

IPhO 1988

Theory problems and solutions · Bad Ischl, Austria · 3 theory problems with complete solutions

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Problem 1: Spectroscopy of Particle Velocities

Basic Data

The absorption and emission of a photon is a reversible process. A good example is to be found in the excitation of an atom from the ground state to a higher energy state and the atoms' subsequent return to the ground state. In such a case we may detect the absorption of a photon from the phenomenon of spontaneous emission or fluorescence. Some of the more modern instrumentation make use of this principle to identify atoms, and also to measure or calculate the value of the velocity in the velocity spectrum of the electron beam.

In an idealised experiment (see fig. 19.1) a single-charged ion travels in the opposite direction to light from a laser source with velocity v . The wavelength of light from the laser source is adjustable. An ion with velocity Zero can be excited to a higher energy state by the application of laser light having a wavelength of $\lambda = 600 \text{ nm}$. If we excite a moving ion, our knowledge on Dopplers' effect tells us that we need to apply laser light of a wavelength other than the value given above.

There is given a velocity spectrum embracing velocity magnitude from $v_1 = 0 \frac{\text{m}}{\text{s}}$ to $v_2 = 6,000 \frac{\text{m}}{\text{s}}$. (see fig. 19.1)

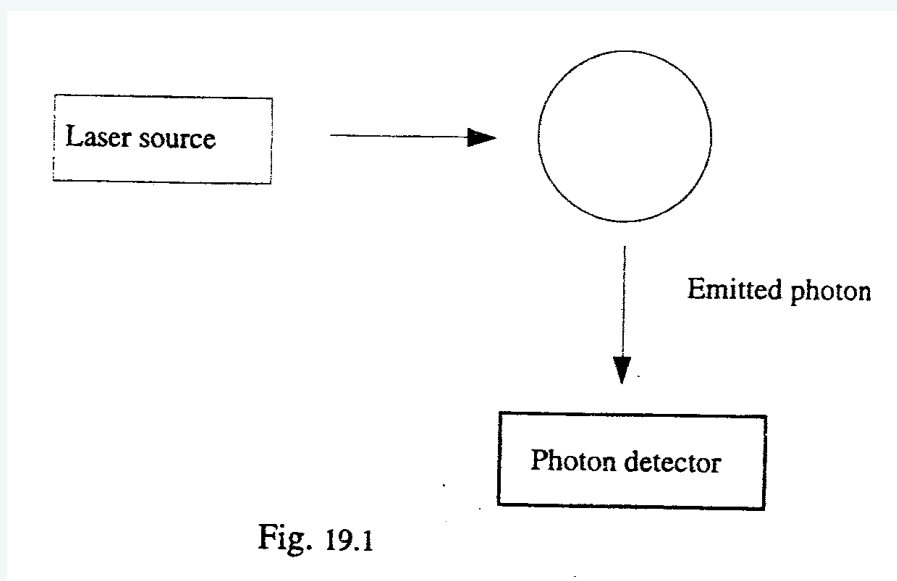


Fig. 19.1

Questions

1.1

1.1.1

What range of wavelength of the laser beam must be used to excite ions of all velocities in the velocity spectrum given above ?

1.1.2

A rigorous analysis of the problem calls for application of the principle from the theory of special relativity

$$\nu' = \nu \cdot \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

Determine the error when the classical formula for Dopplers' effect is used to solve the problem.

1.2

Assuming the ions are accelerated by a potential U before excited by the laser beam, determine the relationship between the width of the velocity spectrum of the ion beam and the accelerating potential. Does the accelerating voltage increase or decrease the velocity spectrum width ?

1.3

Each ion has the value $\frac{e}{m} = 4 \cdot 10^6 \frac{\text{A} \cdot \text{s}}{\text{kg}}$, two energy levels corresponding to wavelength $\lambda^{(1)} = 600 \text{ nm}$ and wavelength $\lambda^{(2)} = \lambda^{(1)} + 10^{-3} \text{ nm}$. Show that lights of the two wavelengths used to excite ions overlap when no accelerating potential is applied. Can accelerating voltage be used to separate the two spectra of laser light used to excite ions so that they no longer overlap ? If the answer is positive, calculate the minimum value of the voltage required.

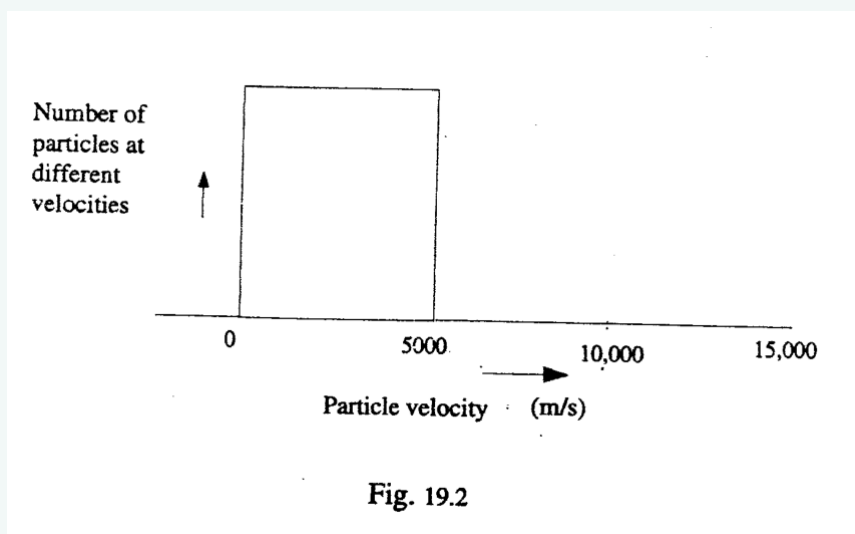


Fig. 19.2

Official English text of the problem.

Answers

- **1.1.1:** The required laser wavelength range is

$$600.000 \text{ nm} \leq \lambda_L \leq 600.012 \text{ nm},$$

with width 0.012 nm.

