

IPhO 1986

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Problem 1: Two slits and a diffraction grating: vector addition of amplitudes

Figure 1.1

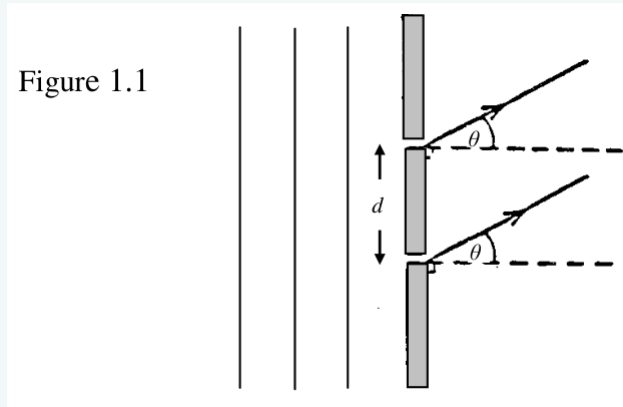


Figure 1.1

A plane monochromatic light wave, wavelength λ and frequency f , is incident normally on two identical narrow slits, separated by a distance d , as indicated in Figure 1.1. The light wave emerging at each slit is given, at a distance x in a direction θ at time t , by

$$y = a \cos[2\pi(ft - x/\lambda)]$$

where the amplitude a is the same for both waves. (Assume x is much larger than d).

(i) Show that the two waves observed at an angle θ to a normal to the slits, have a resultant amplitude A which can be obtained by adding two vectors, each having magnitude a , and each with an associated direction determined by the phase of the light wave.

Verify geometrically, from the vector diagram, that

$$A = 2a \cos \theta$$

where

$$\beta = \frac{\pi}{\lambda} d \sin \theta$$

(ii) The double slit is replaced by a diffraction grating with N equally spaced slits, adjacent slits being separated by a distance d . Use the vector method of adding amplitudes to show that the vector amplitudes, each of magnitude a , form a part of a regular polygon with vertices on a circle of radius R given by

$$R = \frac{a}{2 \sin \beta},$$

Deduce that the resultant amplitude is

$$\frac{a \sin N\beta}{\sin \beta}$$

and obtain the resultant phase difference relative to that of the light from the slit at the edge of the grating.

(iii) Sketch, in the same graph, $\sin N\beta$ and $(1/\sin \beta)$ as a function of β . On a separate graph show how the intensity of the resultant wave varies as a function of β .

(iv) Determine the intensities of the principal intensity maxima.

(v) Show that the number of principal maxima cannot exceed

$$\left(\frac{2d}{\lambda} + 1 \right)$$

(vi) Show that two wavelengths λ and $\lambda + \delta\lambda$, where $\delta\lambda \ll \lambda$, produce principal maxima with an angular separation given by

$$\Delta\theta = \frac{n\Delta\lambda}{d \cos \theta} \quad \text{where } n = 0, \pm 1, \pm 2 \dots \text{etc}$$

Calculate this angular separation for the sodium D lines for which

$$\lambda = 589.0 \text{ nm}, \quad \lambda + \Delta\lambda = 589.6 \text{ nm}, \quad n = 2, \quad \text{and } d = 1.2 \times 10^{-6} \text{ m}.$$

$$\left[\text{reminder: } \cos A + \cos B = 2 \cos \left(\frac{A+B}{2} \right) \cdot \cos \left(\frac{A-B}{2} \right) \right]$$

Official English text of the problem.

Answers

Let I_1 be the intensity from one slit alone, and let

$$\beta = \frac{\pi d \sin \theta}{\lambda}.$$

A signed amplitude coefficient is denoted by A_s ; its absolute value is the physical amplitude.

- (i) The phase difference is 2β , giving

$$A_s = 2a \cos \beta, \quad A = 2a |\cos \beta|.$$

The printed $\cos \theta$ should be $\cos \beta$.

- (ii)

$$R = \frac{a}{2 \sin \beta} \quad (0 < \beta < \pi), \quad A_s = a \frac{\sin N\beta}{\sin \beta}.$$

Generally, $R = a/(2|\sin \beta|)$. For slit phases increasing by 2β , the resultant phase relative to the first edge slit is $(N-1)\beta$ with the signed coefficient above. Using a positive amplitude adds π wherever that coefficient is negative. At $\beta = m\pi$, the physical amplitude is Na .

- (iii) $\sin N\beta$ has amplitude 1 and period $2\pi/N$; $1/\sin \beta$ has vertical asymptotes at $\beta = m\pi$. The intensity is

$$I(\beta) = I_1 \left(\frac{\sin N\beta}{\sin \beta} \right)^2.$$

It is even and has period π , with principal peaks at $m\pi$. Between consecutive principal peaks are $N - 1$ zeros and $N - 2$ secondary maxima.

- (iv)

$$I_{\text{principal}} = N^2 I_1, \quad d \sin \theta = n\lambda.$$

- (v) Counting grazing orders as limiting directions,

$$M = 2 \left\lfloor \frac{d}{\lambda} \right\rfloor + 1 \leq \frac{2d}{\lambda} + 1.$$

- (vi) To first order,

$$\Delta\theta \simeq \frac{n\Delta\lambda}{d \cos \theta}.$$

The angular separation is the magnitude of this signed shift. For the specified sodium lines in second order,

$$\Delta\theta \simeq 5.2 \times 10^{-3} \text{ rad} \simeq 0.30^\circ.$$

Solution

(i)

Normal incidence gives the same incident phase at both slits. The emerging waves are coherent, so their instantaneous displacements add.

