solution

We use coherent, elastic scattering in the far-field—or Fraunhofer—approximation. The slits and lattice sites are treated as identical radiators; their individual diffraction envelope is omitted.

A useful result for N equally spaced radiators with phase difference $2\pi t$ is

$$F_N(t) = \left| \sum_{j=0}^{N-1} e^{2\pi i j t} \right|^2 = \frac{\sin^2(N\pi t)}{\sin^2(\pi t)}.$$

At an integer $t = m$, the limiting value is N^2 : all contributions are in phase. For $N > 1$, the nearest zeros are

$$t = m \pm \frac{1}{N}.$$

a. Diffraction by the two-dimensional grid

Take the grid axes along x, y , with the incident wave along z . An outgoing ray has unit direction

$$\mathbf{s} = (u, v, w), \quad w = \sqrt{1 - u^2 - v^2} > 0.$$

The incident phase is the same at every slit. The outgoing path differences between adjacent slits are $d_1 u$ and $d_2 v$. Hence

$$I(u, v) \propto F_{N_1} \left(\frac{d_1 u}{\lambda} \right) F_{N_2} \left(\frac{d_2 v}{\lambda} \right).$$

Both factors attain their principal maxima when

$$d_1 u = m\lambda, \quad d_2 v = n\lambda.$$

Only orders satisfying $u_m^2 + v_n^2 < 1$ reach a screen in the forward half-space.

A ray in direction $\mathbf{s}$ intersects the plane $z = L$ at

$$X = L \frac{u}{w}, \quad Y = L \frac{v}{w}.$$

Therefore

$$(X_{mn}, Y_{mn}) = \frac{L(u_m, v_n)}{\sqrt{1 - u_m^2 - v_n^2}}.$$

The nearest zero boundaries are

$$u = u_m \pm \frac{\lambda}{N_1 d_1}, \quad v = v_n \pm \frac{\lambda}{N_2 d_2}.$$

These equations, together with the screen mapping above, specify the first-zero boundaries without a small-angle approximation.

For example, define

$$\mathbf{R}(u, v) = \frac{L(u, v)}{\sqrt{1 - u^2 - v^2}}.$$

The two pairs of first-zero points along the natural grid coordinates are

$$\mathbf{R}\left(u_m \pm \frac{\lambda}{N_1 d_1}, v_n\right), \quad \mathbf{R}\left(u_m, v_n \pm \frac{\lambda}{N_2 d_2}\right).$$

Their respective separations give the corresponding screen widths.

For small diffraction angles, $u, v \ll 1$, so $X \simeq Lu$, $Y \simeq Lv$. Thus

$$X_m \simeq \frac{m\lambda L}{d_1}, \quad Y_n \simeq \frac{n\lambda L}{d_2},$$

and

$$\boxed{W_X \simeq \frac{2\lambda L}{N_1 d_1}, \quad W_Y \simeq \frac{2\lambda L}{N_2 d_2}.}$$

b. Diffraction by the tilted finite crystal

Choose crystal unit vectors

$$\mathbf{e}_1 = \hat{\mathbf{x}}, \quad \mathbf{e}_2 = \hat{\mathbf{y}} \cos \Theta + \hat{\mathbf{z}} \sin \Theta, \quad \mathbf{e}_3 = -\hat{\mathbf{y}} \sin \Theta + \hat{\mathbf{z}} \cos \Theta.$$

The two long dimensions are along $\mathbf{e}_1, \mathbf{e}_2$; the thin dimension is along $\mathbf{e}_3$. This takes positive Θ as a tilt of the long crystal axis from $+y$ towards $+z$.

