

IPhO 1989

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Problem 1: Boiling of two immiscible liquids

Consider two liquids A and B insoluble in each other. The pressures p_i ($i = A$ or B) of their saturated vapors obey, to a good approximation, the formula:

$$\ln(p_i/p_0) = \frac{\alpha_i}{T} + \beta_i, \quad i = A \text{ or } B,$$

where p_0 denotes the normal atmospheric pressure, T – the absolute temperature of the vapor, and α_i and β_i ($i = A$ or B) – certain constants depending on the liquid. (The symbol $\ln$ denotes the natural logarithm, i.e. logarithm with base $e = 2.7182818 \dots$)

The values of the ratio p_i/p_0 for the liquids A and B at the temperature 40°C and 90°C are given in Tab. 1.1.

Table 1.1

$t [^\circ\text{C}]$	$p_i/p_0, i = A$	$p_i/p_0, i = B$
40	0.284	0.07278
90	1.476	0.6918

The errors of these values are negligible.

A. Determine the boiling temperatures of the liquids A and B under the pressure p_0 .

B. The liquids A and B were poured into a vessel in which the layers shown in Fig. 1.1 were formed. The surface of the liquid B has been covered with a thin layer of a non-volatile liquid C, which is insoluble in the liquids A and B and vice versa, thereby preventing any free evaporation from the upper surface of the liquid B. The ratio of molecular masses of the liquids A and B (in the gaseous phase) is:

$$\gamma = \mu_A/\mu_B = 8.$$

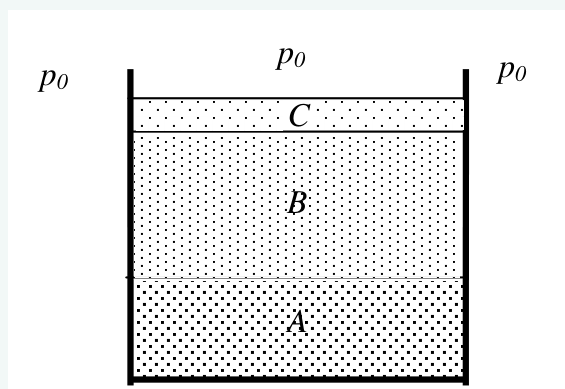


Fig. 1.1

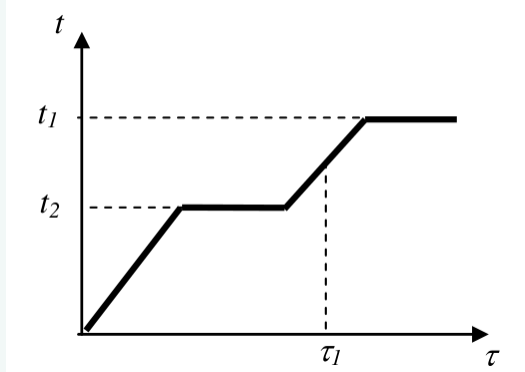


Fig. 1.2

The masses of the liquids A and B were initially the same, each equal to $m = 100\text{g}$. The heights of the layers of the liquids in the vessel and the densities of the liquids are small enough to make the assumption that the pressure in any point in the vessel is practically equal to the normal atmospheric pressure p_0 .

The system of liquids in the vessel is slowly, but continuously and uniformly, heated. It was established that the temperature t of the liquids changed with time τ as shown schematically in the Fig. 1.2.

Determine the temperatures t_1 and t_2 corresponding to the horizontal parts of the diagram and the masses of the liquids A and B at the time τ_1 . The temperatures should be rounded to the nearest degree (in $^{\circ}\text{C}$) and the masses of the liquids should be determined to one-tenth of gram.

REMARK: Assume that the vapors of the liquids, to a good approximation,

(1) obey the Dalton law stating that the pressure of a mixture of gases is equal to the sum of the partial pressures of the gases forming the mixture and

(2) can be treated as perfect gases up to the pressures corresponding to the saturated vapors.

Official English text of the problem.

Answers

- **A.** The normal boiling temperatures are

$$t_A = 76.8^{\circ}\text{C}, \quad t_B = 99.7^{\circ}\text{C}$$

(approximately 77°C and 100°C , respectively).