In the far field, the rays reaching the observation point are effectively parallel. The difference between the two path lengths is therefore

$$\Delta x = d \sin \theta.$$

Consequently, the phase difference has magnitude

$$\delta = \frac{2\pi\Delta x}{\lambda} = \frac{2\pi d \sin \theta}{\lambda} = 2\beta.$$

Choose the reference slit so that the other wave advances in phase by 2β . Writing the reference phase as ψ , the two waves are

$$y_0 = a \cos \psi, \quad y_1 = a \cos(\psi + 2\beta).$$

The given identity gives

$$y_0 + y_1 = 2a \cos \beta \cos(\psi + \beta).$$

To interpret this geometrically, represent each wave by a vector of length a , whose direction equals its phase. Its projection on a fixed axis is the instantaneous wave displacement. Both vectors rotate with angular speed $2\pi f$, maintaining their angular separation 2β . Since projections add, their vector sum represents the resultant wave.

For the vector diagram, draw the two arrows from a common origin, making angles $-\beta$ and $+\beta$ with their phase bisector. Their components perpendicular to this bisector cancel; their components along it add:

$$A_s = a \cos \beta + a \cos \beta = 2a \cos \beta.$$

Thus

$$\boxed{A = 2a |\cos \beta|}.$$

A negative signed coefficient simply represents a reversal of the resultant vector, or an additional phase of π .

The θ in the printed amplitude formula must therefore be replaced by β .

(ii)

Apply the same far-field approximation to the whole grating, neglecting amplitude variation and wavefront curvature across it. Label the slits from one edge so that their phases are

$$\psi, \psi + 2\beta, \dots, \psi + 2(N-1)\beta.$$

Draw the N amplitude vectors head to tail. Every vector has length a , and each turns through the same angle 2β relative to its predecessor. Their endpoints consequently form an equal-chord polygonal chain on a circle.

For the diagram, label the successive endpoints $P_0, P_1, \dots, P_N$ and the circle centre O . Each side $P_j P_{j+1}$ has length a and subtends a directed central angle 2β . Bisecting the corresponding isosceles triangle gives, for $0 < \beta < \pi$,

$$\frac{a}{2} = R \sin \beta.$$

Hence

$$\boxed{R = \frac{a}{2 \sin \beta}}.$$

For arbitrary β , the positive radius is $a/(2|\sin \beta|)$. The chain need not close after the N vectors.

The resultant vector runs directly from P_0 to P_N . These endpoints are separated by a total directed central angle $2N\beta$, so its length is

$$A_N = 2R |\sin N\beta| = a \left| \frac{\sin N\beta}{\sin \beta} \right|.$$

Thus the requested signed coefficient is

$$\boxed{A_s = a \frac{\sin N\beta}{\sin \beta}}.$$

The phase follows from the chord directions. The first small chord has a midpoint radius at angular position β , while the resultant chord has a midpoint radius at $N\beta$. Each chord is perpendicular to its midpoint radius, so their angular difference is

$$N\beta - \beta = (N-1)\beta,$$

up to a reversal of the resultant.

Equivalently, the complete resultant wave is

$$y_{\text{tot}} = a \frac{\sin N\beta}{\sin \beta} \cos[\psi + (N-1)\beta].$$

Therefore, with the chosen edge convention, the phase relative to that edge slit is $(N-1)\beta$ when the signed coefficient is retained. With a nonnegative amplitude, it is

$$\phi = \begin{cases} (N-1)\beta, & \sin N\beta / \sin \beta > 0, \\ (N-1)\beta + \pi, & \sin N\beta / \sin \beta < 0, \end{cases} \quad (\text{mod } 2\pi).$$

Referencing the opposite edge reverses the relative phase modulo 2π . At a zero resultant, phase is undefined.

When $\beta = m\pi$, adjacent phases differ by $2m\pi$. All vectors point in the same direction, giving $A_N = Na$ and zero relative phase modulo 2π . The circle construction then has infinite limiting radius.

(iii)

The time-averaged intensity is proportional to the square of the physical amplitude. Since the one-slit amplitude is a ,

$$\frac{I}{I_1} = \frac{A_N^2}{a^2},$$

and hence

$$\frac{I(\beta)}{I_1} = \left(\frac{\sin N\beta}{\sin \beta} \right)^2.$$

For the first graph, use common horizontal β and vertical function-value axes:

- $\sin N\beta$ oscillates between -1 and $+1$, with period $2\pi/N$. Its zeros are at $\beta = k\pi/N$, and its extrema occur at $\beta = (2k+1)\pi/(2N)$, with values $(-1)^k$.
- $1/\sin \beta$ has period 2π , no zeros, and vertical asymptotes at $\beta = m\pi$. Between 0 and π , it falls from $+\infty$ to its minimum 1 at $\pi/2$, then rises to $+\infty$. The next branch, between π and 2π , is negative, reaching -1 at $3\pi/2$ and tending to $-\infty$ at both ends. These branches repeat.

Although the reciprocal sine diverges at $m\pi$, the product with $\sin N\beta$ has a finite limiting magnitude N .

For the intensity graph, plot I/I_1 . Its principal features are:

Position	I/I_1
$\beta = m\pi$	N^2
$\beta = m\pi + r\pi/N, \quad r = 1, \dots, N-1$	0
$\beta = m\pi + \pi/2, \quad N \text{ even}$	0
$\beta = m\pi + \pi/2, \quad N \text{ odd}$	1

The curve is nonnegative, even, and periodic with period π . Each principal peak is symmetric about $\beta = m\pi$. Its nearest zeros are

$$\beta = m\pi \pm \frac{\pi}{N},$$

so its width between these zeros is $2\pi/N$.

Between neighboring principal peaks there are $N - 1$ zeros, with $N - 2$ smaller secondary peaks between successive zeros. Their exact positions satisfy

$$N \cot(N\beta) = \cot \beta,$$

obtained by differentiating the intensity away from its zeros. Away from the asymptotes, the curve lies beneath the envelope

$$I = \frac{I_1}{\sin^2 \beta}.$$

(iv)

The largest possible resultant of N vectors of length a has length Na . This occurs when all vectors point in the same direction:

$$2\beta = 2n\pi.$$

Thus the principal maxima satisfy

$$\beta = n\pi \iff d \sin \theta = n\lambda.$$

At these directions,

$$A_N = Na,$$

so

$$I_{\text{principal}} = N^2 I_1.$$

Under the stated equal-amplitude model, all principal maxima have this intensity.