The incident and outgoing wavevectors are

$$\mathbf{k}_i = \frac{2\pi}{\lambda} \hat{\mathbf{z}}, \quad \mathbf{k}_f = \frac{2\pi}{\lambda} \mathbf{s}.$$

Define the scattering vector

$$\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i.$$

Unlike the grid in **a**, atoms at different depths also have different incident phases. Both incident and outgoing propagation are accounted for by $\mathbf{q}$. Define

$$p_i = \frac{a}{2\pi} \mathbf{q} \cdot \mathbf{e}_i.$$

Explicitly,

$$\begin{aligned} p_1 &= \frac{a}{\lambda} s_x, \\ p_2 &= \frac{a}{\lambda} [s_y \cos \Theta + (s_z - 1) \sin \Theta], \\ p_3 &= \frac{a}{\lambda} [-s_y \sin \Theta + (s_z - 1) \cos \Theta]. \end{aligned}$$

The complete finite-lattice interference pattern is therefore

$$I(\mathbf{s}) \propto F_{N_0}(p_1) F_{N_0}(p_2) F_{N_1}(p_3).$$

A fully constructive maximum requires

$$p_1 = h, \quad p_2 = k, \quad p_3 = \ell,$$

where h, k, ℓ are integers. Equivalently,

$$\mathbf{q} = \frac{2\pi}{a}(h\mathbf{e}_1 + k\mathbf{e}_2 + \ell\mathbf{e}_3).$$

Writing $r = \lambda/a$, the outgoing direction is

$$\begin{aligned} s_x &= rh, \\ s_y &= r(k \cos \Theta - \ell \sin \Theta), \\ s_z &= 1 + r(k \sin \Theta + \ell \cos \Theta). \end{aligned}$$

Elastic scattering requires $|\mathbf{s}| = 1$. Substitution gives

$$2(k \sin \Theta + \ell \cos \Theta) + r(h^2 + k^2 + \ell^2) = 0.$$

For any allowed forward order, its screen position is

$$X = \frac{Lrh}{1 + r(k \sin \Theta + \ell \cos \Theta)}, \quad Y = \frac{Lr(k \cos \Theta - \ell \sin \Theta)}{1 + r(k \sin \Theta + \ell \cos \Theta)}.$$

These formulas retain the exact outgoing-ray geometry. For small crystal tilt,

$$2(k\Theta + \ell) + r(h^2 + k^2 + \ell^2) \simeq 0,$$

and

$$X \simeq \frac{Lrh}{1 + r(\ell + k\Theta)}, \quad Y \simeq \frac{Lr(k - \ell\Theta)}{1 + r(\ell + k\Theta)}.$$

The first zeros surrounding a fully constructive maximum occur on

$$p_1 = h \pm \frac{1}{N_0}, \quad p_2 = k \pm \frac{1}{N_0}, \quad p_3 = \ell \pm \frac{1}{N_1}.$$

Thus the full widths in scattering-vector components are

$$\Delta q_1 = \Delta q_2 = \frac{4\pi}{N_0 a}, \quad \Delta q_3 = \frac{4\pi}{N_1 a}.$$

Because $N_1 \ll N_0$, the first two factors form narrow peaks, while the third factor supplies a relatively broad envelope. Consequently, a sharp spot can remain visible even when p_3 is not exactly an integer.

For completeness, the position of such a spot can be obtained without requiring exact three-dimensional constructive interference. Set $p_1 = h$, $p_2 = k$. On the forward branch,

$$\zeta_{hk}(\Theta) = \frac{\sqrt{1 - r^2 h^2 - (\sin \Theta + rk)^2} - \cos \Theta}{r}$$

is the resulting value of p_3 . Its position is

$$\mathcal{R}_\Theta(h, k) = \frac{Lr(h, k \cos \Theta - \zeta_{hk} \sin \Theta)}{1 + r(k \sin \Theta + \zeta_{hk} \cos \Theta)}.$$

Its intensity contains the factor $F_{N_1}(\zeta_{hk})$. Fully constructive peaks occur when ζ_{hk} is an integer.

For a well-excited spot, the two pairs of first-zero points are obtained from this mapping by

$$h \longrightarrow h \pm \frac{1}{N_0}, \quad \text{or} \quad k \longrightarrow k \pm \frac{1}{N_0}.$$

Their separations give the screen widths, including the dependence on Θ .