- **B.** To the requested precision,

$$t_1 = 100^{\circ}\text{C}, \quad t_2 = 67^{\circ}\text{C},$$

$$m_A(\tau_1) = 0.0 \text{ g}, \quad m_B(\tau_1) = 95.5 \text{ g}.$$

Solution

A. Normal boiling temperatures

For a pure liquid, boiling occurs when its saturated vapor pressure equals the external pressure. Thus, under pressure p_0 ,

$$p_i(T_{b,i}) = p_0 \implies 0 = \frac{\alpha_i}{T_{b,i}} + \beta_i,$$

so

$$T_{b,i} = -\frac{\alpha_i}{\beta_i}.$$

To determine the constants, let

$$T_L = 313.15 \text{ K}, \quad T_H = 363.15 \text{ K},$$

and denote the corresponding tabulated pressure ratios by $r_{i,L}$ and $r_{i,H}$. Subtracting the two vapor-pressure equations gives

$$\alpha_i = \frac{\ln r_{i,H} - \ln r_{i,L}}{T_H^{-1} - T_L^{-1}}, \quad \beta_i = \ln r_{i,L} - \frac{\alpha_i}{T_L}.$$

Using the supplied data:

Liquid	α_i (K)	β_i
A	-3748.4907	10.711491
B	-5121.6396	13.734914

Therefore,

$$T_{b,A} = 349.9504 \text{ K}, \quad T_{b,B} = 372.8920 \text{ K}.$$

Converting to Celsius,

$$t_A = 76.8^\circ\text{C}, \quad t_B = 99.7^\circ\text{C}.$$

B. The two plateaus and the remaining masses

The slow heating permits a quasistatic treatment using equilibrium vapor pressures. As specified, hydrostatic pressure differences are negligible. The non-volatile layer C prevents free surface evaporation.

At the A–B interface, a vapor bubble in equilibrium with both liquids contains saturated vapor of each. Because the liquids are immiscible, each contributes its own saturated vapor pressure at the common temperature. Dalton's law gives

$$p_{\text{bubble}} = p_A(T) + p_B(T).$$

Consequently, boiling begins at the lower plateau temperature T_2 , determined by

$$p_A(T_2) + p_B(T_2) = p_0.$$

Using the fitted vapor-pressure laws,

$$\exp\left(\frac{\alpha_A}{T_2} + \beta_A\right) + \exp\left(\frac{\alpha_B}{T_2} + \beta_B\right) = 1.$$

Numerical solution gives

$$T_2 = 340.1208 \text{ K}, \quad t_2 = 66.9708^\circ\text{C}.$$

Thus,

$$\boxed{t_2 = 67^\circ\text{C}}.$$

At this temperature,

$$\frac{p_A}{p_0} = 0.7337662, \quad \frac{p_B}{p_0} = 0.2662338.$$

While both liquids remain, their saturated vapor pressures depend only on temperature, so boiling continues at this fixed temperature. The supplied heat provides the latent heat of vaporization.

To determine which liquid is exhausted first, consider a volume V of the escaping vapor mixture. The ideal-gas law for each component gives

$$p_A V = n_A R T_2, \quad p_B V = n_B R T_2,$$

and hence

$$\frac{n_A}{n_B} = \frac{p_A}{p_B}.$$

The molecular-mass ratio is also the molar-mass ratio. Therefore, the positive masses lost by evaporation satisfy

$$\frac{dm_A^{\text{evap}}}{dm_B^{\text{evap}}} = \frac{\mu_A p_A}{\mu_B p_B} = \gamma \frac{p_A}{p_B}.$$