The same result follows by taking the limit:

$$\lim_{\beta \rightarrow n\pi} \frac{\sin N\beta}{\sin \beta} = N(-1)^{n(N-1)},$$

whose square is N^2 .

(v)

A principal maximum of order n requires

$$\sin \theta = \frac{n\lambda}{d}.$$

Since $|\sin \theta| \leq 1$,

$$|n| \leq \frac{d}{\lambda}.$$

The allowed integer orders therefore run from

$$-\left\lfloor \frac{d}{\lambda} \right\rfloor \quad \text{to} \quad +\left\lfloor \frac{d}{\lambda} \right\rfloor.$$

Counting the zero order gives

$$M = 2 \left\lfloor \frac{d}{\lambda} \right\rfloor + 1.$$

Because $\lfloor u \rfloor \leq u$,

$$M \leq \frac{2d}{\lambda} + 1.$$

This count includes any orders exactly at grazing incidence, $\theta = \pm 90^\circ$, as limiting directions. Excluding such directions can only reduce the number.

(vi)

Consider the principal maxima of the same order n for the two wavelengths. Writing $\Delta\lambda = \delta\lambda$, their grating equations are

$$\begin{aligned} d \sin \theta &= n\lambda, \\ d \sin(\theta + \Delta\theta) &= n(\lambda + \Delta\lambda). \end{aligned}$$

For a small angular shift, measured in radians,

$$\sin(\theta + \Delta\theta) \simeq \sin \theta + \Delta\theta \cos \theta.$$

Subtracting the first grating equation from the second gives

$$d \cos \theta \Delta\theta \simeq n\Delta\lambda,$$

hence

$$\Delta\theta \simeq \frac{n\Delta\lambda}{d \cos \theta}.$$

This is a first-order result for existing, nongrazing maxima. For negative n , the signed shift is negative; the angular separation is its absolute value. The zero-order maxima coincide.

For the sodium lines,

$$\Delta\lambda = (589.6 - 589.0) \text{ nm} = 0.6 \times 10^{-9} \text{ m}.$$

At $\lambda = 589.0 \text{ nm}$ in second order,

$$\sin \theta = \frac{2(589.0 \times 10^{-9})}{1.2 \times 10^{-6}} = 0.9816667,$$

so

$$\theta \simeq 79.01^\circ, \quad \cos \theta \simeq 0.19061.$$

Therefore,

$$\Delta\theta \simeq \frac{2(0.6 \times 10^{-9})}{(1.2 \times 10^{-6})(0.19061)} = 5.246 \times 10^{-3} \text{ rad}.$$

To sensible precision,

$$\Delta\theta \simeq 5.2 \times 10^{-3} \text{ rad} \simeq 0.30^\circ.$$

As a numerical check, subtracting the two angles from the unexpanded grating equation gives $5.32 \times 10^{-3} \text{ rad}$, about 1.4% above the first-order estimate.

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 2: Seismic waves and the Earth's liquid core

2. Early this century a model of the earth was proposed in which it was assumed to be a sphere of radius R consisting of a homogeneous isotropic solid mantle down to radius R_c . The core region within radius R_c contained a liquid.

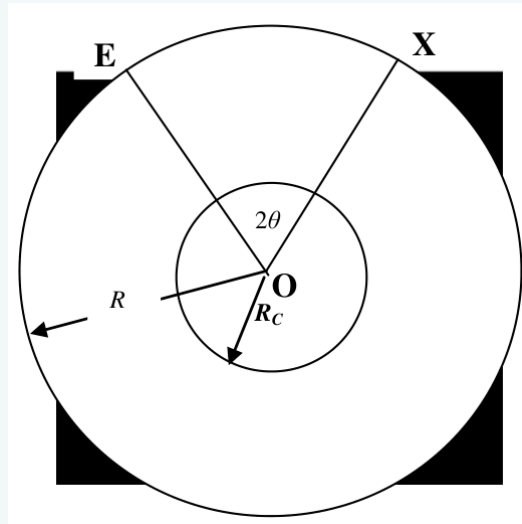


Figure 2.1

The velocities of longitudinal and transverse seismic waves P and S waves respectively, are constant, V_P , and V_S within the mantle. In the core, longitudinal waves have a constant velocity V_{CP} , $V_{CP} < V_P$, and transverse waves are not propagated.

An earthquake at E on the surface of the Earth produces seismic waves that travel through the Earth and are observed by a surface observer who can set up his seismometer at any point X on the Earth's surface. The angular separation between E and X, 2θ given by

$$2\theta = \text{Angle } EOX$$

where O is the centre of the Earth.

- (i) Show that the seismic waves that travel through the mantle in a straight line will arrive at X at a time t (the travel time after the earthquake), is given by

$$t = \frac{2R \sin \theta}{v}, \quad \text{for } \theta > \arccos \left[\frac{R_c}{R} \right],$$

where $v = v_P$ for the P waves and $v = v_S$ for the S waves.

- (ii) For some of the positions of X such that the seismic P waves arrive to the observer after two refractions at the mantle-core interface. Draw the path of such a seismic P wave. Obtain a relation between θ and i , the angle of incidence of the seismic P wave at the mantle-core interface, for P waves.

- (iii) Using the data

R	=	6370 km
R_C	=	3470 km
v_{CP}	=	10.85 km s^{-1}

v_S	=	6.31 km s^{-1}
v_{CP}	=	9.02 km s^{-1}

and the result obtained in (ii), draw a graph of θ against i . Comment on the physical consequences of the form of this graph for observers stationed at different points on the Earth's surface. Sketch the variation of the travel time taken by the P and S waves as a function of θ for $0 \leq \theta \leq 90$ degrees.