The small-angle form makes the physical consequences especially clear. Put

$$\alpha = \frac{X}{L}, \quad \beta = \frac{Y}{L}, \quad \mathbf{s} \simeq \left(\alpha, \beta, 1 - \frac{\alpha^2 + \beta^2}{2} \right).$$

For small Θ, α, β ,

$$p_1 \simeq \frac{\alpha}{r}, \quad p_2 \simeq \frac{\beta}{r}, \quad p_3 \simeq -\frac{\Theta\beta + (\alpha^2 + \beta^2)/2}{r}.$$

The quadratic angular term must be retained in p_3 , since it is the leading longitudinal path difference.

The narrow transverse peaks therefore occur at

$$\alpha_h \simeq hr, \quad \beta_k \simeq kr,$$

with full widths

$$\Delta\alpha \simeq \Delta\beta \simeq \frac{2r}{N_0}.$$

Hence

$$(X, Y) \simeq \left(\frac{h\lambda L}{a}, \frac{k\lambda L}{a} \right), \quad W_X \simeq W_Y \simeq \frac{2\lambda L}{N_0 a}.$$

At these positions,

$$p_3 \simeq -k\Theta - \frac{r}{2}(h^2 + k^2).$$

Thus a thickness-envelope maximum occurs when

$$-k\Theta - \frac{r}{2}(h^2 + k^2) = \ell,$$

and its adjacent zeros occur at $\ell \pm 1/N_1$.

For the central thickness order $\ell = 0$, the strongest spots satisfy

$$h^2 + \left(k + \frac{\Theta}{r} \right)^2 = \left(\frac{\Theta}{r} \right)^2.$$

Finite thickness broadens this circle into a band of appreciably excited lattice spots.

For $k \neq 0$, varying the tilt changes p_3 at the rate

$$\frac{dp_3}{d\Theta} \simeq -k.$$

The full first-zero width in tilt is therefore

$$\Delta\Theta \simeq \frac{2}{N_1|k|}.$$

The two large dimensions determine the narrow spot widths. The thin dimension gives a broader interference envelope and a broader range of crystal orientations over which a spot is appreciably excited.

c. Equivalence with Bragg reflection

Consider a family of atomic planes with coprime integer indices (H, K, M) . Their unit normal and spacing are

$$\hat{\mathbf{n}} = \frac{H\mathbf{e}_1 + K\mathbf{e}_2 + M\mathbf{e}_3}{\sqrt{H^2 + K^2 + M^2}}, \quad d = \frac{a}{\sqrt{H^2 + K^2 + M^2}}.$$

Let θ_B be the glancing angle between the incident beam and these planes. Rays reflected from consecutive planes have path difference

$$2d \sin \theta_B.$$

Constructive interference requires

$$2d \sin \theta_B = p\lambda, \quad p = 1, 2, \dots$$

Specular reflection changes only the component of the wavevector normal to the planes. Therefore $\mathbf{q}$ is normal to them and has magnitude

$$|\mathbf{q}| = 2 \frac{2\pi}{\lambda} \sin \theta_B.$$

Using Bragg's condition,

$$|\mathbf{q}| = \frac{2\pi p}{d}.$$

With a suitable choice of normal direction,

$$\mathbf{q} = \frac{2\pi p}{a} (H\mathbf{e}_1 + K\mathbf{e}_2 + M\mathbf{e}_3).$$

Setting

$$(h, k, \ell) = p(H, K, M)$$

gives exactly

$$\mathbf{q} = \frac{2\pi}{a} (h\mathbf{e}_1 + k\mathbf{e}_2 + \ell\mathbf{e}_3),$$

the three lattice-interference conditions derived in **b**. Enforcing equal incident and outgoing wavelengths then gives the same condition

$$2(k \sin \Theta + \ell \cos \Theta) + \frac{\lambda}{a} (h^2 + k^2 + \ell^2) = 0.$$

As a direct example, take planes normal to $\mathbf{e}_2$. Their spacing is a , and their glancing angle is Θ . Bragg's law gives

$$2a \sin \Theta = p\lambda.$$

In **b**, choose $h = 0$, $k = -p$, $\ell = 0$. The condition becomes

$$-2p \sin \Theta + \frac{\lambda}{a} p^2 = 0,$$

which is identical. The outgoing ray is

$$\mathbf{s} = (0, -\sin 2\Theta, \cos 2\Theta),$$

so

$$X = 0, \quad Y = -L \tan 2\Theta.$$

d. Nearest-neighbour potassium-ion distance

When K^+ and Cl^- scatter identically, the alternating ionic arrangement acts as a simple cubic lattice of identical scattering centres.