- **1.1.2:** The classical formula underestimates the upper wavelength by approximately

$$1.2 \times 10^{-7} \text{ nm}.$$

The maximum relative error in wavelength is 2.0×10^{-10} ; the relative error in the scan width is 1.0×10^{-5} .

- **1.2:** After acceleration, the velocity-spectrum width is

$$\Delta v(U) = \sqrt{v_2^2 + \frac{2eU}{m}} - \sqrt{v_1^2 + \frac{2eU}{m}}.$$

It decreases as the accelerating voltage increases.

- **1.3:** Without acceleration, the two wavelength intervals are approximately

$$[600.000, 600.012] \text{ nm}, \quad [600.001, 600.013] \text{ nm},$$

so they overlap. Acceleration can separate them, with threshold

$$U_{\min} \approx 160 \text{ V}.$$

At the threshold the bands just touch; a slightly larger voltage produces a finite gap.

Solution

Use $c = 3.00 \times 10^8 \text{ m s}^{-1}$. Let λ_0 denote an ion's resonance wavelength at rest, and let λ_L denote the laser wavelength in the laboratory. The idealized spectra are taken to have no intrinsic linewidth.

1.1.1

The ion moves towards the incoming light, so it observes a higher frequency than the laboratory laser frequency. In the classical moving-observer approximation,

$$\nu' = \nu_L \left(1 + \frac{v}{c}\right).$$

Excitation requires the frequency seen by the ion to equal its rest resonance frequency:

$$\nu' = \nu_0 = \frac{c}{\lambda_0}.$$

Since $\nu_L = c/\lambda_L$,

$$\frac{c}{\lambda_0} = \frac{c}{\lambda_L} \left(1 + \frac{v}{c}\right),$$

and therefore

$$\lambda_L(v) = \lambda_0 \left(1 + \frac{v}{c}\right).$$

Thus the laboratory laser wavelength must increase with ion speed. At the endpoints,

$$\lambda_L(0) = 600 \text{ nm},$$

$$\lambda_L(6000) = 600 \left(1 + \frac{6000}{3.00 \times 10^8} \right) \text{ nm} = 600.012 \text{ nm}.$$

The required scan width is consequently

$$\Delta\lambda_L = \frac{\lambda_0(v_2 - v_1)}{c} = 0.012 \text{ nm}.$$

1.1.2

Put $\beta = v/c$. The supplied relativistic formula gives the resonance condition

$$\nu_0 = \nu_L \sqrt{\frac{1 + \beta}{1 - \beta}},$$

so the exact required wavelength is

$$\lambda_{\text{rel}} = \lambda_0 \sqrt{\frac{1 + \beta}{1 - \beta}} = \lambda_0 \frac{1 + \beta}{\sqrt{1 - \beta^2}}.$$

The classical value is $\lambda_{\text{cl}} = \lambda_0(1 + \beta)$. Hence its fractional underestimate, measured relative to the exact wavelength, is

$$\epsilon_\lambda = \frac{\lambda_{\text{rel}} - \lambda_{\text{cl}}}{\lambda_{\text{rel}}} = 1 - \sqrt{1 - \beta^2} \simeq \frac{\beta^2}{2}.$$

This error is greatest at $v_2 = 6000 \text{ m s}^{-1}$, for which

$$\beta_2 = 2.00 \times 10^{-5}, \quad \epsilon_{\lambda, \text{max}} \simeq 2.00 \times 10^{-10}.$$

Equivalently, expanding the exact wavelength,

$$\lambda_{\text{rel}} = \lambda_0 \left(1 + \beta + \frac{\beta^2}{2} + O(\beta^3) \right),$$

gives the maximum absolute correction

$$\delta\lambda_{\text{max}} \simeq \frac{\lambda_0 \beta_2^2}{2} = 600 \text{ nm} \times 2.00 \times 10^{-10} = 1.20 \times 10^{-7} \text{ nm}.$$

Thus the relativistic upper endpoint is approximately

$$600.01200012 \text{ nm}.$$

The lower endpoint has no correction because $v_1 = 0$. Therefore the absolute error in the scan width is also $1.20 \times 10^{-7} \text{ nm}$, corresponding to

$$\frac{\delta(\Delta\lambda_L)}{\Delta\lambda_L} \simeq \frac{1.20 \times 10^{-7}}{0.012} = 1.00 \times 10^{-5}$$

or 0.001%.

1.2

Let u be an ion's speed after acceleration. Every singly charged ion gains the same kinetic energy eU . Using nonrelativistic energy conservation,

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + eU,$$

so

$$u(v, U) = \sqrt{v^2 + \frac{2eU}{m}}.$$

This function increases with the initial speed v , so the endpoints of the accelerated distribution are

$$u_1 = \sqrt{v_1^2 + \frac{2eU}{m}}, \quad u_2 = \sqrt{v_2^2 + \frac{2eU}{m}}.$$

Consequently,

$$\Delta v(U) = u_2 - u_1 = \sqrt{v_2^2 + \frac{2eU}{m}} - \sqrt{v_1^2 + \frac{2eU}{m}}.$$

Rationalizing makes its dependence on voltage especially clear:

$$\Delta v(U) = \frac{v_2^2 - v_1^2}{\sqrt{v_2^2 + 2eU/m} + \sqrt{v_1^2 + 2eU/m}}.$$

The numerator is constant and the denominator increases with U . Thus acceleration shifts the whole distribution towards higher speeds while **decreasing its width**.

