

IPhO 1987

Theory problems and solutions · Jena, German Democratic Republic · 3 theory problems with complete solutions

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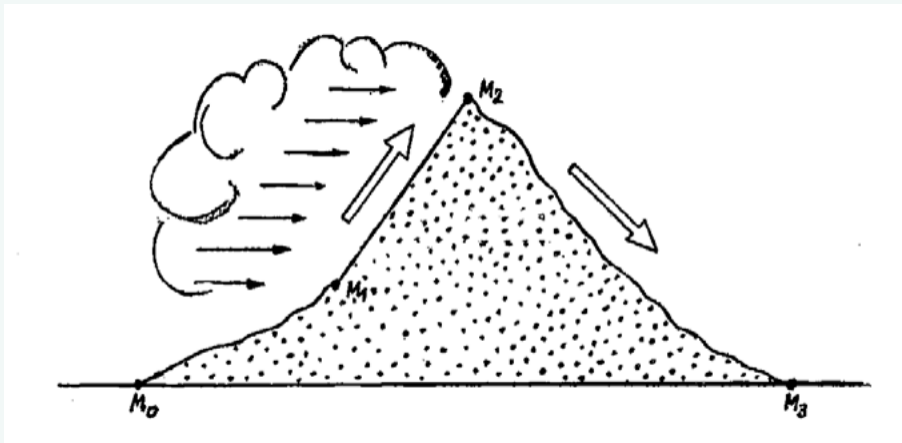
Problem 1: Ascending moist air

Moist air is streaming adiabatically across a mountain range as indicated in the figure.

Equal atmospheric pressures of 100 kPa are measured at meteorological stations M_0 and M_3 and a pressure of 70 kPa at station M_2 . The temperature of the air at M_0 is 20°C .

As the air is ascending, cloud formation sets in at 84.5 kPa.

Consider a quantity of moist air ascending the mountain with a mass of 2000 kg over each square meter. This moist air reaches the mountain ridge (station M_2) after 1500 seconds. During that rise an amount of 2.45 g of water per kilogram of air is precipitated as rain.



1. Determine temperature T_1 at M_1 where the cloud ceiling forms.
2. What is the height h_1 (at M_1) above station M_0 of the cloud ceiling assuming a linear decrease of atmospheric density?
3. What temperature T_2 is measured at the ridge of the mountain range?
4. Determine the height of the water column (precipitation level) precipitated by the air stream in 3 hours, assuming a homogeneous rainfall between points M_1 and M_2 .
5. What temperature T_3 is measured in the back of the mountain range at station M_3 ? Discuss the state of the atmosphere at station M_3 in comparison with that at station M_0 .

Hints and Data

The atmosphere is to be dealt with as an ideal gas. Influences of the water vapour on the specific heat capacity and the atmospheric density are to be neglected; the same applies to the temperature dependence of the specific latent heat of vaporisation. The temperatures are to be

determined to an accuracy of 1 K, the height of the cloud ceiling to an accuracy of 10 m and the precipitation level to an accuracy of 1 mm.

Specific heat capacity of the atmosphere in the pertaining temperature range:

$$c_p = 1005 \text{ J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$$

Atmospheric density for p_0 and T_0 at station M_0 :

$$\rho_0 = 1.189 \text{ kg} \cdot \text{m}^{-3}$$

Specific latent heat of vaporisation of the water within the volume of the cloud:

$$L_v = 2500 \text{ kJ} \cdot \text{kg}^{-1}$$

$$\frac{c_p}{c_v} = \chi = 1.4 \quad \text{and} \quad g = 9.81 \text{ m} \cdot \text{s}^{-2}$$

Official English text of the problem.

Answers

1. $T_1 \approx 279 \text{ K}$, or 6°C .
2. $h_1 \approx 1410 \text{ m}$.
3. $T_2 \approx 271 \text{ K}$, or approximately -2°C , to the requested accuracy.
4. The precipitation level in 3 hours is approximately **35 mm**.
5. $T_3 \approx 300 \text{ K}$, or approximately 27°C . The air is about **7 K** warmer than at M_0 and is drier: the **foehn effect**.