Numerically,

$$\gamma \frac{p_A}{p_B} = 8 \frac{0.7337662}{0.2662338} = 22.0488.$$

This ratio remains constant throughout the lower plateau. Since the initial masses are equal, **A is exhausted first**. When its entire 100 g has evaporated, the mass of B lost is

$$\Delta m_B = \frac{100}{22.0488} \text{ g} = 4.5354 \text{ g}.$$

The remaining liquid masses are therefore

$$m_A = 0, \quad m_B = 100 - 4.5354 = 95.4646 \text{ g}.$$

Once A is exhausted, B alone has insufficient vapor pressure to boil at T_2 , since

$$p_B(T_2) = 0.2662338 p_0 < p_0.$$

Boiling stops and the temperature rises until B reaches its normal boiling temperature. The upper plateau is therefore

$$t_1 = t_B = 99.7420^\circ\text{C},$$

which rounds to

$$\boxed{t_1 = 100^\circ\text{C}}.$$

The marked time τ_1 lies on the rising segment between the two plateaus. At this stage A has disappeared, B has not yet resumed boiling, and layer C prevents free surface evaporation. Thus the liquid masses remain those at the end of the lower plateau:

$$m_A(\tau_1) = 0.0 \text{ g}, \quad m_B(\tau_1) = 95.5 \text{ g}.$$

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Three gravitating masses rotating as a rigid body

Three non-collinear points P_1 , P_2 and P_3 , with known masses m_1 , m_2 and m_3 , interact with one another through their mutual gravitational forces only; they are isolated in free space and do not interact with any other bodies. Let σ denote the axis going through the center-of-mass of the three masses, and perpendicular to the triangle $P_1P_2P_3$. What conditions should the angular velocities ω of the system (around the axis σ) and the distances:

$$P_1P_2 = a_{12}, \quad P_2P_3 = a_{23}, \quad P_1P_3 = a_{13},$$

fulfill to allow the shape and size of the triangle $P_1P_2P_3$ unchanged during the motion of the system, i.e. under what conditions does the system rotate around the axis σ as a rigid body?

Official English text of the problem.

Answers

- **Problem 2:** The triangle must be equilateral:

$$a_{12} = a_{23} = a_{13} = a > 0.$$

All three masses must rotate with the same constant angular velocity, whose magnitude satisfies

$$\omega^2 = \frac{G(m_1 + m_2 + m_3)}{a^3}.$$

Either sense of rotation is possible. These conditions are necessary and sufficient for any positive masses; the masses need not be equal.

Solution

Problem 2

Use Newton's laws of motion and Newtonian gravitation for point masses. Define

$$M = m_1 + m_2 + m_3.$$

There is no external force, so the center of mass moves uniformly. Work in an inertial frame in which it is at rest, and let $\mathbf{r}_i$ be the position of P_i relative to it. Then

$$\sum_{i=1}^3 m_i \mathbf{r}_i = \mathbf{0}.$$

For rigid rotation about the perpendicular axis σ , all three masses have a common angular velocity ω . Their angular momentum about the center of mass is

$$\mathbf{L} = I\boldsymbol{\omega}, \quad I = \sum_{i=1}^3 m_i |\mathbf{r}_i|^2.$$

The gravitational forces are internal and central, so their total torque vanishes and $\mathbf{L}$ is conserved. Because a rigid triangle has constant I , $\boldsymbol{\omega}$ is constant. Consequently,

$$\ddot{\mathbf{r}}_i = -\omega^2 \mathbf{r}_i.$$

Newton's second law, after dividing by m_i , therefore requires

$$-\omega^2 \mathbf{r}_i = G \sum_{j \neq i} \frac{m_j}{a_{ij}^3} (\mathbf{r}_j - \mathbf{r}_i). \quad (1)$$

The center-of-mass relation also gives

$$\sum_{j \neq i} m_j (\mathbf{r}_j - \mathbf{r}_i) = -M \mathbf{r}_i. \quad (2)$$

Substituting (2) into the left-hand side of (1) and rearranging,

$$\sum_{j \neq i} m_j \left(\frac{G}{a_{ij}^3} - \frac{\omega^2}{M} \right) (\mathbf{r}_j - \mathbf{r}_i) = \mathbf{0}. \quad (3)$$

At each vertex, the two vectors pointing toward the other vertices are linearly independent, since the three points are non-collinear. Thus both coefficients in (3) must vanish. With positive masses, this implies, for every pair,

$$\frac{G}{a_{ij}^3} = \frac{\omega^2}{M}.$$

Hence

$$a_{12} = a_{23} = a_{13} = a, \quad \omega^2 = \frac{GM}{a^3}.$$

This proves necessity.