For the given $v_1 = 0$,

$$\Delta v(U) = \sqrt{v_2^2 + \frac{2eU}{m}} - \sqrt{\frac{2eU}{m}}.$$

When $2eU/m \gg v_2^2$, this becomes

$$\Delta v(U) \simeq \frac{v_2^2}{2\sqrt{2eU/m}} \propto U^{-1/2}.$$

1.3

Write

$$\lambda_1 = 600 \text{ nm}, \quad \lambda_2 = 600.001 \text{ nm}, \quad \delta\lambda = \lambda_2 - \lambda_1 = 0.001 \text{ nm}.$$

Without acceleration, each rest wavelength produces a laser excitation interval

$$\mathcal{I}_i(0) = \left[\lambda_i, \lambda_i \left(1 + \frac{v_2}{c} \right) \right].$$

Numerically,

$$\begin{aligned} \mathcal{I}_1(0) &= [600.000, 600.012] \text{ nm}, \\ \mathcal{I}_2(0) &\simeq [600.001, 600.013] \text{ nm}. \end{aligned}$$

Their common interval is approximately

$$[600.001, 600.012] \text{ nm},$$

so the two spectra clearly overlap.

After acceleration, the intervals become

$$\mathcal{I}_i(U) = \left[\lambda_i \left(1 + \frac{u_1}{c} \right), \lambda_i \left(1 + \frac{u_2}{c} \right) \right].$$

Since $\lambda_2 > \lambda_1$, separation occurs when the upper edge of the first interval lies below the lower edge of the second. The limiting condition is

$$\lambda_1 \left(1 + \frac{u_2}{c}\right) = \lambda_2 \left(1 + \frac{u_1}{c}\right).$$

Equivalently,

$$u_2 - u_1 = \frac{c \delta \lambda}{\lambda_1} \left(1 + \frac{u_1}{c}\right).$$

To leading Doppler order, $u_1/c \ll 1$, so the required limiting velocity width is

$$d \equiv \frac{c \delta \lambda}{\lambda_1} = \frac{(3.00 \times 10^8)(0.001)}{600} \text{ m s}^{-1} = \boxed{500 \text{ m s}^{-1}}.$$

Acceleration can achieve this because it continuously narrows the velocity distribution.

With $v_1 = 0$, energy conservation also gives

$$u_2^2 - u_1^2 = v_2^2.$$

At the separation threshold, $u_2 - u_1 \simeq d$, and therefore

$$u_2 + u_1 \simeq \frac{v_2^2}{d} = \frac{6000^2}{500} = 72,000 \text{ m s}^{-1}.$$

Solving for the two speeds,

$$u_1 \simeq \frac{72,000 - 500}{2} = 35,750 \text{ m s}^{-1},$$

$$u_2 \simeq 36,250 \text{ m s}^{-1}.$$

Finally, using $e/m = 4 \times 10^6 \text{ C kg}^{-1}$,

$$U_{\min} \simeq \frac{u_1^2}{2(e/m)} = \frac{(35,750)^2}{8 \times 10^6} \text{ V} \approx \boxed{160 \text{ V}}.$$

The resulting speeds satisfy $u/c \approx 1.2 \times 10^{-4}$, consistent with the approximations used. Retaining the small factor $1 + u_1/c$ in the classical endpoint condition does not change the reported voltage.

At the limiting voltage the two spectral edges coincide. Strict separation requires a voltage slightly above this threshold.

The solution was checked against the official solution.

Problem 2: Maxwell's Wheel

Introduction

A cylindrical wheel of uniform density, having the mass $M = 0,40 \text{ kg}$, the radius $R = 0,060 \text{ m}$ and the thickness $d = 0,010 \text{ m}$ is suspended by means of two light strings of the same length from the ceiling. Each string is wound around the axle of the wheel. Like the strings, the mass of the axle is negligible. When the wheel is turned manually, the strings are wound up until the centre of mass is raised $1,0 \text{ m}$ above the floor. If the wheel is allowed to move downward vertically under the pulling

force of the gravity, the strings are unwound to the full length of the strings and the wheel reaches the lowest point. The strings then begin to wound in the opposite sense resulting in the wheel being raised upwards.

Analyze and answer the following questions, assuming that the strings are in vertical position and the points where the strings touch the axle are directly below their respective suspending points (see fig. 19.5).