Solution

All temperatures in thermodynamic relations are absolute temperatures. Set

$$T_0 = 293.15 \text{ K}, \quad \kappa = \frac{\chi - 1}{\chi} = \frac{2}{7}.$$

The specific gas constant of air satisfies

$$R = c_p - c_v, \quad \frac{R}{c_p} = \kappa.$$

We use the given approximation that water does not appreciably alter the air's heat capacities or density. Condensed water is treated as being removed as rain.

1. Temperature at the onset of cloud formation

Before reaching M_1 , no condensation occurs. The rising air therefore follows the dry adiabatic relation, even though it contains water vapour.

For an ideal gas, the first law for an adiabatic change gives

$$c_p dT = \frac{RT}{p} dp.$$

Consequently,

$$\frac{dT}{T} = \kappa \frac{dp}{p}, \quad Tp^{-\kappa} = \text{constant}.$$

Thus

$$T_1 = T_0 \left(\frac{p_1}{p_0} \right)^\kappa = 293.15 \left(\frac{84.5}{100} \right)^{2/7} = 279.38 \text{ K}.$$

Therefore,

$$\boxed{T_1 \approx 279 \text{ K} \approx 6^\circ \text{C}.$$

2. Height of the cloud formation level

First determine the density at M_1 . From $p = \rho RT$,

$$\frac{\rho_1}{\rho_0} = \frac{p_1}{p_0} \frac{T_0}{T_1}.$$

Hence

$$\rho_1 = 1.189 \left(\frac{84.5}{100} \right) \left(\frac{293.15}{279.38} \right) = 1.0542 \text{ kg m}^{-3}.$$

Hydrostatic equilibrium requires

$$\frac{dp}{dh} = -\rho g.$$

With density decreasing linearly between M_0 and M_1 , its average over this height interval is

$$\bar{\rho} = \frac{\rho_0 + \rho_1}{2}.$$

Integrating the hydrostatic equation therefore gives

$$p_0 - p_1 = gh_1 \frac{\rho_0 + \rho_1}{2}.$$

Thus

$$h_1 = \frac{2(p_0 - p_1)}{g(\rho_0 + \rho_1)} = \frac{2(100000 - 84500)}{9.81(1.189 + 1.0542)} = 1408.7 \text{ m}.$$

To the requested accuracy of 10 m,

$$\boxed{h_1 \approx 1410 \text{ m}.$$

3. Temperature at the mountain ridge

During the cloudy ascent, condensation releases latent heat. Although the parcel exchanges no heat with its surroundings, this internal conversion of latent energy reduces its cooling.

The condensed mass per kilogram of air is

$$\Delta r = 2.45 \times 10^{-3} \text{ kg kg}^{-1},$$

so the latent energy released per kilogram of air is

$$Q_\ell = L_v \Delta r = (2.500 \times 10^6)(2.45 \times 10^{-3}) = 6125 \text{ J kg}^{-1}.$$

Without condensation, the ridge temperature would be

$$T_{2,\text{dry}} = T_0 \left(\frac{p_2}{p_0} \right)^\kappa = 293.15 \left(\frac{70}{100} \right)^{2/7} = 264.75 \text{ K.}$$

The latent energy corresponds to a temperature increase

$$\frac{Q_\ell}{c_p} = \frac{6125}{1005} = 6.095 \text{ K.}$$

Adding this increase at the final pressure gives

$$T_2 \simeq T_{2,\text{dry}} + \frac{Q_\ell}{c_p} = 270.84 \text{ K.}$$

To check that this approximation meets the specified accuracy, let dr be an increment of condensed water per kilogram of air. The first law during condensation is

$$c_p dT - \frac{RT}{p} dp = L_v dr.$$

Multiplying by the integrating factor $p^{-\kappa}$ and integrating gives

$$T_2 = T_1 \left(\frac{p_2}{p_1} \right)^\kappa + \frac{L_v}{c_p} \int_0^{\Delta r} \left(\frac{p_2}{p(r)} \right)^\kappa dr.$$

Condensation occurs at pressures between p_1 and p_2 , so

$$\left(\frac{p_2}{p_1} \right)^\kappa \leq \left(\frac{p_2}{p(r)} \right)^\kappa \leq 1.$$

Therefore,

$$270.52 \text{ K} \leq T_2 \leq 270.84 \text{ K.}$$

The simple latent-heat correction overestimates T_2 by at most 0.32 K, well within the required accuracy. Thus

$$\boxed{T_2 \approx 271 \text{ K} \quad (\text{approximately } -2^\circ\text{C}).}$$