To establish sufficiency, take an equilateral configuration of side a . The gravitational acceleration of any mass is

$$\ddot{\mathbf{r}}_i = \frac{G}{a^3} \sum_{j \neq i} m_j (\mathbf{r}_j - \mathbf{r}_i) = -\frac{GM}{a^3} \mathbf{r}_i.$$

Thus its resultant gravitational force points toward the center of mass and provides exactly the centripetal acceleration required when

$$\omega^2 = \frac{GM}{a^3}.$$

Choose the initial velocities in the center-of-mass frame as

$$\dot{\mathbf{r}}_i(0) = \boldsymbol{\omega} \times \mathbf{r}_i(0), \quad \boldsymbol{\omega} \parallel \boldsymbol{\sigma}.$$

The corresponding uniform rotation preserves the equilateral triangle and satisfies Newton's equations at every time. The conditions are therefore sufficient as well. An arbitrary uniform motion of the center of mass may be added.

The solution was checked against the official solution.

Problem 3: From an electron microscope to a proton microscope

The problem concerns investigation of transforming the electron microscope with magnetic guiding of the electron beam (which is accelerated with the potential difference $U = 511 \text{ kV}$) into a proton microscope (in which the proton beam is accelerated with the potential difference $-U$). For this purpose, solve the following two problems:

A. An electron after leaving a device, which accelerated it with the potential difference U , falls into a region with an inhomogeneous field $\mathbf{B}$ generated with a system of stationary coils $L_1, L_2, \dots, L_n$. The known currents in the coils are $i_1, i_2, \dots, i_n$, respectively.

What should the currents $i'_1, i'_2, \dots, i'_n$ in the coils $L_1, L_2, \dots, L_n$ be, in order to guide the proton (initially accelerated with the potential difference $-U$) along the same trajectory (and in the same direction) as that of the electron?

HINT: The problem can be solved by finding a condition under which the equation describing the trajectory is the same in both cases. It may be helpful to use the relation:

$$\mathbf{p} \frac{d}{dt} \mathbf{p} = \frac{1}{2} \frac{d}{dt} \mathbf{p}^2 = \frac{1}{2} \frac{d}{dt} p^2.$$

B. How many times would the resolving power of the above microscope increase or decrease if the electron beam were replaced with the proton beam? Assume that the resolving power of the microscope (i.e. the smallest distance between two point objects whose circular images can be just separated) depends only on the wave properties of the particles.

Assume that the velocities of the electrons and protons before their acceleration are zero, and that there is no interaction between own magnetic moment of either electrons or protons and the magnetic field. Assume also that the electromagnetic radiation emitted by the moving particles can be neglected.

NOTE: Very often physicists use 1 electron-volt (1 eV), and its derivatives such as 1 keV or 1 MeV, as a unit of energy. 1 electron-volt is the energy gained by the electron that passed the potential difference equal to 1 V.

Perform the calculations assuming the following data:

Rest energy of electron:	$E_e = m_e c^2 = 511 \text{ keV}$
Rest energy of proton:	$E_p = m_p c^2 = 938 \text{ MeV}$

Official English text of the problem.

Answers

- A.** Reverse every coil current and multiply its magnitude by approximately 35.0:

$$i'_k = -\sqrt{\frac{2E_p + eU}{2E_e + eU}} i_k \simeq -35.0 i_k, \quad k = 1, \dots, n.$$

- **B.** The minimum resolvable distance becomes

$$\frac{d_p}{d_e} = \frac{\lambda_p}{\lambda_e} \simeq 0.0286.$$

Thus the minimum distance—the quantity called “resolving power” in the statement—**decreases by a factor of 35.0**. The ability to resolve fine detail improves by that factor; resolving power defined conventionally as $1/d$ increases by 35.0.