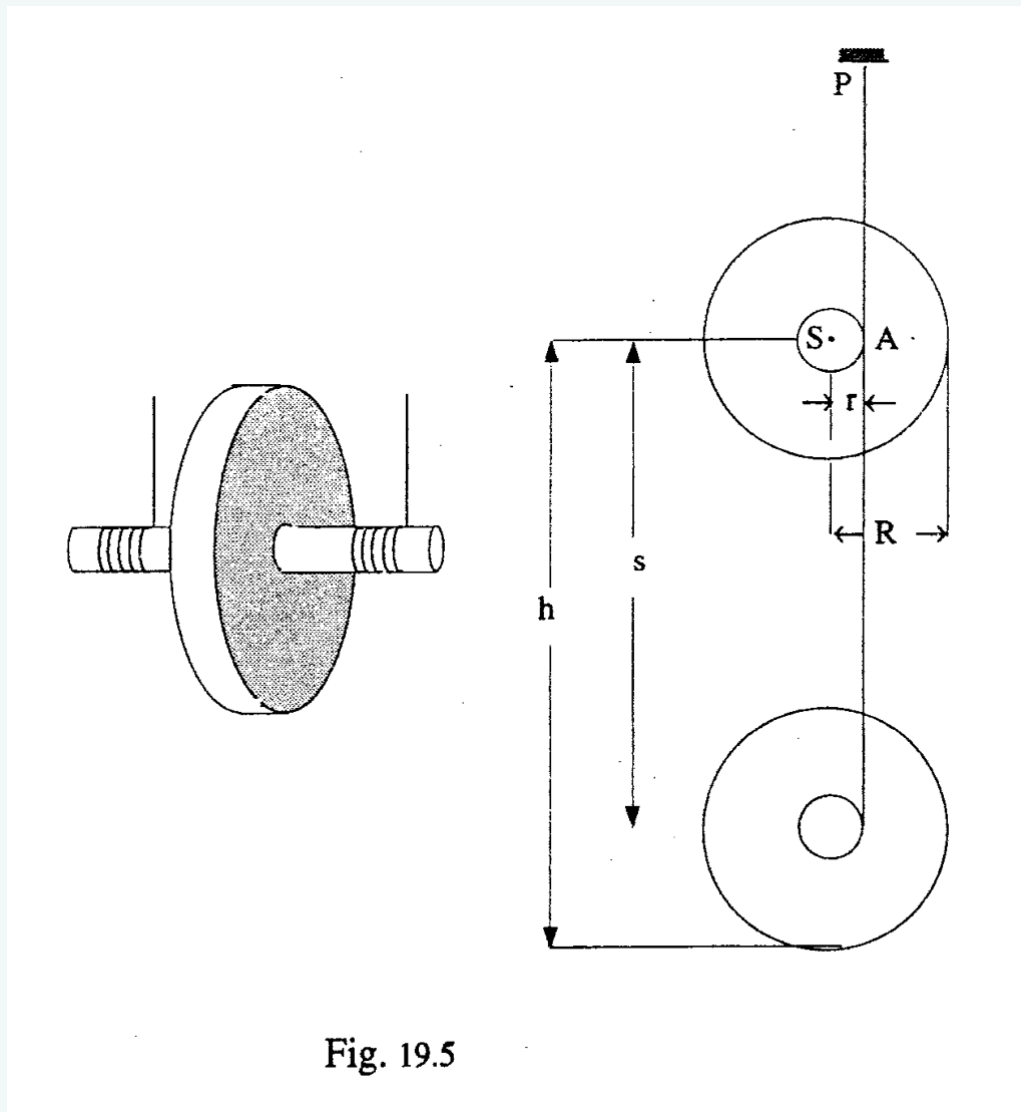


Fig. 19.5

Fig. 19.5

Questions

2.1

Determine the angular speed of the wheel when the centre of mass of the wheel covers the vertical distance s .

2.2

Determine the kinetic energy of the linear motion of the centre of mass E_r after the wheel travels a distance $s = 0,50$ m, and calculate the ratio between E_r and the energy in any other form in this problem up to this point.

Radius of the axle = 0,0030 m

2.3

Determine the tension in the string while the wheel is moving downward.

2.4

Calculate the angular speed ω' as a function of the angle Φ when the strings begin to unwind themselves in opposite sense as depicted in fig. 19.6.

Sketch a graph of variables which describe the motion (in cartesian system which suits the problem) and also the speed of the centre of mass as a function of Φ .

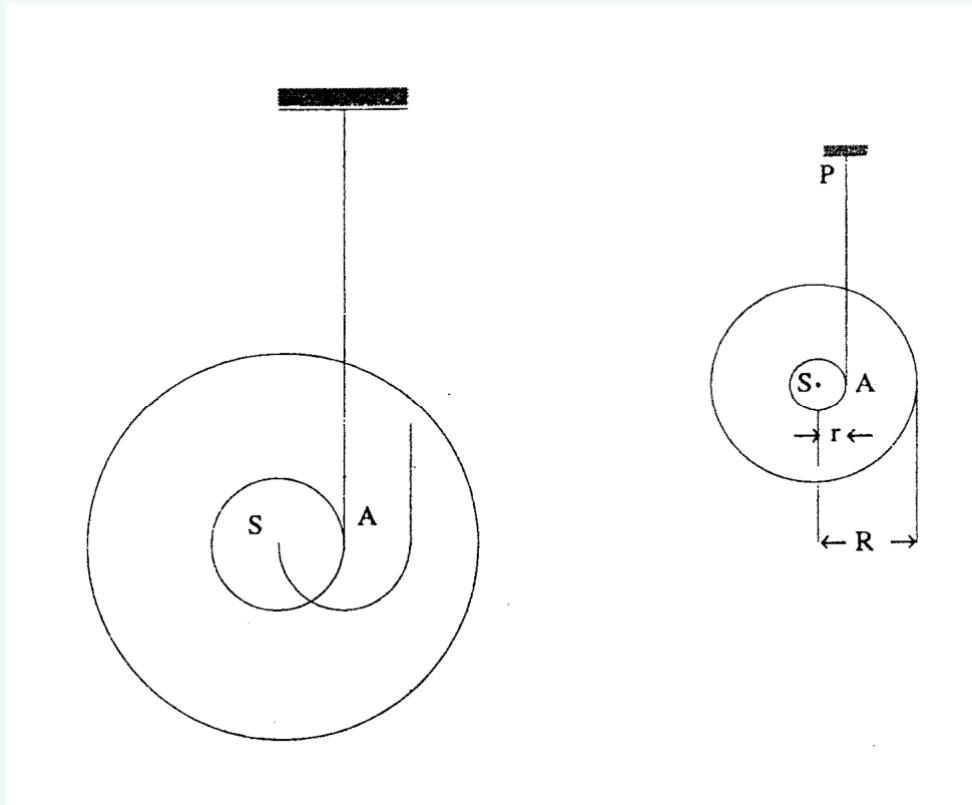


Fig. 19.6

2.5

If the string can withstand a maximum tension $T_m = 10 \text{ N}$, find the maximum length of the string which may be unwound without breaking by the wheel.

Official English text of the problem.

Answers

Here $r = 0.0030 \text{ m}$, and numerical evaluations use $g = 9.81 \text{ m s}^{-2}$. For the reversal, let ℓ be the preceding straight descent and define $\Phi = 0$ at entry, $\Phi = \pi/2$ at the bottom, and $\Phi = \pi$ at exit. Parts 2.4–2.5 use the fixed-contact, semicircular reversal depicted in Fig. 19.6; this is an idealization: except at $\Phi = \pi/2$, the required contact force has a horizontal component, which vertical strings cannot supply.

• 2.1

$$\omega(s) = \sqrt{\frac{4gs}{R^2 + 2r^2}}.$$

At $s = 0.50 \text{ m}$, $\omega = 73.6 \text{ rad s}^{-1}$.

• 2.2

$$E_r = \frac{Mgs}{201} = 9.76 \times 10^{-3} \text{ J} \simeq 9.8 \text{ mJ}.$$

With gravitational potential energy measured from the floor,

$$E_r : E_{\text{rot}} : U = 1 : 200 : 201.$$

Thus $E_r/E_{\text{rot}} = 1/200$, and $E_r/(Mgs) = 1/201$.