Let b denote the nearest-neighbour K–Cl separation. The effective simple cubic lattice spacing is b . Its largest reflecting-plane spacing is b , so the smallest nonzero diffraction angle comes from first-order reflection with

$$2b \sin \theta_B = \lambda.$$

Random crystal orientations produce a cone of scattered rays with scattering angle $2\theta_B$. A perpendicular photographic plate therefore records a ring of radius

$$R = L \tan(2\theta_B).$$

Using the given data,

$$2\theta_B = \arctan\left(\frac{0.053}{0.10}\right) = 27.9236^\circ,$$

and hence

$$\theta_B = 13.9618^\circ.$$

No small-angle approximation is needed here.

Thus

$$b = \frac{0.15 \text{ nm}}{2 \sin 13.9618^\circ} = 0.31085 \text{ nm}.$$

The species alternate along each simple cubic axis. Two nearest potassium ions therefore lie across a face diagonal of a cube of side b , giving

$$d_{\text{K-K}} = \sqrt{2} b = \sqrt{2}(0.31085 \text{ nm}) = 0.43961 \text{ nm}.$$

To two significant figures,

$$d_{\text{K-K}} = 0.44 \text{ nm} = 4.4 \times 10^{-10} \text{ m}.$$

The solution was checked against the official solution.

Problem 2: Electric experiments in the magnetosphere of the earth

In May 1991 the spaceship Atlantis will be placed in orbit around the earth. We shall assume that this orbit will be circular and that it lies in the earth's equatorial plane.

At some predetermined moment the spaceship will release a satellite S, which is attached to a conducting rod of length L . We suppose that the rod is rigid, has negligible mass, and is covered by an electrical insulator. We also neglect all friction. Let α be the angle that the rod makes to the line between the Atlantis and the centre of the earth. (see Fig. 1).

S also lies in the equatorial plane.

Assume that the mass of the satellite is much smaller than that of the Atlantis, and that L is much smaller than the radius of the orbit.

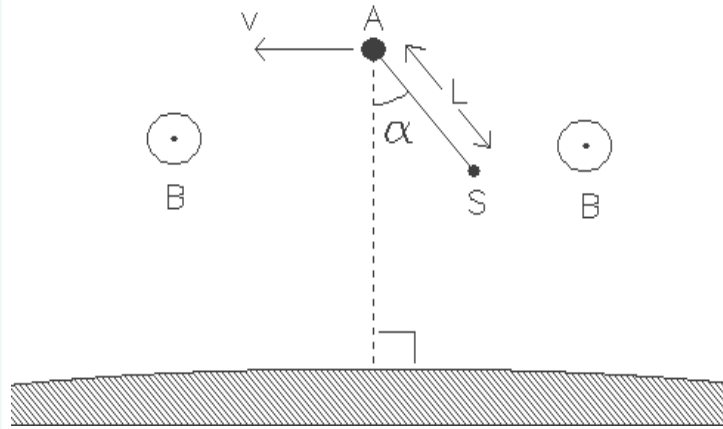


Figure 1 The spaceship Atlantis (A) with a satellite (S) in an orbit around the earth. The orbit lies in the earth's equatorial plane. The magnetic field (B) is perpendicular to the diagram and is directed towards the reader.

a_1 - Deduce for which value(s) of α the configuration of the spaceship and satellite remain unchanged (with respect to the earth)? In other words, for which value(s) of α is α constant?

a_2 - Discuss the stability of the equilibrium for each case.

Suppose that, at a given moment, the rod deviates from the stable configuration by a small angle. The system will begin to swing like a pendulum.

b - Express the period of the swinging in terms of the period of revolution of the system around the earth.