Solution

A. Coil currents

Let $e > 0$ denote the elementary charge, so that $q_e = -e$ and $q_p = +e$. After acceleration, the motion in the magnetic field obeys the relativistic Lorentz-force equation

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B}, \quad \mathbf{p} = \gamma m\mathbf{v}.$$

Taking the scalar product with $\mathbf{p}$, which is parallel to $\mathbf{v}$, gives

$$\frac{1}{2} \frac{d(p^2)}{dt} = \mathbf{p} \cdot \frac{d\mathbf{p}}{dt} = q\mathbf{p} \cdot (\mathbf{v} \times \mathbf{B}) = 0.$$

Hence the momentum magnitude p remains constant. This holds even when the magnetic field is inhomogeneous, under the stated neglect of magnetic-moment interactions and radiation.

To describe the spatial trajectory independently of the particle’s speed, let s be arc length increasing in the direction of motion, and let

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}$$

be the unit tangent. Then $\mathbf{p} = p\mathbf{t}$, $\mathbf{v} = v\mathbf{t}$, and $ds/dt = v$. Consequently,

$$pv \frac{d\mathbf{t}}{ds} = qv\mathbf{t} \times \mathbf{B},$$

so the trajectory equations are

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}, \quad \frac{d\mathbf{t}}{ds} = \frac{q}{p} \mathbf{t} \times \mathbf{B}(\mathbf{r}).$$

For the electron in $\mathbf{B}$ and the proton in a new field $\mathbf{B}'$, these equations coincide if

$$\frac{e}{p_p} \mathbf{B}' = -\frac{e}{p_e} \mathbf{B},$$

or

$$\mathbf{B}'(\mathbf{r}) = -\frac{p_p}{p_e} \mathbf{B}(\mathbf{r}).$$

With the same entry point and direction, both particles therefore follow the same oriented spatial trajectory.

The magnetic field of the coil system is linear in its currents. Multiplying every current by the same factor multiplies the field everywhere by that factor. Thus the required choice is

$$i'_k = -\frac{p_p}{p_e} i_k.$$

It remains to calculate the momentum ratio. Both particles start from rest and gain the same kinetic energy,

$$K = eU = 511 \text{ keV} = 0.511 \text{ MeV}.$$

The electron must be treated relativistically because $K = E_e$. We use the exact relativistic expression for both particles:

$$(K + mc^2)^2 = p^2 c^2 + m^2 c^4.$$

Therefore,

$$pc = \sqrt{K(K + 2mc^2)},$$

and

$$\frac{p_p}{p_e} = \sqrt{\frac{K + 2E_p}{K + 2E_e}}.$$

Numerically, using energies in MeV,

$$\begin{aligned} p_e c &= \sqrt{0.511(0.511 + 2 \times 0.511)} \simeq 0.885 \text{ MeV}, \\ p_p c &= \sqrt{0.511(0.511 + 2 \times 938)} \simeq 31.0 \text{ MeV}. \end{aligned}$$

The ratio is

$$\frac{p_p}{p_e} = \sqrt{\frac{1876.511}{1.533}} = 34.9868 \dots$$

Hence

$$\boxed{i'_k \simeq -35.0 i_k \quad (k = 1, \dots, n).}$$

The negative sign is essential: reversing the magnetic field compensates for the reversal of the particle's charge.

B. Resolving power

The de Broglie wavelength is

$$\lambda = \frac{h}{p}.$$

For a diffraction-limited microscope with a fixed aperture geometry, the minimum resolvable object separation is proportional to the wavelength. More explicitly, it has the form

$$d_{\min} = C \frac{\lambda}{\sin \alpha},$$

where α is the aperture half-angle and C depends on the resolution criterion.

The field scaling found in part A preserves the beam trajectories and therefore the geometrical factors. Thus

$$\frac{d_p}{d_e} = \frac{\lambda_p}{\lambda_e} = \frac{p_e}{p_p} = \sqrt{\frac{K + 2E_e}{K + 2E_p}} = \frac{1}{34.9868 \dots}.$$

Therefore,

$$\boxed{d_p \simeq 0.0286 d_e.}$$

The proton microscope can distinguish points approximately 35.0 times closer together. Under the statement's definition, its "resolving power," meaning the smallest resolvable distance, decreases by this factor. The conventional reciprocal resolving power increases by the same factor.

The solution was checked against the official solution.