(iv) After an earthquake an observer measures the time delay between the arrival of the S wave, following the P wave, as 2 minutes 11 seconds. Deduce the angular separation of the earthquake from the observer using the data given in Section (iii).

(v) The observer in the previous measurement notices that some time after the arrival of the P and S waves there are two further recordings on the seismometer separated by a time interval of 6 minutes 37 seconds. Explain this result and verify that it is indeed associated with the angular separation determined in the previous section.

Official English text of the problem.

Answers

- (i) The direct-wave travel time is

$$t = \frac{2R \sin \theta}{v}, \quad 0 \leq \theta \leq \arccos(R_c/R).$$

Equality describes the limiting tangent ray. The printed inequality is reversed.

- (ii) Put $a = R_c/R$, $b = v_{CP}/v_P$. The ray follows three straight segments, refracting towards the normal on entering the core and away from it on leaving. Its half angular sweep is

$$\Phi(i) = \frac{\pi}{2} + i - \arcsin(a \sin i) - \arcsin(b \sin i).$$

Ordinarily $\theta = \Phi$; allowing rays that emerge beyond the antipode, the smaller angular separation is given by

$$\theta = \min\{\Phi, \pi - \Phi\}.$$

- (iii) Direct P and S rays end at $\theta = 56.99^\circ$. Core-transmitted P rays first appear at

$$\theta_{\min} = 75.58^\circ, \quad i_{\min} = 55.02^\circ.$$

Thus the P-wave shadow is approximately

$$113.99^\circ < 2\theta < 151.15^\circ,$$

while S waves have no direct or core-transmitted arrival for $2\theta > 113.99^\circ$. The refracted P travel-time branches meet at 20.789 min; their antipodal arrival times are 21.733 and 23.397 min. The complete graph descriptions are given below.

- (iv) The earthquake–observer angular separation is

$$2\theta \approx 17.8^\circ.$$

- (v) The later signals are P and S waves reflected at the core boundary. Their calculated arrival times are 551.3 s and 947.9 s after the earthquake, giving

$$\Delta t_{\text{ref}} = 396.65 \text{ s} \approx 6 \text{ min } 37 \text{ s}.$$

Solution

We use the physically consistent reading of the numerical data:

$$v_P = 10.85 \text{ km s}^{-1}, \quad v_S = 6.31 \text{ km s}^{-1}, \quad v_{CP} = 9.02 \text{ km s}^{-1}.$$

The data table in (iii) prints the label v_{CP} twice; the first value so labelled, 10.85 km s^{-1} , is v_P . The inequality printed in (i) is also reversed; see part (i).

We use geometrical ray theory: rays are straight within each homogeneous region, refraction obeys Snell's law, and reflection obeys equality of incidence and reflection angles. Diffraction, mode conversion and additional reflections at the Earth's surface are neglected.

(i)

Let M be the midpoint of the chord EX . The triangle OEM is right-angled, with $\angle EOM = \theta$. Hence

$$EM = R \sin \theta, \quad OM = R \cos \theta.$$

The direct path length and travel time are therefore

$$EX = 2R \sin \theta, \quad t = \frac{2R \sin \theta}{v}.$$

The closest point of the chord to the centre is M . Thus the ray avoids the core only if

$$R \cos \theta \geq R_c,$$

or

$$\theta \leq \arccos(R_c/R).$$

Strict inequality gives a path entirely within the mantle; equality is the tangent limit. For larger θ , the chord intersects the core, so the mantle-only formula no longer describes the actual ray.

(ii)

Denote the core entry and exit points by A and B . Draw the path

$$E \longrightarrow A \longrightarrow B \longrightarrow X,$$

with EA and BX in the mantle and AB a chord of the core. Draw the radial normals OA and OB .

Let r be the refracted angle inside the core, measured from the normal. Snell's law gives

$$\frac{\sin i}{v_P} = \frac{\sin r}{v_{CP}},$$

so

$$r = \arcsin \left(\frac{v_{CP}}{v_P} \sin i \right).$$

Since $v_{CP} < v_P$, we have $r < i$: the ray bends towards the normal on entering the core. By symmetry, its incidence angle at the exit is r , and it emerges into the mantle at angle i .

Let $\alpha = \angle OEA$. In triangle EOA , the angle at A is $\pi - i$. The sine rule gives

$$\frac{\sin \alpha}{R_c} = \frac{\sin i}{R},$$

hence

$$\alpha = \arcsin\left(\frac{R_c}{R} \sin i\right), \quad \angle EOA = i - \alpha.$$

The isosceles triangle AOB gives

$$\angle AOB = \pi - 2r.$$

The total angular sweep of the ray about O is consequently

$$2\Phi = 2(i - \alpha) + \pi - 2r.$$

Therefore

$$\Phi(i) = \frac{\pi}{2} + i - \arcsin\left(\frac{R_c}{R} \sin i\right) - \arcsin\left(\frac{v_{CP}}{v_P} \sin i\right).$$

For a sweep no greater than π , $\theta = \Phi$. If the ray emerges just beyond the antipode, its smaller source–observer angle is $2\pi - 2\Phi$, giving

$$\theta = \min\{\Phi, \pi - \Phi\}.$$

The requested drawing should therefore show a symmetric broken ray, with equal mantle legs, a straight core chord, and angles i and $r < i$ marked at the entry boundary.