4. Precipitation level in 3 hours

The air mass associated with each square metre is

$$\Sigma = 2000 \text{ kg m}^{-2}.$$

During one stated 1500 s interval, it supplies a rainfall mass per unit area of

$$\Sigma \Delta r = 2000(2.45 \times 10^{-3}) = 4.90 \text{ kg m}^{-2}.$$

For the steady stream, the rainfall accumulated in 3 h = 10800 s is therefore

$$\frac{m_w}{A} = 4.90 \frac{10800}{1500} = 35.28 \text{ kg m}^{-2}.$$

Using the usual liquid-water density

$$\rho_w \simeq 1000 \text{ kg m}^{-3},$$

the corresponding water-column height is

$$H = \frac{m_w/A}{\rho_w} = \frac{35.28}{1000} = 0.03528 \text{ m.}$$

Homogeneous rainfall gives this same precipitation level throughout the specified rain region:

$$H \approx 35 \text{ mm in 3 hours.}$$

5. Temperature and atmospheric conditions at M_3

With the condensed water removed as rain, the descending air becomes unsaturated as compression raises its temperature. Its descent is therefore dry adiabatic:

$$T_3 = T_2 \left(\frac{p_3}{p_2} \right)^\kappa.$$

Using the approximation from part 3,

$$T_3 \simeq 270.84 \left(\frac{100}{70} \right)^{2/7} = 299.90 \text{ K.}$$

The more precise bounds from part 3 imply

$$299.54 \text{ K} \leq T_3 \leq 299.90 \text{ K,}$$

so the uncertainty from the condensation history remains below 1 K.

Hence

$$T_3 \approx 300 \text{ K} \approx 27^\circ \text{C.}$$

Compared with M_0 , the air at M_3 :

- is approximately 7 K warmer;
- has lost 2.45 g of water per kilogram of air;
- has lower relative humidity, owing both to the water loss and to its higher temperature.

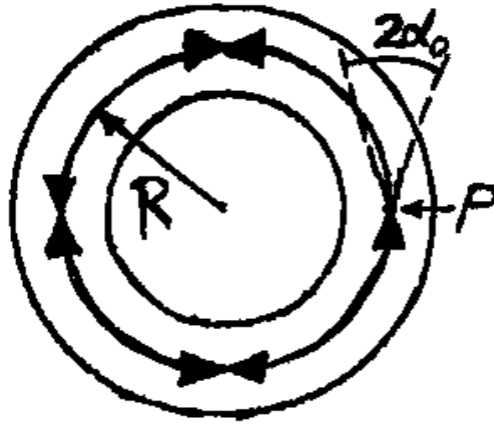
Latent heat released during ascent reduces the cooling, while the removed rain cannot produce corresponding evaporative cooling during descent. The resulting warm, dry descending wind is the **foehn effect**.

The solution was checked against the official solution.

Problem 2: Electrons in a magnetic field

A beam of electrons emitted by a point source P enters the magnetic field $\vec{B}$ of a toroidal coil (toroid) in the direction of the lines of force. The angle of the aperture of the beam $2 \cdot \alpha_0$ is assumed to be small ($2 \cdot \alpha_0 \ll 1$). The injection of the electrons occurs on the mean radius R of the toroid with acceleration voltage V_0 .

Neglect any interaction between the electrons. The magnitude of $\vec{B}$, B, is assumed to be constant.



1. To guide the electron in the toroidal field a homogeneous magnetic deflection field $\vec{B}_1$ is required. Calculate $\vec{B}_1$ for an electron moving on a circular orbit of radius R in the torus.
2. Determine the value of $\vec{B}$ which gives four focussing points separated by $\pi/2$ as indicated in the diagram.

Note: When considering the electron paths you may disregard the curvature of the magnetic field.

3. The electron beam cannot stay in the toroid without a deflection field $\vec{B}_1$, but will leave it with a systematic motion (drift) perpendicular to the plane of the toroid.

- a) Show that the radial deviation of the electrons from the injection radius is finite.
- b) Determine the direction of the drift velocity.

Note: The angle of aperture of the electron beam can be neglected. Use the laws of conservation of energy and of angular momentum.