• 2.3 The tension in **each** string during ordinary downward unwinding is

$$T = \frac{MgR^2}{2(R^2 + 2r^2)} = 1.95 \text{ N} \simeq 2.0 \text{ N}.$$

• 2.4

$$\omega'(\Phi) = \sqrt{\frac{4g(\ell + r \sin \Phi)}{R^2 + 2r^2}}, \quad 0 \leq \Phi \leq \pi.$$

Taking the entry position of the centre as the origin, with x rightward and y upward,

$$x = r(1 - \cos \Phi), \quad y = -r \sin \Phi,$$

$$v_x = r\omega' \sin \Phi, \quad v_y = -r\omega' \cos \Phi, \quad v = r\omega'.$$

The trajectory is a lower semicircle. The speed is symmetric about $\pi/2$ and reaches its maximum there.

• 2.5 The strength-limited unwound length of **each** string is

$$\ell_{\text{max}} = \frac{R^2 + 2r^2}{4gr} \left(\frac{2T_m}{M} - g \right) - r = 1.23 \text{ m} \approx 1.2 \text{ m}.$$

The floor imposes a shorter limit in the stated setup:

$$\ell \leq h - R - r = 0.937 \text{ m} \approx 0.94 \text{ m}.$$

At that limiting clearance, the peak tension is only 8.1 N per string.

Solution

We assume release from rest, inextensible strings, no slipping during winding or unwinding, and negligible dissipative losses.

The wheel's moment of inertia about its axle is

$$I = \frac{1}{2}MR^2 = 7.2 \times 10^{-4} \text{ kg m}^2.$$

The thickness d does not enter this moment of inertia. It is convenient to define

$$J = I + Mr^2 = M \left(\frac{R^2}{2} + r^2 \right).$$

For the given dimensions,

$$\frac{I}{Mr^2} = \frac{R^2}{2r^2} = 200, \quad \frac{J}{Mr^2} = 201.$$

2.1

When an angle $d\theta$ unwinds, the centre descends by $ds = r d\theta$. Therefore

$$v = r\omega.$$

Conservation of mechanical energy after a descent s gives

$$Mgs = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(I + Mr^2)\omega^2.$$

Consequently,

$$\omega(s) = \sqrt{\frac{2Mgs}{I + Mr^2}} = \sqrt{\frac{4gs}{R^2 + 2r^2}}.$$

Numerically,

$$\omega(s) = (104.1 \text{ rad s}^{-1}) \sqrt{\frac{s}{1 \text{ m}}}.$$

2.2

The translational kinetic energy is

$$E_r = \frac{1}{2}Mv^2 = \frac{1}{2}Mr^2\omega^2 = \frac{Mr^2}{J}Mgs = \frac{Mgs}{201}.$$

At $s = 0.50 \text{ m}$,

$$Mgs = 0.40(9.81)(0.50) = 1.962 \text{ J},$$

so

$$E_r = 0.009761 \text{ J} \simeq 9.8 \text{ mJ}.$$

The rotational energy is

$$E_{\text{rot}} = \frac{1}{2}I\omega^2,$$

and hence

$$\frac{E_r}{E_{\text{rot}}} = \frac{Mr^2}{I} = \frac{1}{200}.$$

Its numerical value is

$$E_{\text{rot}} = 1.95224 \text{ J}.$$

The centre is now $h - s = 0.50 \text{ m}$ above the floor. Taking the floor as the zero of gravitational potential energy,

$$U = Mg(h - s) = 1.962 \text{ J}.$$

Thus the three energies present are in the ratio

$$E_r : E_{\text{rot}} : U = 1 : 200 : 201.$$

The initial energy was $Mgh = 3.924 \text{ J}$, and the energy check is

$$0.009761 + 1.952239 + 1.962 = 3.924 \text{ J}.$$

The translational energy is $1/201$ of the gravitational energy released. If “energy in any other form” means the combined remaining rotational and gravitational energies, the corresponding ratio is

$$\frac{E_r}{E_{\text{rot}} + U} = \frac{1}{401}.$$

2.3

Let T denote the tension in each string and take downward acceleration a as positive. Newton’s second law gives

$$Mg - 2T = Ma.$$

The two tensions produce the torque

$$2Tr = I\alpha,$$

while the unwinding condition gives $a = r\alpha$. Therefore

$$2T = \frac{I}{r^2}a,$$

and

$$a = \frac{Mg}{M + I/r^2} = \frac{g}{201} = 0.0488 \text{ m s}^{-2}.$$

It follows that

$$T = \frac{M(g - a)}{2} = \frac{Mg}{2} \frac{200}{201} = 1.95224 \text{ N}.$$

This tension is constant during the straight descent. The much larger force during reversal must be considered separately.

2.4

We interpret the curved trajectory in Fig. 19.6 as a short reversal during which the contact line through A is fixed and the centre S describes a semicircle of radius r about it. This imposed geometry is the reversal idealization used here.