(iii)

The dimensionless ratios are

$$a = \frac{R_c}{R} = 0.544741, \quad b = \frac{v_{CP}}{v_P} = 0.831336.$$

Thus

$$\Phi(i) = \frac{\pi}{2} + i - \arcsin(a \sin i) - \arcsin(b \sin i).$$

Differentiating, with angles in radians,

$$\frac{d\Phi}{di} = 1 - \frac{a \cos i}{\sqrt{1 - a^2 \sin^2 i}} - \frac{b \cos i}{\sqrt{1 - b^2 \sin^2 i}}.$$

At the endpoints,

$$\Phi'(0) = 1 - a - b = -0.376077, \quad \Phi'(\pi/2) = 1.$$

Each subtracted term decreases with i , so Φ' increases and has exactly one zero. Numerically,

$$i_{\min} = 55.017^\circ, \quad \Phi_{\min} = \theta_{\min} = 75.577^\circ.$$

To obtain the corresponding travel-time graph, the length of either mantle leg is

$$\ell_m = \sqrt{R^2 - R_c^2 \sin^2 i} - R_c \cos i.$$

This follows by projecting the outer and inner radii onto the mantle ray, whose perpendicular distance from O is $R_c \sin i$.

The core chord has length

$$\ell_c = 2R_c \cos r = 2R_c \sqrt{1 - b^2 \sin^2 i}.$$

Consequently, the core-transmitted P travel time is

$$T(i) = \frac{2}{v_P} \left(\sqrt{R^2 - R_c^2 \sin^2 i} - R_c \cos i \right) + \frac{2R_c}{v_{CP}} \sqrt{1 - b^2 \sin^2 i}.$$

The following points specify both requested graphs:

i (degrees)	Half sweep Φ (degrees)	Physical θ (degrees)	Core-transmitted P time (min)
0	90.00	90.00	21.733
20	82.74	82.74	21.504
40	77.20	77.20	21.003
55.02	75.58	75.58	20.789
70	77.84	77.84	21.171
80	82.60	82.60	22.028
89.23	90.00	90.00	23.397
90	90.76	89.24	23.538

The Φ -versus- i curve begins at 90° , falls smoothly to its minimum, then rises to 90.7568° . When plotting the physical smaller angle θ , fold the last portion at 90° : θ reaches 90° at $i = 89.2322^\circ$, then decreases slightly to 89.2432° at grazing incidence.

The direct-wave cutoff is

$$\theta_d = \arccos(R_c/R) = 56.993^\circ.$$

The physical consequences are therefore:

- For $0 \leq \theta < 56.993^\circ$, direct P and S waves reach the observer.
- For $56.993^\circ < \theta < 75.577^\circ$, neither direct nor core-transmitted P rays reach the observer. This is the P-wave shadow zone.
- For $\theta > 56.993^\circ$, there is no direct S ray, and the liquid core supplies no transmitted S ray.
- At $\theta = 75.577^\circ$, two refracted P paths merge. Since nearby incident rays emerge at nearly the same angular separation, ray energy is concentrated there: this is a caustic.
- For most of $75.577^\circ < \theta < 90^\circ$, two refracted P paths give two distinct arrivals.

Using the given numbers exactly, there is a small additional feature near the antipode. For

$$89.2432^\circ < \theta < 90^\circ,$$

a third refracted path reaches the observer by passing beyond the antipodal direction. At the antipode, the two oblique branches have equal travel times.

For the **travel-time sketch**, draw the direct branches

$$t_{P,\text{dir}} = 1174.19 \sin \theta \text{ s}, \quad t_{S,\text{dir}} = 2019.02 \sin \theta \text{ s},$$

only over $0 \leq \theta \leq 56.993^\circ$. Both start at the origin and increase with downward curvature; the S curve lies above the P curve. Their endpoints are

$$\begin{aligned} t_{P,\text{dir}}(\theta_d) &= 984.68 \text{ s} = 16.411 \text{ min}, \\ t_{S,\text{dir}}(\theta_d) &= 1693.16 \text{ s} = 28.219 \text{ min}. \end{aligned}$$

The principal core-transmitted P branches begin together at

$$(\theta, T) = (75.577^\circ, 20.789 \text{ min})$$

and rise towards 90° . They meet with a common tangent, since differentiation gives

$$\frac{dT}{di} = \frac{2R_c \sin i}{v_P} \frac{d\Phi}{di}, \quad \frac{dT}{d\Phi} = \frac{2R_c \sin i}{v_P}.$$

The smaller- i branch arrives earlier and flattens at the antipode, where the ray follows a diameter:

$$T(0) = \frac{2(R - R_c)}{v_P} + \frac{2R_c}{v_{CP}} = 21.733 \text{ min}.$$

The other principal branch reaches 23.397 min there. The small third branch decreases from 23.538 min at $\theta = 89.243^\circ$ to 23.397 min at 90° .

For a sketch including the later reflected arrivals discussed in part (v), also draw

$$t_{v,\text{ref}}(\theta) = \frac{2}{v} \sqrt{R^2 + R_c^2 - 2RR_c \cos \theta}, \quad 0 \leq \theta \leq \theta_d.$$

These curves increase from 8.909 min for P and 15.320 min for S at $\theta = 0$, and meet their respective direct curves tangentially at θ_d .

(iv)

For direct P and S waves the path length is the same, so their arrival-time difference is

$$\Delta t = 2R \sin \theta \left(\frac{1}{v_S} - \frac{1}{v_P} \right).$$

Here

$$\Delta t = 2(60) + 11 = 131 \text{ s}.$$

Therefore

$$\sin \theta = \frac{131}{2(6370) \left(\frac{1}{6.31} - \frac{1}{10.85} \right)} = \frac{131}{844.8239} = 0.1551.$$

Thus

$$\theta = 8.9204^\circ, \quad \boxed{2\theta = 17.8408^\circ \approx 17.8^\circ}.$$

Since $\theta < \theta_d$, the mantle-only path assumption is consistent.

The individual arrival times are

$$t_{P,\text{dir}} = 182.07 \text{ s}, \quad t_{S,\text{dir}} = 313.07 \text{ s}.$$

(v)

The core boundary can reflect both P- and S-wave energy. These reflected waves take longer paths through the mantle and therefore produce later arrivals.