In Fig. 1 the magnetic field of the earth is perpendicular to the diagram and is directed towards the reader. Due to the orbital velocity of the rod, a potential difference arises between its ends. The environment (the magnetosphere) is a rarefied, ionised gas with a very good electrical conductivity. Contact with the ionised gas is made by means of electrodes in A (the Atlantis) and S (the satellite). As a consequence of the motion, a current, I , flows through the rod.

c_1 - In which direction does the current flow through the rod? (Take $\alpha = 0$)

Data:

- the period of the orbit	$T = 5,4 \cdot 10^3 \text{ s}$
- the length of the rod	$L = 2,0 \cdot 10^4 \text{ m}$
- magnetic field strength of the earth at the height of the satellite	$B = 5,0 \cdot 10^{-5} \text{ Wb} \cdot \text{m}^{-2}$
- the mass of the shuttle Atlantis	$m = 1,0 \cdot 10^5 \text{ kg}$

Next, a current source inside the shuttle is included in the circuit, which maintains a net direct current of 0.1 A in the opposite direction.

c_2 - How long must this current be maintained to change the altitude of the orbit by 10 m.

Assume that α remains zero. Ignore all contributions from currents in the magnetosphere.

- Does the altitude decrease or increase?

Answers

- **a₁:** The rod must be **radial or tangential**. To leading order in L/R , the directed angles are

$$\alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \pmod{2\pi}.$$

- **a₂:** The radial equilibria, $\alpha = 0, \pi$, are **stable**. The tangential equilibria, $\alpha = \pi/2, 3\pi/2$, are **unstable**.
- **b:** The small-oscillation period is

$$T_{\text{swing}} = \frac{T}{\sqrt{3}} \simeq 3.1 \times 10^3 \text{ s}.$$

- **c₁:** Conventional current flows **from the satellite S to Atlantis A**.
- **c₂:** The altitude **increases**, and the required time, in the circular-orbit approximation, is

$$t = \frac{\pi m \Delta h}{IBLT} \simeq 5.8 \times 10^3 \text{ s} \simeq 1.6 \text{ h}.$$

Solution

a₁ — Equilibrium orientations

Let R be the orbital radius of Atlantis, $\mu = GM_E$, and

$$\Omega = \frac{2\pi}{T}.$$

Since Atlantis follows a circular orbit,

$$\Omega^2 = \frac{\mu}{R^3}.$$

The satellite's much smaller mass allows us to neglect its effect on Atlantis's motion.

Use a frame centred on Atlantis and rotating with the orbit. Let $\mathbf{e}_x$ point towards the earth's centre, and let $\mathbf{e}_y$ point to the right in Fig. 1. The satellite's displacement from Atlantis is

$$\boldsymbol{\rho} = L \cos \alpha \mathbf{e}_x + L \sin \alpha \mathbf{e}_y.$$

For a general small displacement $x\mathbf{e}_x + y\mathbf{e}_y$, expansion of the earth's gravitational field to first order in L/R gives

$$\mathbf{g}_S - \mathbf{g}_A = \frac{\mu}{R^3} (2x \mathbf{e}_x - y \mathbf{e}_y).$$

The additional centrifugal acceleration is

$$\Omega^2 (x \mathbf{e}_x + y \mathbf{e}_y).$$

Their sum is therefore

$$\mathbf{a}_{\text{eff}} = 3\Omega^2 x \mathbf{e}_x.$$

The unit vector perpendicular to the rod, in the direction of increasing α , is

$$\mathbf{e}_\alpha = -\sin \alpha \mathbf{e}_x + \cos \alpha \mathbf{e}_y.$$

The rod's force acts along the rod. The Coriolis force also acts along the rod, because the relative velocity $L\dot{\alpha} \mathbf{e}_\alpha$ is perpendicular to it. Neither contributes to the tangential equation.