Let ℓ be the straight descent before the reversal. At entry,

$$\omega_0^2 = \frac{2Mg\ell}{J}.$$

Choose the entry position of S as the origin, with x to the right and y upward. Then $A = (r, 0)$. Define Φ to increase from 0 to π as the centre passes beneath A . The coordinates are

$$x = r(1 - \cos \Phi), \quad y = -r \sin \Phi.$$

They satisfy

$$(x - r)^2 + y^2 = r^2, \quad y \leq 0.$$

Since this is rotation about A ,

$$\dot{\Phi} = \omega', \quad v = r\omega'.$$

The moment of inertia about the contact line is $J = I + Mr^2$. The centre has descended an additional distance $r \sin \Phi$, so conservation of energy gives

$$\frac{1}{2}J\omega'^2 = \frac{1}{2}J\omega_0^2 + Mgr \sin \Phi.$$

Therefore

$$\omega'(\Phi) = \sqrt{\omega_0^2 + \frac{2Mgr}{J} \sin \Phi} = \sqrt{\frac{4g(\ell + r \sin \Phi)}{R^2 + 2r^2}}.$$

Differentiating the coordinates,

$$v_x = r\omega' \sin \Phi, \quad v_y = -r\omega' \cos \Phi.$$

Thus

$$v(\Phi) = r\omega'(\Phi) = v_0 \sqrt{1 + \frac{r}{\ell} \sin \Phi}, \quad v_0 = r\omega_0.$$

The angular acceleration, useful for the force calculation, is

$$\ddot{\Phi} = \frac{Mgr}{J} \cos \Phi.$$

The requested sketches can be specified as follows:

Φ	x	y	v_x	v_y	Speed
0	0	0	0	$-v_0$	v_0
$\pi/2$	r	$-r$	$v_{\max}$	0	$v_{\max}$
π	$2r$	0	0	$+v_0$	v_0

Here

$$v_{\max} = v_0 \sqrt{1 + \frac{r}{\ell}}, \quad \omega'_{\max} = \omega_0 \sqrt{1 + \frac{r}{\ell}}.$$

The spatial path consists of a vertical descent, a lower semicircle of radius $r = 3.0 \text{ mm}$, and a vertical ascent displaced horizontally by $2r = 6.0 \text{ mm}$. As functions of Φ , x increases smoothly from 0 to $2r$, while y is a negative half-sine with minimum $-r$. Both v and ω' form symmetric arches, with equal endpoint values and their maxima at $\pi/2$.

At the bottom, the **vertical velocity is zero**, but the **speed is maximal**: in this semicircular model, the centre is then moving horizontally.

2.5

At the midpoint of the reversal, the centre lies directly below A . Its tangential acceleration vanishes because $\cos \Phi = 0$, and its acceleration is vertically upward:

$$a_{\text{bottom}} = \frac{v_{\max}^2}{r} = r\omega_{\max}'^2.$$

The vertical force equation is therefore

$$2T_{\text{peak}} - Mg = Mr\omega'_{\text{max}}{}^2.$$

Using the result of 2.4,

$$\omega'_{\text{max}}{}^2 = \frac{2Mg(\ell + r)}{J},$$

we obtain

$$T_{\text{peak}} = \frac{Mg}{2} + \frac{M^2gr(\ell + r)}{J}.$$

The bottom is indeed the point of maximum constraint load in the fixed- $\mathcal{A}$ model. To check this, put

$$\epsilon = \frac{Mr^2}{J}, \quad b = \frac{r\omega_0^2}{g}, \quad u = \sin \Phi.$$

Resolving the total contact reaction F into radial and tangential components gives

$$\frac{F^2}{M^2g^2} = b^2 + (1 - \epsilon)^2 + 2b(1 + 2\epsilon)u + 3\epsilon(2 + \epsilon)u^2.$$

All coefficients of u and u^2 are nonnegative, so the maximum occurs at $u = 1$, where the reaction is vertical and shared equally by the strings.

Setting the peak tension in each string equal to T_m ,

$$2T_m = Mg + \frac{2M^2gr(\ell_{\text{max}} + r)}{J}.$$

Hence

$$\ell_{\text{max}} = \frac{J(2T_m - Mg)}{2M^2gr} - r = \frac{R^2 + 2r^2}{4gr} \left(\frac{2T_m}{M} - g \right) - r.$$

Numerically,

$$\ell_{\text{max}} = \frac{0.060^2 + 2(0.0030)^2}{4(9.81)(0.0030)} \left(\frac{20}{0.40} - 9.81 \right) - 0.0030 = 1.2322 \text{ m}.$$

Thus the string-strength limit is approximately

$$\ell_{\text{max}} \simeq 1.2 \text{ m}.$$

There is also a geometric restriction from the floor. During the reversal, the minimum centre height is $h - \ell - r$. To keep the wheel above the floor,

$$h - \ell - r \geq R,$$

so

$$\ell \leq h - R - r = 0.937 \text{ m} \simeq 0.94 \text{ m}.$$

Substituting this length into the peak-tension formula gives

$$T_{\text{peak}} = 8.08 \text{ N} < 10 \text{ N}.$$

Consequently, the floor limits the available unwinding before the string-strength limit is reached.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 3: Recombination of Positive and Negative Ions in Ionized Gas

Introduction

A gas consists of positive ions of some element (at high temperature) and electrons. The positive ion belongs to an atom of unknown mass number Z . It is known that this ion has only one electron in the shell (orbit).

Let this ion be represented by the symbol $A^{(Z-1)+}$

Constants:

Constant	Value
electric field constant	$\varepsilon_0 = 8,85 \cdot 10^{-12} \frac{\text{A} \cdot \text{s}}{\text{V} \cdot \text{m}}$
elementary charge	$e = \pm 1,602 \cdot 10^{-19} \text{ A} \cdot \text{s}$
	$q^2 = \frac{e^2}{4 \cdot \pi \cdot \varepsilon_0} = 2,037 \cdot 10^{-28} \text{ J} \cdot \text{m}$
Planck's constant	$\hbar = 1,054 \cdot 10^{-34} \text{ J} \cdot \text{s}$
(rest) mass of an electron	$m_e = 9,108 \cdot 10^{-31} \text{ kg}$
Bohr's atomic radius	$r_B = \frac{\hbar}{m \cdot q^2} = 5,92 \cdot 10^{-11} \text{ m}$
Rydberg's energy	$E_R = \frac{q^2}{2 \cdot r_B} = 2,180 \cdot 10^{-18} \text{ J}$
(rest) mass of a proton	$m_p \cdot c^2 = 1,503 \cdot 10^{-10} \text{ J}$

Questions:

3.1

Assume that the ion which has just one electron left the shell.

$A^{(Z-1)+}$ is in the ground state.