Let C be the reflection point on the core boundary, on the nearer radial bisector of $\angle EOX$. The path is

$$E \longrightarrow C \longrightarrow X.$$

Reflection symmetry about OC makes the incidence and reflection angles equal, and $EC = CX$. Since $\angle EOC = \theta$, the cosine rule gives

$$EC^2 = R^2 + R_c^2 - 2RR_c \cos \theta.$$

Thus the reflected path length is

$$L_{\text{ref}} = 2\sqrt{R^2 + R_c^2 - 2RR_c \cos \theta}.$$

For $\theta = 8.9204^\circ$,

$$L_{\text{ref}} = 5981.54 \text{ km}.$$

The reflected arrival times are consequently

$$\begin{aligned} t_{P,\text{ref}} &= \frac{5981.54}{10.85} = 551.29 \text{ s}, \\ t_{S,\text{ref}} &= \frac{5981.54}{6.31} = 947.95 \text{ s}. \end{aligned}$$

Both occur after the direct S arrival at 313.07 s, as required.

Their separation is

$$\begin{aligned} \Delta t_{\text{ref}} &= L_{\text{ref}} \left(\frac{1}{v_S} - \frac{1}{v_P} \right) \\ &= 396.65 \text{ s} \\ &= 6 \text{ min } 36.65 \text{ s}. \end{aligned}$$

Rounded to the nearest second,

$$\boxed{\Delta t_{\text{ref}} = 6 \text{ min } 37 \text{ s}},$$

which verifies the interpretation and the angular separation found in part (iv).

The solution was checked against the official solution. Where this solution differs from the official solution, the official solution contains an error or a misprint.

Problem 3: Normal modes of masses joined by springs (ring and chain)

Three particles, each of mass m , are in equilibrium and joined by unstretched massless springs, each with Hooke's Law spring constant k . They are constrained to move in a circular path as indicated in Figure 3.1.

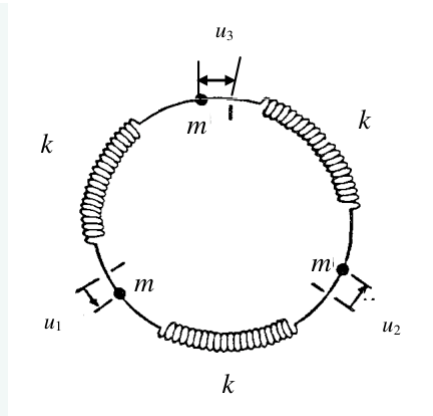


Figure 3.1

(i) If each mass is displaced from equilibrium by small displacements u_1 , u_2 and u_3 respectively, write down the equation of motion for each mass.

(ii) Verify that the system has simple harmonic solutions of the form

$$u_n = a_n \cos \omega t,$$

with accelerations, $(-\omega^2 u_n)$ where a_n ($n = 1, 2, 3$) are constant amplitudes, and ω , the angular frequency, can have 3 possible values,

$$\omega_o \sqrt{3}, \omega_o \sqrt{3} \text{ and } 0. \quad \text{where } \omega_o^2 = \frac{k}{m}.$$

(iii) The system of alternate springs and masses is extended to N particles, each mass m is joined by springs to its neighbouring masses. Initially the springs are unstretched and in equilibrium. Write down the equation of motion of the n th mass ($n = 1, 2 \dots N$) in terms of its displacement and those of the adjacent masses when the particles are displaced from equilibrium.

$$u_n(t) = a_s \sin \left(\frac{2ns\pi}{N} + \phi \right) \cos \omega_s t,$$

are oscillatory solutions where $s = 1, 2, \dots N$, $n = 1, 2, \dots N$ and where ϕ is an arbitrary phase, providing the angular frequencies are given by

$$\omega_s = 2\omega_o \sin \left(\frac{s\pi}{N} \right),$$

where a_s ($s = 1, \dots, N$) are constant amplitudes independent of n .

State the range of possible frequencies for a chain containing an infinite number of masses.

(iv) Determine the ratio

$$u_n / u_{n+1}$$

for large N , in the two cases:

(a) low frequency solutions

(b) $\omega = \omega_{\max}$, where $\omega_{\max}$ is the maximum frequency solution.

Sketch typical graphs indicating the displacements of the particles against particle number along the chain at time t for cases (a) and (b).

(v) If one of the masses is replaced by a mass $m' \ll m$ estimate any major change one would expect to occur to the angular frequency distribution.

Describe qualitatively the form of the frequency spectrum one would predict for a diatomic chain with alternate masses m and m' on the basis of the previous result.

Reminder

$$\begin{aligned}\sin(A + B) &= \sin A \cos B + \cos A \sin B \\ \sin A + \sin B &= 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \\ 2 \sin^2 A &= 1 - \cos 2A\end{aligned}$$

Official English text of the problem.

Answers

- (i)

$$\begin{aligned}m\ddot{u}_1 &= k(u_2 + u_3 - 2u_1), \\ m\ddot{u}_2 &= k(u_3 + u_1 - 2u_2), \\ m\ddot{u}_3 &= k(u_1 + u_2 - 2u_3).\end{aligned}$$

- (ii)

$$\omega = \sqrt{\frac{3k}{m}}, \quad \sqrt{\frac{3k}{m}}, \quad 0.$$

The nonzero frequency has two independent modes satisfying $a_1 + a_2 + a_3 = 0$. The zero mode has $a_1 = a_2 = a_3$.

- (iii)

$$m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n), \quad u_{n+N} = u_n,$$

$$\omega_s = 2\sqrt{\frac{k}{m}} \sin\left(\frac{s\pi}{N}\right).$$

For an infinite chain,

$$0 \leq \omega \leq 2\sqrt{\frac{k}{m}}.$$

In ordinary frequency, $0 \leq f \leq \sqrt{k/m}/\pi$.