Consequently,

$$L\ddot{\alpha} = \mathbf{a}_{\text{eff}} \cdot \mathbf{e}_\alpha = -3\Omega^2 L \sin \alpha \cos \alpha,$$

or

$$\boxed{\ddot{\alpha} + 3\Omega^2 \sin \alpha \cos \alpha = 0}.$$

A constant angle requires

$$\sin \alpha \cos \alpha = 0,$$

so

$$\boxed{\alpha = \frac{n\pi}{2}, \quad n \in \mathbb{Z}}.$$

Thus the rod is either radial or tangential. These angles use the stated approximation $L \ll R$.

a₂ — Stability

Write

$$\alpha = \alpha_0 + \delta, \quad |\delta| \ll 1,$$

where α_0 is an equilibrium angle. Linearizing the equation of motion gives

$$\ddot{\delta} + 3\Omega^2 \cos(2\alpha_0) \delta = 0.$$

For the radial configurations, $\alpha_0 = 0, \pi$,

$$\ddot{\delta} + 3\Omega^2 \delta = 0.$$

The acceleration restores the rod towards equilibrium, so both radial configurations are **stable**.

For the tangential configurations, $\alpha_0 = \pi/2, 3\pi/2$,

$$\ddot{\delta} - 3\Omega^2 \delta = 0.$$

This equation admits an exponentially growing displacement,

$$\delta \propto e^{\sqrt{3}\Omega t},$$

so both tangential configurations are **unstable**.

b — Period of small oscillations

Near either stable equilibrium, the equation found above is that of simple harmonic motion:

$$\ddot{\delta} + \omega_{\text{swing}}^2 \delta = 0, \quad \omega_{\text{swing}} = \sqrt{3}\Omega.$$

Hence

$$\boxed{T_{\text{swing}} = \frac{2\pi}{\sqrt{3}\Omega} = \frac{T}{\sqrt{3}}}.$$

Using the given orbital period,

$$T_{\text{swing}} = \frac{5.4 \times 10^3}{\sqrt{3}} \text{ s} \simeq \boxed{3.1 \times 10^3 \text{ s}}.$$

c₁ — Direction of the induced current

The magnetic force on a positive charge moving with the rod is

$$\mathbf{F}_B = q \mathbf{v} \times \mathbf{B}.$$

In Fig. 1, $\mathbf{v}$ points left and $\mathbf{B}$ points out of the page. Therefore $\mathbf{v} \times \mathbf{B}$ points upwards.

For $\alpha = 0$, S lies below A. The motional emf therefore drives positive charges along the rod from S towards A:

$$\text{Conventional current flows from S to A.}$$

c₂ — Time required and direction of altitude change

The current source reverses the current, so it flows from A to S, downwards in the figure. The magnetic force on the rod is

$$\mathbf{F} = I \boldsymbol{\ell} \times \mathbf{B},$$

where $\boldsymbol{\ell}$ points along the current. Downwards crossed with out of the page points left, along the orbital velocity.

Since the rod is perpendicular to the field,

$$F = IBL = (0.1)(5.0 \times 10^{-5})(2.0 \times 10^4) = \boxed{0.10 \text{ N}}.$$

This forward force supplies energy to the orbit.

Treat the weak thrust as producing a gradual change of a nearly circular orbit. The total orbiting mass is approximately the given shuttle mass m . For a circular orbit,

$$v^2 = \frac{\mu}{R},$$

and the total mechanical energy is

$$E = \frac{1}{2}mv^2 - \frac{\mu m}{R} = -\frac{\mu m}{2R}.$$

The magnetic force supplies power Fv , so

$$Fv = \frac{dE}{dt} = \frac{\mu m}{2R^2} \frac{dR}{dt}.$$

Substituting $v = \Omega R$ and $\mu = \Omega^2 R^3$,

$$F\Omega R = \frac{m\Omega^2 R}{2} \frac{dR}{dt},$$

which gives

$$\frac{dR}{dt} = \frac{2F}{m\Omega} = \frac{FT}{\pi m}.$$

The change of altitude equals the change of orbital radius. Over a rise of only 10 m, we may treat B , F , and T as constant. Thus

$$t = \frac{\pi m \Delta h}{FT} = \frac{\pi m \Delta h}{IBLT}.$$

Numerically,

$$t = \frac{\pi(1.0 \times 10^5)(10)}{(0.10)(5.4 \times 10^3)} \text{ s} = 5.82 \times 10^3 \text{ s}.$$

Therefore,

$$t \simeq 5.8 \times 10^3 \text{ s} \simeq 97 \text{ min} \simeq 1.6 \text{ h}.$$

Because the force adds orbital energy, E becomes less negative and R increases:

The altitude increases by 10 m.