Data:

$$\frac{e}{m} = 1.76 \cdot 10^{11} \text{ C} \cdot \text{kg}^{-1}; \quad V_0 = 3 \text{ kV}; \quad R = 50 \text{ mm}$$

Official English text of the problem.

Answers

Let $e > 0$ denote the magnitude of the electron charge. Take $\hat{\phi}$ along the injected beam and the toroidal field, and define $\hat{z} = \hat{r} \times \hat{\phi}$, pointing out of the page for the injection shown.

- 1. The homogeneous deflection field is perpendicular to the toroid:

$$\boxed{\vec{B}_1 = \frac{mv_0}{eR} \hat{z} = 3.69 \times 10^{-3} \text{ T } \hat{z}}, \quad v_0 = \sqrt{\frac{2eV_0}{m}} = 3.25 \times 10^7 \text{ m s}^{-1}.$$

- 2. In the prescribed approximation,

$$\boxed{B = \frac{4mv_0}{eR} = 4B_1 = 1.48 \times 10^{-2} \text{ T}}.$$

- **3(a).** The radial deviation is bounded:

$$0 \leq r - R \leq \frac{mv_0}{eB}.$$

With the field from part 2, this bound is 12.5 mm. The actual maximum deviation in the constant- B model is 5.37 mm.

- **3(b).** The drift is **into the page**, opposite to the deflection field required in part 1:

$$\vec{v}_{\text{drift}} \parallel -\hat{\mathbf{z}}.$$

Solution

1. Deflection field

Using nonrelativistic mechanics, acceleration through the voltage V_0 gives

$$\frac{1}{2}mv_0^2 = eV_0,$$

hence

$$v_0 = \sqrt{2 \left(\frac{e}{m} \right) V_0} = \sqrt{2(1.76 \times 10^{11})(3.00 \times 10^3)} = 3.25 \times 10^7 \text{ m s}^{-1}.$$

For the reference electron on the circular orbit, the velocity is parallel to the toroidal field. Consequently, that field produces no force on this electron.

The deflection field must supply the inward centripetal force. For $\vec{B}_1 = B_1 \hat{\mathbf{z}}$, the electron's Lorentz force is

$$\vec{F} = -e\vec{v} \times \vec{B}_1 = -ev_0 B_1 \hat{\mathbf{r}},$$

which points inward everywhere around the circle. Thus

$$ev_0 B_1 = \frac{mv_0^2}{R}.$$

Therefore

$$\vec{B}_1 = \frac{mv_0}{eR} \hat{\mathbf{z}},$$

with magnitude

$$B_1 = \frac{3.2496 \times 10^7}{(1.76 \times 10^{11})(0.050)} = 3.69 \times 10^{-3} \text{ T}.$$

The required field is homogeneous and points out of the plane of the diagram.

2. Four focussing points

For this part, neglect the curvature of the guiding field as instructed. Locally, an electron follows a helix about the field direction. If its initial angle to the field is α , then

$$v_{\parallel} = v_0 \cos \alpha \simeq v_0, \quad v_{\perp} = v_0 \sin \alpha \simeq v_0 \alpha.$$

The transverse motion has cyclotron angular frequency and period

$$\omega = \frac{eB}{m}, \quad T = \frac{2\pi}{\omega} = \frac{2\pi m}{eB}.$$

Although the gyration radius depends on α , the period does not. After one complete gyration, every electron returns to its initial transverse position.

The distance along the beam to the next focus is therefore, to first order in the small aperture angle,

$$\ell = v_{\parallel} T \simeq v_0 T = \frac{2\pi m v_0}{eB}.$$

Differences in $v_{\parallel}$ enter only at order α^2 , which is neglected.

Four equally spaced focussing points require

$$\ell = \frac{2\pi R}{4} = \frac{\pi R}{2}.$$

Equating these distances gives

$$\frac{2\pi m v_0}{eB} = \frac{\pi R}{2},$$

so

$$B = \frac{4m v_0}{eR} = 4B_1 = 1.48 \times 10^{-2} \text{ T}.$$

The successive refocussings occur at azimuthal advances $\pi/2, \pi, 3\pi/2, 2\pi$; the last coincides with P. Thus there are four distinct focussing positions around the toroid, including P.