- (iv)

(a) $u_n/u_{n+1} \simeq +1$, away from displacement nodes. The graph is a slowly varying sinusoid sampled at the particle positions.

(b) $u_n/u_{n+1} = -1$ at the upper band edge. Displacements alternate between equal positive and negative values. This is exact for even N ; an odd ring approaches this behaviour locally for large N .

- (v) A single light mass produces one high-frequency mode, concentrated on that mass:

$$\omega_{\text{defect}} \simeq \sqrt{\frac{2k}{m'}} \gg 2\sqrt{\frac{k}{m}}.$$

An alternating chain has an acoustic band and an optical band separated by a gap. For $m' < m$, their exact limiting ranges are

$$\begin{array}{ll} \text{Acoustic:} & 0 \leq \omega \leq \sqrt{\frac{2k}{m}}, \\ \text{Optical:} & \sqrt{\frac{2k}{m'}} \leq \omega \leq \sqrt{2k \left(\frac{1}{m} + \frac{1}{m'} \right)}. \end{array}$$

Solution

(i)

Measure all displacements as arc lengths in the same sense around the ring. In the small-displacement approximation, a spring's extension is the difference between the displacements of its ends. Hooke's law therefore gives a tangential restoring force proportional to that difference. The constraint force has no tangential component.

For particle 1, the two springs exert forces $k(u_2 - u_1)$ and $k(u_3 - u_1)$. Newton's second law gives

$$m\ddot{u}_1 = k(u_2 - u_1) + k(u_3 - u_1).$$

Applying the same reasoning to particles 2 and 3,

$$\begin{array}{l} m\ddot{u}_1 = k(u_2 + u_3 - 2u_1), \\ m\ddot{u}_2 = k(u_3 + u_1 - 2u_2), \\ m\ddot{u}_3 = k(u_1 + u_2 - 2u_3). \end{array}$$

(ii)

Substitute $u_n = a_n \cos \omega t$, so that $\ddot{u}_n = -\omega^2 u_n$. Define

$$\lambda = \frac{\omega^2}{\omega_o^2}, \quad \omega_o^2 = \frac{k}{m}, \quad S = a_1 + a_2 + a_3.$$

Each equation becomes

$$\lambda a_n = 3a_n - S.$$

Adding the three equations gives

$$\lambda S = 0.$$

There are two possibilities.

If $\lambda = 0$, the individual equations require

$$a_1 = a_2 = a_3.$$

All masses move through the same arc length, so no spring changes its length. Thus

$$\omega = 0.$$

This is the rigid-rotation mode. Strictly, it is not an oscillation: its general motion is $u_1 = u_2 = u_3 = A + Bt$. The prescribed cosine form represents its static-displacement case.

If $\lambda \neq 0$, then $S = 0$, and a nonzero amplitude requires

$$\lambda = 3.$$

Therefore

$$\omega = \sqrt{3} \omega_o = \sqrt{\frac{3k}{m}}.$$

The condition $a_1 + a_2 + a_3 = 0$ leaves two independent amplitudes. For example, two independent mode shapes are

$$(a_1, a_2, a_3) \propto (1, -1, 0), \quad (a_1, a_2, a_3) \propto (1, 1, -2).$$

Consequently, the three normal modes have frequencies

$$\sqrt{3} \omega_o, \quad \sqrt{3} \omega_o, \quad 0,$$

with the nonzero frequency occurring twice.

(iii)

Particle n experiences the forces

$$k(u_{n+1} - u_n) \quad \text{and} \quad -k(u_n - u_{n-1}).$$

Hence

$$m\ddot{u}_n = k(u_{n+1} + u_{n-1} - 2u_n).$$

The ring imposes periodic boundary conditions,

$$u_0 = u_N, \quad u_{N+1} = u_1.$$

Write

$$q = \frac{2\pi s}{N}, \quad u_n = a_s \sin(nq + \phi) \cos \omega_s t.$$

This expression is periodic because $Nq = 2\pi s$. Using the supplied addition identity,

$$\begin{aligned} u_{n+1} + u_{n-1} &= a_s [\sin(nq + \phi + q) + \sin(nq + \phi - q)] \cos \omega_s t \\ &= 2 \cos q u_n. \end{aligned}$$

The equation of motion therefore becomes

$$-m\omega_s^2 u_n = 2k(\cos q - 1)u_n = -4k \sin^2 \left(\frac{q}{2} \right) u_n.$$

For a nontrivial mode,

$$\omega_s^2 = 4 \frac{k}{m} \sin^2 \left(\frac{\pi s}{N} \right).$$

Since $\sin(\pi s/N) \geq 0$ for the stated range $s = 1, \dots, N$,

$$\omega_s = 2\omega_o \sin \left(\frac{\pi s}{N} \right).$$

As the chain becomes infinite, the allowed wave numbers become continuous. The sine factor takes every value between zero and one, giving

$$0 \leq \omega \leq 2\omega_o = 2\sqrt{\frac{k}{m}}.$$

Zero is the rigid-displacement endpoint; genuine oscillations have $\omega > 0$. Since $f = \omega/(2\pi)$,

$$0 \leq f \leq \frac{1}{\pi} \sqrt{\frac{k}{m}}.$$

With SI values of k and m , angular frequencies are in rad s^{-1} , and ordinary frequencies are in Hz .

(iv)

At times and positions where the denominator is nonzero, the common time factor cancels:

$$\frac{u_n}{u_{n+1}} = \frac{\sin(nq + \phi)}{\sin(nq + \phi + q)}.$$

The same expression applies directly to the ratio of modal amplitudes.