In the lowest energy state, the square of the average distance of the electron from the nucleus or r^2 with components along x-, y- and z-axis being $(\Delta x)^2$, $(\Delta y)^2$ and $(\Delta z)^2$ respectively and

$$r_0^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

and also the square of the average momentum by

$$p_0^2 = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2,$$

whereas

$$\Delta p_x \geq \frac{\hbar}{2 \cdot \Delta x}, \quad \Delta p_y \geq \frac{\hbar}{2 \cdot \Delta y} \quad \text{and} \quad \Delta p_z \geq \frac{\hbar}{2 \cdot \Delta z}.$$

Write inequality involving $(p_0)^2 \cdot (r_0)^2$ in a complete form.

3.2

The ion represented by $A^{(Z-1)+}$ may capture an additional electron and consequently emits a photon.

Write down an equation which is to be used for calculation the frequency of an emitted photon.

3.3

Calculate the energy of the ion $A^{(Z-1)+}$ using the value of the lowest energy. The calculation should be approximated based on the following principles:

3.3.A

The potential energy of the ion should be expressed in terms of the average value of $\frac{1}{r}$.

(ie. $\frac{1}{r_0}$; r_0 is given in the problem).

3.3.B

In calculating the kinetic energy of the ion, use the average value of the square of the momentum given in 3.1 after being simplified by

$$(p_0)^2 \cdot (r_0)^2 \approx (\hbar)^2$$

3.4

Calculate the energy of the ion $A^{(Z-2)+}$ taken to be in the ground state, using the same principle as the calculation of the energy of $A^{(Z-1)+}$. Given the average distance of each of the two electrons in the outermost shell (same as r_0 given in 3.3) denoted by r_1 and r_2 , assume the average distance between the two electrons is given by $r_1 + r_2$ and the average value of the square of the momentum of each electron obeys the principle of uncertainty ie.

$$p_1^2 \cdot r_1^2 \approx \hbar^2 \quad \text{and} \quad p_2^2 \cdot r_2^2 \approx \hbar^2$$

hint: Make use of the information that in the ground state $r_1 = r_2$

3.5

Consider in particular the ion $A^{(Z-2)+}$ is at rest in the ground state when capturing an additional electron and the captured electron is also at rest prior to the capturing. Determine the numerical value of Z , if the frequency of the emitted photon accompanying electron capturing is $2,057 \cdot 10^{17}$ rad/s. Identify the element which gives rise to the ion.

Official English text of the problem.

Answers

Define the energy scale

$$\mathcal{R} = \frac{m_e (q^2)^2}{2\hbar^2}.$$

With consistent constants, this is the Rydberg energy E_R . The printed constants are inconsistent, so their numerical consequences are distinguished below.

- 3.1

$$p_0^2 r_0^2 \geq \frac{\hbar^2}{4} [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right] \geq \frac{9}{4} \hbar^2.$$

- 3.2

$$h\nu = \hbar\omega = E_i - E_f + K_i + K_e - K_f,$$

where E_i, E_f are internal electronic energies and the K 's are translational kinetic energies.

- 3.3

$$E_1 = -Z^2 \mathcal{R} \quad (\text{conventionally, } -Z^2 E_R).$$

- 3.3.A

$$U_1(r_0) \approx -\frac{Zq^2}{r_0}.$$

- 3.3.B

$$T_1(r_0) \approx \frac{\hbar^2}{2m_e r_0^2}, \quad r_{0,\min} = \frac{\hbar^2}{m_e Z q^2}.$$

- 3.4

$$E_2 = -2 \left(Z - \frac{1}{4} \right)^2 \mathcal{R}, \quad r_1 = r_2 = \frac{\hbar^2}{m_e q^2 (Z - \frac{1}{4})}.$$

- 3.5 For the intended capture of a **second** electron,

$$Z = \frac{1}{2} + \sqrt{\frac{\hbar\omega}{\mathcal{R}} + \frac{1}{8}}$$

with negligible recoil. The quoted E_R gives $Z = 3.673$; the printed $q^2, m_e, \hbar$ give $Z = 4.088$. Both have the nearest integer

$$Z = 4, \quad \text{beryllium (Be).}$$

This is a conditional, approximate identification. As printed, part 3.5 describes capture of a third electron, whose energy has not been specified.

Solution

The charge states imply a nucleus of charge $+Ze$. We treat its mass as large compared with the electron mass, neglect its motion when calculating electronic energies, and use nonrelativistic electron kinetic energies. Electronic binding energies are measured relative to the nucleus and separated electrons.

3.1

Squaring and adding the three uncertainty inequalities gives

$$p_0^2 \geq \frac{\hbar^2}{4} \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right].$$

Multiplication by the specified r_0^2 gives the complete expression

$$p_0^2 r_0^2 \geq \frac{\hbar^2}{4} \left[3 + \frac{(\Delta x)^2}{(\Delta y)^2} + \frac{(\Delta y)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta z)^2} + \frac{(\Delta z)^2}{(\Delta y)^2} + \frac{(\Delta z)^2}{(\Delta x)^2} + \frac{(\Delta x)^2}{(\Delta z)^2} \right].$$

Each reciprocal pair is at least 2, since $u + u^{-1} \geq 2$ for $u > 0$. Hence

$$p_0^2 r_0^2 \geq \frac{9}{4} \hbar^2.$$

Equality requires both equal spatial uncertainties,

$$\Delta x = \Delta y = \Delta z,$$

and saturation of all three component uncertainty relations. Spherical symmetry alone does not establish saturation.

3.2

The radiative capture reaction is



Let E_i, E_f denote the initial and final internal electronic energies. Let K_i, K_e be the initial ion and electron kinetic energies, and K_f the final ion kinetic energy. Conservation of energy gives

$$E_i + K_i + K_e = E_f + K_f + \hbar\omega.$$

Thus

$$\hbar\omega = h\nu = E_i - E_f + K_i + K_e - K_f, \quad \omega = 2\pi\nu.$$

The common nuclear and electron rest energies cancel. Momentum conservation determines the final ion's recoil. If the initial kinetic energies and recoil are negligible, this reduces to

$$\hbar\omega \approx E_i - E_f.$$

3.3

We now minimize the approximate total electronic energy required by the problem.