3(a). Finite radial deviation without the deflection field

Now remove $\vec{B}_1$ and neglect the beam aperture. Use cylindrical coordinates about the toroid's symmetry axis:

$$\vec{B} = B\hat{\phi}, \quad \vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{z}.$$

Initially,

$$r = R, \quad \dot{r} = 0, \quad \dot{z} = 0, \quad r\dot{\phi} = v_0.$$

A magnetic field does no work, so the speed remains constant:

$$\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 = v_0^2. \quad (1)$$

The magnetic force has no azimuthal component. Therefore angular momentum about the symmetry axis is conserved:

$$mr^2\dot{\phi} = mRv_0, \quad r\dot{\phi} = \frac{Rv_0}{r}. \quad (2)$$

Since the azimuthal speed cannot exceed the total speed, equation (2) immediately implies

$$r \geq R. \quad (3)$$

To obtain an upper bound, use the axial component of the Lorentz force:

$$-e\vec{v} \times \vec{B} = eB\dot{z}\hat{r} - eB\dot{r}\hat{z}.$$

Thus

$$m\ddot{z} = -eB\dot{r}.$$

Because B is constant, integration with the initial conditions gives

$$\dot{z} = -\frac{eB}{m}(r - R) = -\omega(r - R). \quad (4)$$

Substituting equations (2) and (4) into energy conservation yields

$$\dot{r}^2 + \frac{R^2 v_0^2}{r^2} + \omega^2(r - R)^2 = v_0^2. \quad (5)$$

Every term on the left is nonnegative. Consequently,

$$\omega^2(r - R)^2 \leq v_0^2.$$

Together with equation (3), this proves

$$0 \leq r - R \leq \frac{v_0}{\omega} = \frac{mv_0}{eB}.$$

For the field found in part 2, $\omega = 4v_0/R$, giving the finite bound

$$r - R \leq \frac{R}{4} = 12.5 \text{ mm}.$$

Equation (5) also gives the actual outer turning point. Set $\dot{r} = 0$ and write $x = (r - R)/R$. Then

$$16x^2 + \frac{1}{(1+x)^2} = 1.$$

Besides the initial turning point $x = 0$, the positive root satisfies

$$16x(1+x)^2 = x + 2,$$

giving

$$x = 0.107403, \quad r_{\max} - R = 5.37 \text{ mm}.$$

3(b). Direction of the drift

Equations (3) and (4) give

$$\dot{z} = -\omega(r - R) \leq 0.$$

Thus the axial velocity never points in the $+\hat{\mathbf{z}}$ direction.

The motion actually leaves $r = R$: initially the magnetic force vanishes because the velocity is parallel to $\vec{B}$, and the radial equation gives

$$\ddot{r}(0) = \frac{v_0^2}{R} > 0.$$

Hence $r > R$ shortly after injection and $\dot{z} < 0$. As the radial motion oscillates between its turning points, the axial displacement continues to accumulate in the negative direction:

$$\langle \dot{z} \rangle = -\omega \langle r - R \rangle < 0.$$

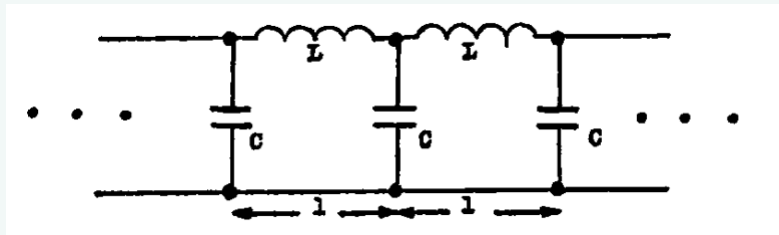
Therefore the systematic drift is

$$\text{into the page, opposite to } \vec{B}_1.$$

The solution was checked against the official solution.

Problem 3: Infinite LC-grid

When sine waves propagate in an infinite LC-grid (see the figure below) the phase of the ac-voltage across two successive capacitors differs by Φ .



- Determine how Φ depends on ω , L and C (ω is the angular frequency of the sine wave).
- Determine the velocity of propagation of the waves if the length of each unit is ℓ .
- State under what conditions the propagation velocity of the waves is almost independent of ω . Determine the velocity in this case.
- Suggest a simple mechanical model which is an analogue to the above circuit and derive equations which establish the validity of your model.