The solution was checked against the official solution.

Problem 3: The rotating neutron star

A 'millisecond pulsar' is a source of radiation in the universe that emits very short pulses with a period of one to several milliseconds. This radiation is in the radio range of wavelengths; and a suitable radio receiver can be used to detect the separate pulses and thereby to measure the period with great accuracy.

These radio pulses originate from the surface of a particular sort of star, the so-called neutron star. These stars are very compact: they have a mass of the same order of magnitude as that of the sun, but their radius is only a few tens of kilometers. They spin very quickly. Because of the fast rotation, a neutron star is slightly flattened (oblate). Assume the axial cross-section of the surface to be an ellipse with almost equal axes. Let r_p be the polar and r_e the equatorial radii; and let us define the flattening factor by:

$$\epsilon = \frac{(r_e - r_p)}{r_p}$$

Consider a neutron star with

a mass of	$2.0 \cdot 10^{30} \text{ kg},$
an average radius of	$1.0 \cdot 10^4 \text{ m},$
and a rotation period of	$2.0 \cdot 10^{-2} \text{ s}.$

a - Calculate the flattening factor, given that the gravitational constant is $6.67 \cdot 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$.

In the long run (over many years) the rotation of the star slows down, due to energy loss, and this leads to a decrease in the flattening. The star has however a solid crust that floats on a liquid interior. The solid crust resists a continuous adjustment to equilibrium shape. Instead, starquakes occur with sudden changes in the shape of the crust towards equilibrium. During and after such a star-quake the angular velocity is observed to change according to figure 1.

angular velocity
of the crust (s^{-1})

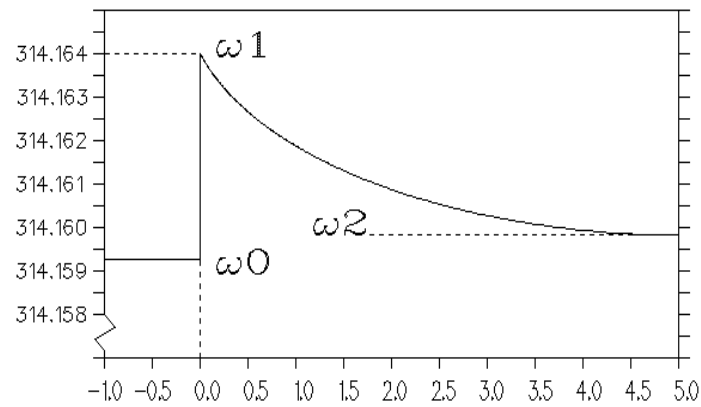


Figure 1

time (days) -->

A sudden change in the shape of the crust of a neutron star results in a sudden change of the angular velocity.

b - Calculate the average radius of the liquid interior, using the data of Fig. 1. Make the approximation that the densities of the crust and the interior are the same. (Ignore the change in shape of the interior).

Official English text of the problem.

Answers

- a.** In the spherical gravitational-field approximation,

$$\epsilon \simeq 3.7 \times 10^{-4}$$

(dimensionless).

- b.** The average radius of the liquid interior is approximately

$$r_{\text{liquid}} \simeq 9.7 \times 10^3 \text{ m} = 9.7 \text{ km}.$$

Solution

a. Flattening factor

Let M , R , and T denote the mass, average radius, and rotation period. The angular velocity is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2.0 \times 10^{-2}} \simeq 314.16 \text{ s}^{-1}.$$

For the equilibrium shape, model the star as a rotating fluid and approximate its gravitational potential by the spherical expression $-GM/r$. In the frame rotating with the star, the effective potential per unit mass is

$$\Phi_{\text{eff}} = -\frac{GM}{r} - \frac{1}{2}\omega^2 s^2,$$

where s is the distance from the rotation axis.