3.3.A

The attractive Coulomb potential is

$$U(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Zq^2}{r}.$$

Using the prescribed approximation to its expectation value,

$$U_1 \approx -Zq^2 \left\langle \frac{1}{r} \right\rangle \approx -\frac{Zq^2}{r_0}.$$

3.3.B

The electron has zero mean momentum in the stationary ground state, so its mean squared momentum is represented by p_0^2 . The stipulated estimate

$$p_0^2 r_0^2 \approx \hbar^2$$

therefore gives

$$T_1 \approx \frac{p_0^2}{2m_e} \approx \frac{\hbar^2}{2m_e r_0^2}.$$

This is a deliberately coarse replacement of the uncertainty result: it discards the numerical coefficient from 3.1 and is not an equality derived from that bound.

The estimated total energy is consequently

$$E_1(r_0) = \frac{\hbar^2}{2m_e r_0^2} - \frac{Zq^2}{r_0}.$$

Its stationary point satisfies

$$\frac{dE_1}{dr_0} = -\frac{\hbar^2}{m_e r_0^3} + \frac{Zq^2}{r_0^2} = 0,$$

giving

$$r_{0,\min} = \frac{\hbar^2}{m_e Z q^2}.$$

The second derivative there is $\hbar^2/(m_e r_0^4) > 0$, so this is a minimum. Substitution yields

$$T_{1,\min} = \frac{m_e Z^2 (q^2)^2}{2\hbar^2}, \quad U_{1,\min} = -\frac{m_e Z^2 (q^2)^2}{\hbar^2},$$

and hence

$$E_1 = -\frac{m_e Z^2 (q^2)^2}{2\hbar^2} = -Z^2 \mathcal{R}.$$

With consistent Bohr and Rydberg constants, these are the familiar forms

$$r_{0,\min} = \frac{r_B}{Z}, \quad E_1 = -Z^2 E_R.$$

3.4

There are now two electron kinetic energies, two attractive electron–nucleus terms, and one repulsive electron–electron term. Under the prescribed separation approximation,

$$E_2(r_1, r_2) = \frac{\hbar^2}{2m_e} \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) - Zq^2 \left(\frac{1}{r_1} + \frac{1}{r_2} \right) + \frac{q^2}{r_1 + r_2}.$$

Using the given ground-state condition $r_1 = r_2 = r$,

$$E_2(r) = \frac{\hbar^2}{m_e r^2} - \frac{2Zq^2}{r} + \frac{q^2}{2r} = \frac{\hbar^2}{m_e r^2} - \frac{2q^2(Z - \frac{1}{4})}{r}.$$

Minimization gives

$$-\frac{2\hbar^2}{m_e r^3} + \frac{2q^2(Z - \frac{1}{4})}{r^2} = 0,$$

so

$$r = \frac{\hbar^2}{m_e q^2 (Z - \frac{1}{4})}.$$

At this radius,

$$E_2 = -\frac{m_e (q^2)^2}{\hbar^2} \left(Z - \frac{1}{4} \right)^2 = -2\mathcal{R} \left(Z - \frac{1}{4} \right)^2.$$

Thus, in this model, electron repulsion replaces Z by the effective value $Z - \frac{1}{4}$ in each electron's ground-state energy.

3.5

To connect this part with the energies calculated in 3.3 and 3.4, the capture must be interpreted as

$$A^{(Z-1)+} + e^- \longrightarrow A^{(Z-2)+} + \gamma.$$

The literal charge state printed in 3.5 instead describes a different process: capture of a third electron, whose energy is not calculated in this problem.

For the intended process, the released binding energy is

$$\begin{aligned} \Delta E &= E_1 - E_2 \\ &= \left[2 \left(Z - \frac{1}{4} \right)^2 - Z^2 \right] \mathcal{R} \\ &= \left(Z^2 - Z + \frac{1}{8} \right) \mathcal{R}. \end{aligned}$$

Although both incoming particles are at rest, the final ion must recoil. If its mass is M , momentum conservation gives

$$p_{\text{ion}} = p_\gamma = \frac{\hbar\omega}{c}.$$

Consequently,

$$\Delta E = \hbar\omega + \frac{(\hbar\omega)^2}{2Mc^2}.$$

The supplied frequency is an **angular frequency**, so the photon energy is

$$E_\gamma = \hbar\omega = (1.054 \times 10^{-34})(2.057 \times 10^{17}) = \boxed{2.168 \times 10^{-17} \text{ J}}.$$

Using $Mc^2 \geq m_p c^2$, the fractional recoil correction is bounded by

$$\frac{K_{\text{recoil}}}{E_\gamma} = \frac{E_\gamma}{2Mc^2} \leq \frac{2.168 \times 10^{-17}}{2(1.503 \times 10^{-10})} = 7.21 \times 10^{-8}.$$

It is entirely negligible at the precision of the data.

Therefore

$$Z^2 - Z + \frac{1}{8} = \frac{\hbar\omega}{\mathcal{R}},$$

whose physical root is

$$Z = \frac{1}{2} + \sqrt{\frac{\hbar\omega}{\mathcal{R}} + \frac{1}{8}}.$$

The inconsistent constants lead to two numerical estimates:

Constants used for the energy scale	$\mathcal{R}$	Estimated Z
Quoted Rydberg energy E_{R}	$2.180 \times 10^{-18} \text{ J}$	3.673
Printed $q^2, m_e, \hbar$, through $m_e(q^2)^2/(2\hbar^2)$	$1.701 \times 10^{-18} \text{ J}$	4.088

Both estimates have the nearest integer 4. Accordingly, under the intended capture interpretation and the approximate model, the inferred element is

$$Z = 4, \quad \text{beryllium,}$$

with capture



The supplied text does not support an exact, internally consistent integer determination.

The solution was checked against the official solution.