Formulae:

$$\cos \alpha - \cos \beta = -2 \cdot \sin \left(\frac{\alpha + \beta}{2} \right) \cdot \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \cdot \cos \left(\frac{\alpha + \beta}{2} \right) \cdot \sin \left(\frac{\alpha - \beta}{2} \right)$$

Official English text of the problem.

Answers

- a)** The dispersion relation is

$$\omega^2 LC = 4 \sin^2 \frac{\Phi}{2}.$$

Taking the principal phase lag $0 \leq \Phi \leq \pi$,

$$\Phi = 2 \arcsin \left(\frac{\omega \sqrt{LC}}{2} \right).$$

A real phase lag requires $0 \leq \omega \leq \omega_c = 2/\sqrt{LC}$.

- **b)** The phase velocity of the sinusoidal wave is

$$v_{\text{ph}} = \frac{\omega \ell}{\Phi} = \frac{\omega \ell}{2 \arcsin(\omega \sqrt{LC}/2)}.$$

For completeness, the group velocity of a narrow wave packet is

$$v_{\text{g}} = \frac{\ell}{\sqrt{LC}} \sqrt{1 - \frac{\omega^2 LC}{4}}.$$

- **c)** For $\omega \sqrt{LC} \ll 1$, equivalently $\Phi \ll 1$, the propagation velocity is approximately independent of frequency:

$$v_{\text{ph}} \simeq v_{\text{g}} \simeq \frac{\ell}{\sqrt{LC}}.$$

- **d)** An infinite chain of identical masses m , separated by ℓ and joined by identical massless springs of stiffness κ , is an analogue. Its equation

$$m \ddot{x}_n = \kappa(x_{n-1} - 2x_n + x_{n+1})$$

matches the circuit equation

$$L \ddot{q}_n = \frac{1}{C}(q_{n-1} - 2q_n + q_{n+1}),$$

where $\dot{q}_n$ is the current through the n -th inductor. Thus the correspondence is $m \leftrightarrow L$, $\kappa \leftrightarrow 1/C$.

Solution

Assume ideal, identical inductors and capacitors and negligible resistance. Number the capacitors by the integer n . Let $u_n(t)$ be the voltage across capacitor n , and let $i_n(t)$ flow through the inductor from node n to node $n + 1$.

a)

Kirchhoff's current law at node n gives

$$C \frac{du_n}{dt} = i_{n-1} - i_n.$$

The voltage–current relation for each inductor gives

$$L \frac{di_n}{dt} = u_n - u_{n+1}.$$

Differentiating the first equation and substituting the inductor relations,

$$LC \frac{d^2 u_n}{dt^2} = (u_{n-1} - u_n) - (u_n - u_{n+1}),$$

so

$$LC \ddot{u}_n = u_{n-1} - 2u_n + u_{n+1}.$$

For a sinusoidal travelling wave with phase lag Φ per cell, write

$$u_n(t) = U \cos(\omega t - n\Phi).$$

The neighbouring voltages satisfy

$$u_{n-1} + u_{n+1} = 2u_n \cos \Phi,$$

while $\ddot{u}_n = -\omega^2 u_n$. Therefore

$$-\omega^2 LC = 2 \cos \Phi - 2,$$

or

$$\cos \Phi = 1 - \frac{\omega^2 LC}{2}.$$

Using $1 - \cos \Phi = 2 \sin^2(\Phi/2)$,

$$\omega^2 LC = 4 \sin^2 \frac{\Phi}{2}.$$

On the principal branch,

$$\Phi = 2 \arcsin \left(\frac{\omega \sqrt{LC}}{2} \right).$$

Reversing the propagation direction reverses the signed phase lag.

Since $\sin^2(\Phi/2) \leq 1$, real travelling-wave modes are possible only within the frequency band bounded by

$$\omega_c = \frac{2}{\sqrt{LC}}.$$

Above this cutoff, there is no propagating mode with real Φ .

b)

A constant phase satisfies

$$\omega t - n\Phi = \text{constant}.$$

Consequently, the time for the same phase to advance by one cell is

$$\Delta t = \frac{\Phi}{\omega}.$$

As the cell length is ℓ , the phase velocity is

$$v_{\text{ph}} = \frac{\ell}{\Delta t} = \frac{\omega \ell}{\Phi} = \frac{\omega \ell}{2 \arcsin(\omega \sqrt{LC}/2)}.$$

Equivalently, the wave number is $k = \Phi/\ell$, and $v_{\text{ph}} = \omega/k$.