The equilibrium surface is an equipotential: otherwise matter would flow along it. At the pole $s = 0$, while at the equator $s = r_e$. Hence

$$-\frac{GM}{r_p} = -\frac{GM}{r_e} - \frac{1}{2}\omega^2 r_e^2,$$

or

$$GM \left(\frac{1}{r_p} - \frac{1}{r_e} \right) = \frac{1}{2}\omega^2 r_e^2.$$

Using the definition of ϵ ,

$$\frac{1}{r_p} - \frac{1}{r_e} = \frac{r_e - r_p}{r_p r_e} = \frac{\epsilon}{r_e},$$

so that

$$\epsilon = \frac{\omega^2 r_e^3}{2GM}.$$

Since the flattening is small, we may replace r_e by the average radius R in this expression:

$$\epsilon \simeq \frac{\omega^2 R^3}{2GM} = \frac{2\pi^2 R^3}{GMT^2}.$$

Numerically,

$$\epsilon \simeq \frac{2\pi^2 (1.0 \times 10^4)^3}{(6.67 \times 10^{-11})(2.0 \times 10^{30})(2.0 \times 10^{-2})^2} = 3.699 \times 10^{-4}.$$

Thus

$$\boxed{\epsilon \simeq 3.7 \times 10^{-4}}.$$

This corresponds to $r_e - r_p \simeq \epsilon R \simeq 3.7 \text{ m}$, consistent with nearly equal radii.

b. Radius of the liquid interior

Before the quake, assume that the crust and liquid interior rotate together at angular velocity ω_0 . During the sudden deformation, the crust's moment of inertia decreases. There is insufficient time for appreciable angular momentum transfer to the liquid, so the crust spins up to ω_1 , while the liquid initially remains at ω_0 .

Over the following days, internal friction transfers angular momentum from the crust to the liquid until both rotate at ω_2 . External angular momentum loss is negligible over this interval compared with the secular evolution over years.

Let I_c be the crust's moment of inertia after the quake and I_l that of the liquid. The liquid's moment of inertia is unchanged, as stipulated. Conservation of angular momentum during the relaxation gives

$$I_c \omega_1 + I_l \omega_0 = (I_c + I_l) \omega_2.$$

Rearranging,

$$I_c(\omega_1 - \omega_2) = I_l(\omega_2 - \omega_0).$$

Consequently,

$$\boxed{\frac{I_l}{I_c + I_l} = \frac{\omega_1 - \omega_2}{\omega_1 - \omega_0}}.$$

The required conserved quantity is angular momentum; rotational kinetic energy is dissipated during the relaxation.

To relate the moments of inertia to radii, neglect the small oblateness and use the common density ρ specified in the question. A spherical shell of radius r and thickness dr has

$$dm = 4\pi\rho r^2 dr, \quad dI = \frac{2}{3}r^2 dm.$$

Thus a uniform sphere of radius a has

$$I(a) = \frac{8\pi\rho}{3} \int_0^a r^4 dr = \frac{8\pi\rho}{15} a^5.$$

Writing the liquid radius as r_l , we obtain

$$\frac{I_l}{I_c + I_l} = \frac{r_l^5}{R^5}.$$

Therefore,

$$r_l = R \left(\frac{\omega_1 - \omega_2}{\omega_1 - \omega_0} \right)^{1/5}.$$

Approximate readings from the graph are

$$\omega_0 \simeq 314.1593 \text{ s}^{-1}, \quad \omega_1 \simeq 314.1640 \text{ s}^{-1}, \quad \omega_2 \simeq 314.1599 \text{ s}^{-1}.$$

These give

$$\left(\frac{r_l}{R} \right)^5 \simeq \frac{0.0041}{0.0047} \simeq 0.872,$$

and hence

$$r_l \simeq (1.0 \times 10^4)(0.872)^{1/5} \simeq 9.73 \times 10^3 \text{ m}.$$

To sensible precision,

$$r_l \simeq 9.7 \text{ km}.$$

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.