(a) Low frequency

Choose the representative of q close to zero. Then

$$\omega = 2\omega_o \left| \sin \frac{q}{2} \right| \simeq \omega_o |q|,$$

so low frequency means $|q| \ll 1$. Writing $\theta = nq + \phi$,

$$\frac{u_n}{u_{n+1}} = \frac{1}{\cos q + \cot \theta \sin q} \simeq 1 - q \cot \theta.$$

Thus, away from nodes,

$$\frac{u_n}{u_{n+1}} \simeq +1.$$

Neighbouring masses have almost equal displacements and move approximately in phase. Near a node, the displacements themselves are small, so their ratio need not be close to one.

For the requested sketch, plot a sinusoid with wavelength $L \gg 1$, measured in particle spacings, and mark its values at integer particle numbers. A representative profile is

$$u_n = C(t) \sin \left(\frac{2\pi(n - n_0)}{L} \right).$$

Its key values are:

$n - n_0$	0	$L/4$	$L/2$	$3L/4$	L
$u_n/C(t)$	0	+1	0	-1	0

There are many particles between successive nodes, and neighbouring points usually have very similar heights.

(b) Maximum frequency

At the upper band edge, $q = \pi$, giving

$$\omega_{\max} = 2\omega_o.$$

The displacement is

$$u_n = a_s \sin(n\pi + \phi) \cos \omega_{\max} t = (-1)^n a_s \sin \phi \cos \omega_{\max} t.$$

Consequently,

$$u_{n+1} = -u_n, \quad \frac{u_n}{u_{n+1}} = -1.$$

Every neighbouring pair moves in opposite directions with equal amplitudes.

The sketch consists of alternating points at $+C(t)$ and $-C(t)$:

$n - n_0$	0	1	2	3	4
$u_n/C(t)$	+1	-1	+1	-1	+1

The spatial period is two particle spacings. If neighbouring points are connected as a visual guide, they form a zigzag.

This pattern fits an even ring exactly. For odd finite N , the largest allowed wave number is $\pi \pm \pi/N$, and

$$\omega_{\max} = 2\omega_o \cos\left(\frac{\pi}{2N}\right).$$

Its displacement pattern alternates in sign with a slowly varying envelope. For large N , the neighbouring-displacement ratio is approximately -1 away from envelope nodes.

(v)

First replace just one mass by $m' \ll m$. A rapid vibration of this light mass causes comparatively little motion of its heavy neighbours. To leading order, treat those neighbours as fixed.

Displacing the light mass by x produces a restoring force $-kx$ from each spring. Therefore

$$m'\ddot{x} \simeq -2kx,$$

giving

$$\omega_{\text{defect}} \simeq \sqrt{\frac{2k}{m'}}.$$

Relative to the original upper band edge,

$$\frac{\omega_{\text{defect}}}{2\omega_o} \simeq \sqrt{\frac{m}{2m'}} \gg 1.$$

The approximation is self-consistent. For a neighbouring heavy mass with amplitude u_1 , the leading high-frequency balance is

$$-m\omega_{\text{defect}}^2 u_1 \simeq kx,$$

so

$$\frac{u_1}{x} \simeq -\frac{m'}{2m}.$$

Its motion is small and opposite to that of the light mass. Motion farther away is smaller still: the high-frequency mode is localized around the defect.

Thus one mode moves far above the original frequency band. The other frequencies are perturbed but remain on the scale $\sqrt{k/m}$; in an infinite chain the extended-mode band remains $0 \leq \omega \leq 2\omega_o$. The zero-frequency rigid-displacement mode also remains.

For an **alternating chain**, the repeated light masses give a whole high-frequency band. The spectrum separates into:

- An **acoustic band**, containing low-frequency collective motion. At long wavelengths, neighbouring heavy and light masses move approximately together.
- An **optical band**, in which the light masses oscillate rapidly, with much smaller motion of the heavy masses. At long wavelengths, the two types of mass move in opposite directions.

The band edges can be derived directly. Let x_j be the displacement of heavy mass j , and y_j that of the light mass between x_j and x_{j+1} . Then

$$\begin{aligned} m\ddot{x}_j &= k(y_{j-1} + y_j - 2x_j), \\ m'\ddot{y}_j &= k(x_j + x_{j+1} - 2y_j). \end{aligned}$$

For a mode with phase change Q per two-particle cell, use

$$\begin{aligned} x_j &= X \cos(jQ + \phi) \cos \omega t, \\ y_j &= Y \cos\left((j + \frac{1}{2})Q + \phi\right) \cos \omega t. \end{aligned}$$

The amplitude equations are

$$\begin{aligned} (2k - m\omega^2)X &= 2k \cos(Q/2)Y, \\ (2k - m'\omega^2)Y &= 2k \cos(Q/2)X. \end{aligned}$$

A nonzero solution requires

$$(2k - m\omega^2)(2k - m'\omega^2) = 4k^2 \cos^2(Q/2),$$

or

$$mm'\omega^4 - 2k(m + m')\omega^2 + 4k^2 \sin^2(Q/2) = 0.$$

Its two roots are

$$\omega_{\pm}^2 = k \left(\frac{1}{m} + \frac{1}{m'} \right) \pm k \sqrt{\left(\frac{1}{m} + \frac{1}{m'} \right)^2 - \frac{4}{mm'} \sin^2(Q/2)}.$$

As $|Q|$ increases from 0 to π , the lower root increases and the upper root decreases. For $m' < m$, their ranges are therefore

$$0 \leq \omega_- \leq \sqrt{\frac{2k}{m}},$$

and

$$\sqrt{\frac{2k}{m'}} \leq \omega_+ \leq \sqrt{2k \left(\frac{1}{m} + \frac{1}{m'} \right)}.$$

There are no modes in the gap

$$\sqrt{\frac{2k}{m}} < \omega < \sqrt{\frac{2k}{m'}}.$$

For $m' \ll m$, the optical band is narrow relative to its high frequency, approximately $\sqrt{2k/m'}$, and is widely separated from the acoustic band. A finite periodic chain has discrete frequencies within these bands; they become continuous in the infinite-chain limit.

The solution was checked against the official solution.