The problem specifies sine waves, so this phase velocity is the requested velocity. If one instead follows the envelope of a narrow wave packet, its velocity is the group velocity. From

$$\omega(k) = \frac{2}{\sqrt{LC}} \sin \frac{k\ell}{2}, \quad 0 \leq k\ell \leq \pi,$$

we obtain

$$v_g = \frac{d\omega}{dk} = \frac{\ell}{\sqrt{LC}} \cos \frac{\Phi}{2} = \frac{\ell}{\sqrt{LC}} \sqrt{1 - \frac{\omega^2 LC}{4}}.$$

At the cutoff, adjacent capacitor voltages oscillate in antiphase, $\Phi = \pi$, and the group velocity vanishes.

c)

When the phase change across one cell is small,

$$\Phi \ll 1,$$

we may use

$$\sin \frac{\Phi}{2} \simeq \frac{\Phi}{2}.$$

The dispersion relation then becomes

$$\omega^2 LC \simeq \Phi^2, \quad \Phi \simeq \omega \sqrt{LC}.$$

Thus the required frequency condition is

$$\omega \sqrt{LC} \ll 1,$$

and

$$v_{ph} \simeq \frac{\ell}{\sqrt{LC}}.$$

The group velocity has the same limit. Physically, this is the long-wavelength regime: many cells lie within one wavelength, so the discrete grid behaves approximately as a continuous transmission line.

For example, the leading correction to the phase velocity is

$$v_{ph} = \frac{\ell}{\sqrt{LC}} \left[1 - \frac{\omega^2 LC}{24} + O((\omega^2 LC)^2) \right].$$

Hence its frequency dependence is small under the stated condition. Since LC has units of s^2 , $\ell/\sqrt{LC}$ has units of velocity.

d)

Consider an infinite straight chain of identical masses m , with equilibrium separation ℓ . Connect each pair of neighbouring masses by a massless spring of stiffness κ . The masses move longitudinally without friction, and the springs obey Hooke's law.

Let $x_n(t)$ be the displacement of mass n from equilibrium. The right-hand spring exerts the force

$$\kappa(x_{n+1} - x_n),$$

and the left-hand spring exerts

$$-\kappa(x_n - x_{n-1}).$$

Newton's second law therefore gives

$$m\ddot{x}_n = \kappa(x_{n-1} - 2x_n + x_{n+1}).$$

To establish a direct circuit correspondence, define $q_n(t)$ as the charge that has passed through inductor n , so that

$$i_n = \dot{q}_n.$$

Taking quantities relative to equilibrium, charge conservation at capacitor n gives

$$Cu_n = q_{n-1} - q_n.$$

The inductor equation consequently becomes

$$L\ddot{q}_n = u_n - u_{n+1} = \frac{q_{n-1} - 2q_n + q_{n+1}}{C}.$$

Thus

$$\boxed{L\ddot{q}_n = \frac{1}{C}(q_{n-1} - 2q_n + q_{n+1})},$$

which has precisely the same form as the mechanical equation under

$$\boxed{x_n \leftrightarrow q_n, \quad m \leftrightarrow L, \quad \kappa \leftrightarrow \frac{1}{C}}.$$

The energy expressions confirm this correspondence:

$$E_{\text{LC}} = \sum_n \left[\frac{1}{2} L \dot{q}_n^2 + \frac{(q_{n-1} - q_n)^2}{2C} \right],$$

$$E_{\text{mech}} = \sum_n \left[\frac{1}{2} m \dot{x}_n^2 + \frac{1}{2} \kappa (x_{n-1} - x_n)^2 \right].$$

Inductive energy corresponds to kinetic energy, and capacitor energy corresponds to spring potential energy.

Finally, substituting $x_n = A \cos(\omega t - n\Phi)$ into the mechanical equation yields

$$\omega^2 = \frac{4\kappa}{m} \sin^2 \frac{\Phi}{2},$$

identical to the LC dispersion relation when $\kappa/m = 1/(LC)$.

The solution was checked against the official solution.